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ON AN IDENTITY FOR THE CHEBYCHEV FUNCTIONAL AND SOME RAMIFICATIONS

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ABSTRACT. An identity for the Chebychev functional is presented in which a Riemann-Stieltjes integral is involved. This allows bounds for the functional to be obtained for functions that are of bounded variation, Lipschitzian and monotone. Some applications are presented to produce bounds for moments of functions about a general point γ and for moment generating functions.

Key words and phrases: Chebychev functional, Bounds, Riemann-Stieltjes, Moments, Moment Generating Function.

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1. INTRODUCTION

For two measurable functions $f, g : [a, b] \to \mathbb{R}$, define the functional, which is known in the literature as Chebychev's functional, by

$$T(f,g) := \mathcal{M}(fg) - \mathcal{M}(f)\,\mathcal{M}(g), \tag{1.1}$$

where the integral mean is given by

$$\mathcal{M}(f) = \frac{1}{b-a}\int_a^b f(x)\,dx. \tag{1.2}$$

The integrals in (1.1) are assumed to exist.

Further, the weighted Chebychev functional is defined by

$$\mathfrak{T}(f,g;p) := \mathfrak{M}(f,g;p) - \mathfrak{M}(f;p)\,\mathfrak{M}(g;p), \tag{1.3}$$

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where the weighted integral mean is given by

$$\mathfrak{M}(f;p)=\frac{\int_a^b p(x)\,f(x)\,dx}{\int_a^b p(x)\,dx}. \tag{1.4}$$

We note that,

$$\mathfrak{T}(f,g;1)\equiv T(f,g)$$

and

$$\mathfrak{M}(f;1)\equiv\mathcal{M}(f).$$

It is the aim of this article to obtain bounds on the functionals (1.1) and (1.3) in terms of one of the functions, say f, being of bounded variation, Lipschitzian or monotonic nondecreasing.

This is accomplished by developing identities involving a Riemann-Stieltjes integral. These identities seem to be new. The main results are obtained in Section 2, while in Section 3 bounds for moments about a general point γ are obtained for functions of bounded variation, Lipschitzian and monotonic. In a previous article, Cerone and Dragomir [2] obtained bounds in terms of the $\|f'\|_p$, $p\geq 1$ where it necessitated the differentiability of the function f. There is no need for such assumptions in the work covered by the current development. A further application is given in Section 4 in which the moment generating function is approximated.

2. AN IDENTITY FOR THE CHEBYCHEV FUNCTIONAL

It is worthwhile noting that a number of identities relating to the Chebychev functional already exist. The reader is referred to [7] Chapters IX and X. Korkine's identity is well known, see [7, p. 296] and is given by

$$T(f,g)=\frac{1}{2(b-a)^2}\int_a^b\int_a^b (f(x)-f(y))(g(x)-g(y))\,dxdy. \tag{2.1}$$

It is identity (2.1) that is often used to prove an inequality of Grüss for functions bounded above and below, [7].

The Grüss inequality is given by

$$|T(f,g)|\leq\frac{1}{4}(\Phi_f-\phi_f)(\Phi_g-\phi_g) \tag{2.2}$$

where $\phi_f\leq f(x)\leq\Phi_f$ for $x\in[a,b]$.

If we let $S(f)$ be an operator defined by

$$S(f)(x):=f(x)-\mathcal{M}(f), \tag{2.3}$$

which shifts a function by its integral mean, then the following identity holds. Namely,

$$T(f,g)=T(S(f),g)=T(f,S(g))=T(S(f),S(g)), \tag{2.4}$$

and so

$$T(f,g)=\mathcal{M}(S(f)\,g)=\mathcal{M}(fS(g))=\mathcal{M}(S(f)\,S(g)) \tag{2.5}$$

since $\mathcal{M}(S(f))=\mathcal{M}(S(g))=0$.

For the last term in (2.4) or (2.5) only one of the functions needs to be shifted by its integral mean. If the other were to be shifted by any other quantity, the identities would still hold. A weighted version of (2.5) related to $\mathfrak{T}(f,g)=\mathcal{M}((f(x)-\kappa)\,S(g))$ for κ arbitrary was given by Sonin [8] (see [7, p. 246]).

The interested reader is also referred to Dragomir [5] and Fink [6] for extensive treatments of the Grüss and related inequalities.

The following lemma presents an identity for the Chebychev functional that involves a Riemann-Stieltjes integral.

Lemma 2.1. *Let $f, g : [a,b] \to \mathbb{R}$, where f is of bounded variation and g is continuous on $[a,b]$, then*

$$T(f,g) = \frac{1}{(b-a)^2}\int_a^b \psi(t)\,df(t), \tag{2.6}$$

where

$$\psi(t) = (t-a)A(t,b) - (b-t)A(a,t) \tag{2.7}$$

with

$$A(a,b) = \int_a^b g(x)\,dx. \tag{2.8}$$

Proof. From (2.6) integrating the Riemann-Stieltjes integral by parts produces

$$\begin{aligned}\frac{1}{(b-a)^2}\int_a^b \psi(t)\,df(t) &= \frac{1}{(b-a)^2}\left\{\psi(t)f(t)\Big]_a^b - \int_a^b f(t)\,d\psi(t)\right\}\\ &= \frac{1}{(b-a)^2}\left\{\psi(b)f(b) - \psi(a)f(a) - \int_a^b f(t)\psi'(t)\,dt\right\}\end{aligned}$$

since $\psi(t)$ is differentiable. Thus, from (2.7), $\psi(a) = \psi(b) = 0$ and so

$$\begin{aligned}\frac{1}{(b-a)^2}\int_a^b \psi(t)\,df(t) &= \frac{1}{(b-a)^2}\int_a^b [(b-a)g(t) - A(a,b)]f(t)\,dt\\ &= \frac{1}{b-a}\int_a^b [g(t) - \mathcal{M}(g)]f(t)\,dt\\ &= \mathcal{M}(fS(g))\end{aligned}$$

from which the result (2.6) is obtained on noting identity (2.5). □

The following well known lemmas will prove useful and are stated here for lucidity.

Lemma 2.2. *Let $g, v : [a,b] \to \mathbb{R}$ be such that g is continuous and v is of bounded variation on $[a,b]$. Then the Riemann-Stieltjes integral $\int_a^b g(t)\,dv(t)$ exists and is such that*

$$\left|\int_a^b g(t)\,dv(t)\right| \le \sup_{t\in[a,b]}|g(t)| \bigvee_a^b (v), \tag{2.9}$$

where $\bigvee_a^b (v)$ is the total variation of v on $[a,b]$.

Lemma 2.3. *Let $g, v : [a,b] \to \mathbb{R}$ be such that g is Riemann-integrable on $[a,b]$ and v is $L-$Lipschitzian on $[a,b]$. Then*

$$\left|\int_a^b g(t)\,dv(t)\right| \le L\int_a^b |g(t)|\,dt \tag{2.10}$$

with v is $L-$Lipschitzian if it satisfies

$$|v(x) - v(y)| \le L|x-y|$$

for all $x, y \in [a,b]$.

Lemma 2.4. *Let $g, v : [a,b] \to \mathbb{R}$ be such that g is continuous on $[a,b]$ and v is monotonic nondecreasing on $[a,b]$. Then*

$$\left| \int_a^b g(t)\, dv(t) \right| \leq \int_a^b |g(t)|\, dv(t)\,. \tag{2.11}$$

It should be noted that if v is nonincreasing then $-v$ is nondecreasing.

Theorem 2.5. *Let $f, g : [a,b] \to \mathbb{R}$, where f is of bounded variation and g is continuous on $[a,b]$. Then*

$$(b-a)^2 |T(f,g)| \leq \begin{cases} \sup\limits_{t\in[a,b]} |\psi(t)| \bigvee\limits_a^b (f)\,, & \\ L \int_a^b |\psi(t)|\, dt, & \text{for } f\ L-\text{Lipschitzian}, \\ \int_a^b |\psi(t)|\, df(t)\,, & \text{for } f \text{ monotonic nondecreasing}, \end{cases} \tag{2.12}$$

where $\bigvee\limits_a^b (f)$ is the total variation of f on $[a,b]$.

Proof. Follows directly from Lemmas 2.1 – 2.4. That is, from the identity (2.6) and (2.9) – (2.11). ∎

The following lemma gives an identity for the weighted Chebychev functional that involves a Riemann-Stieltjes integral.

Lemma 2.6. *Let $f, g, p : [a,b] \to \mathbb{R}$, where f is of bounded variation and g, p are continuous on $[a,b]$. Further, let $P(b) = \int_a^b p(x)\, dx > 0$, then*

$$\mathfrak{T}(f,g;p) = \frac{1}{P^2(b)} \int_a^b \Psi(t)\, df(t)\,, \tag{2.13}$$

where $\mathfrak{T}(f,g;p)$ is as given in (1.3),

$$\Psi(t) = P(t)\,\bar{G}(t) - \bar{P}(t)\, G(t) \tag{2.14}$$

with

$$\begin{cases} P(t) = \int_a^t p(x)\, dx, & \bar{P}(t) = P(b) - P(t) \\ \text{and} & \\ G(t) = \int_a^t p(x)\, g(x)\, dx, & \bar{G}(t) = G(b) - G(t)\,. \end{cases} \tag{2.15}$$

Proof. The proof follows closely that of Lemma 2.1.

We first note that $\Psi(t)$ may be represented in terms of only $P(\cdot)$ and $G(\cdot)$. Namely,

$$\Psi(t) = P(t)\, G(b) - P(b)\, G(t)\,. \tag{2.16}$$

It may further be noticed that $\Psi(a) = \Psi(b) = 0$. Thus, integrating from (2.13) and using either (2.14) or (2.16) gives

$$\begin{aligned}
\frac{1}{P^2(b)} \int_a^b \Psi(t)\, df(t) &= \frac{-1}{P^2(b)} \int_a^b f(t)\, d\Psi(t) \\
&= \frac{1}{P^2(b)} \int_a^b \left[P(b) G'(t) - P'(t) G(b)\right] f(t)\, dt \\
&= \frac{1}{P(b)} \int_a^b \left[p(t) g(t) - \frac{G(b)}{P(b)} p(t)\right] f(t)\, dt \\
&= \frac{1}{P(b)} \int_a^b p(t) g(t) f(t)\, dt - \frac{G(b)}{P(b)} \cdot \frac{1}{P(b)} \int_a^b p(t) f(t)\, dt \\
&= \mathfrak{M}(f,g;p) - \mathfrak{M}(g;p)\,\mathfrak{M}(f;p) \\
&= \mathfrak{T}(f,g;p),
\end{aligned}$$

where we have used the fact that

$$\frac{G(b)}{P(b)} = \mathfrak{M}(g;p).$$

□

Theorem 2.7. *Let the conditions of Lemma 2.6 on f, g and p continue to hold. Then*

$$P^2(b)\,|\mathfrak{T}(f,g;p)| \le \begin{cases} \sup\limits_{t\in[a,b]} |\Psi(t)| \bigvee\limits_a^b (f), \\[2ex] L \int_a^b |\Psi(t)|\, dt, & \textit{for } f\ L-\textit{Lipschitzian}, \\[2ex] \int_a^b |\Psi(t)|\, df(t), & \textit{for } f \textit{ monotonic nondecreasing.} \end{cases} \tag{2.17}$$

where $\mathfrak{T}(f,g;p)$ is as given by (1.3) and $\Psi(t) = P(t)G(b) - P(b)G(t)$, with $P(t) = \int_a^t p(x)\,dx$, $G(t) = \int_a^t p(x) g(x)\,dx$.

Proof. The proof uses Lemmas 2.1 – 2.4 and follows closely that of Theorem 2.5. □

Remark 2.8. If we take $p(x) \equiv 1$ in the above results involving the weighted Chebychev functional, then the results obtained earlier for the unweighted Chebychev functional are recaptured.

Grüss type inequalities obtained from bounds on the Chebychev functional have been applied in a variety of areas including in obtaining perturbed rules in numerical integration, see for example [4]. In the following section the above work will be applied to the approximation of moments. For other related results see also [1] and [3].

Remark 2.9. If f is differentiable then the identity (2.6) would become

$$T(f,g) = \frac{1}{(b-a)^2} \int_a^b \psi(t) f'(t)\, dt \tag{2.18}$$

and so

$$(b-a)^2 |T(f,g)| \le \begin{cases} \|\psi\|_1 \|f'\|_\infty, & f' \in L_\infty[a,b]; \\[2ex] \|\psi\|_q \|f'\|_p, & f' \in L_p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \|\psi\|_\infty \|f'\|_1, & f' \in L_1[a,b]; \end{cases}$$

where the Lebesgue norms $\|\cdot\|$ are defined in the usual way as

$$\|g\|_p := \left(\int_a^b |g(t)|^p \, dt\right)^{\frac{1}{p}}, \qquad \text{for } g \in L_p[a,b], \ p \geq 1, \ \frac{1}{p} + \frac{1}{q} = 1$$

and

$$\|g\|_\infty := ess \sup_{t\in[a,b]} |g(t)|, \qquad \text{for } g \in L_\infty[a,b].$$

The identity for the weighted integral means (2.13) and the corresponding bounds (2.17) will not be examined further here.

Theorem 2.10. *Let $g : [a,b] \to \mathbb{R}$ be absolutely continuous on $[a,b]$ then for*

$$D(g;a,t,b) := \mathcal{M}(g;t,b) - \mathcal{M}(g;a,t), \tag{2.19}$$

$$|D(g;a,t,b)| \leq \begin{cases} \left(\dfrac{b-a}{2}\right)\|g'\|_\infty, & g' \in L_\infty[a,b]; \\[2ex] \left[\dfrac{(t-a)^q + (b-t)^q}{q+1}\right]^{\frac{1}{q}} \|g'\|_p, & g' \in L_p[a,b], \\ & p > 1, \ \frac{1}{p} + \frac{1}{q} = 1; \\ \|g'\|_1, & g' \in L_1[a,b]; \\[2ex] \bigvee_a^b(g), & g \ \textit{of bounded variation;} \\[2ex] \left(\dfrac{b-a}{2}\right)L, & g \ \textit{is } L-\textit{Lipschitzian.} \end{cases} \tag{2.20}$$

Proof. Let the kernel $r(t,u)$ be defined by

$$r(t,u) := \begin{cases} \dfrac{u-a}{t-a}, & u \in [a,t], \\[2ex] \dfrac{b-u}{b-t}, & u \in (t,b] \end{cases} \tag{2.21}$$

then a straight forward integration by parts argument of the Riemann-Stieltjes integral over each of the intervals $[a,t]$ and $(t,b]$ gives the identity

$$\int_a^b r(t,u)\, dg(u) = D(g;a,t,b). \tag{2.22}$$

Now for g absolutely continuous then

$$D(g;a,t,b) = \int_a^b r(t,u)\, g'(u)\, du \tag{2.23}$$

and so

$$|D(g;a,t,b)| \leq ess \sup_{u\in[a,b]} |r(t,u)| \int_a^b |g'(u)|\, du, \quad \text{for } g' \in L_1[a,b],$$

where from (2.21)

$$ess \sup_{u\in[a,b]} |r(t,u)| = 1 \tag{2.24}$$

and so the third inequality in (2.20) results. Further, using the Hölder inequality gives

$$|D(g;a,t,b)| \leq \left(\int_a^b |r(t,u)|^q du\right)^{\frac{1}{q}} \left(\int_a^b |g'(t)|^p dt\right)^{\frac{1}{p}} \quad (2.25)$$
$$\text{for } p > 1,\ \frac{1}{p}+\frac{1}{q}=1,$$

where explicitly from (2.21)

$$\begin{aligned}\left(\int_a^b |r(t,u)|^q du\right)^{\frac{1}{q}} &= \left[\int_a^t \left(\frac{u-a}{t-a}\right)^q du + \int_t^b \left(\frac{b-u}{b-t}\right)^q du\right]^{\frac{1}{q}} \\ &= \left[(t-a)^q+(b-t)^q\right]^{\frac{1}{q}} \left(\int_0^1 u^q du\right)^{\frac{1}{q}} \\ &= \left[\frac{(t-a)^q+(b-t)^q}{q+1}\right]^{\frac{1}{q}}.\end{aligned} \quad (2.26)$$

Also

$$|D(g;a,t,b)| \leq ess \sup_{u\in[a,b]} |g'(u)| \int_a^b |r(t,u)|\, du, \quad (2.27)$$

and so from (2.26) with $q=1$ gives the first inequality in (2.20).

Now, for $g(u)$ of bounded variation on $[a,b]$ then from Lemma 2.2, equation (2.9) and identity (2.22) gives

$$|D(g;a,t,b)| \leq ess \sup_{u\in[a,b]} |r(t,u)| \bigvee_a^b (g)$$

producing the fourth inequality in (2.20) on using (2.24). From (2.10) and (2.22) we have, by associating g with v and $r(t,\cdot)$ with $g(\cdot)$,

$$|D(g;a,t,b)| \leq L \int_a^b |r(t,u)|\, du$$

and so from (2.26) with $q=1$ gives the final inequality in (2.20). □

Remark 2.11. The results of Theorem 2.10 may be used to obtain bounds on $\psi(t)$ since from (2.7) and (2.19)

$$\psi(t) = (t-a)(b-t)\, D(g;a,t,b).$$

Hence, upper bounds on the Chebychev functional may be obtained from (2.12) and (2.18) for general functions g. The following two sections investigate the exact evaluation (2.12) for specific functions for $g(\cdot)$.

3. RESULTS INVOLVING MOMENTS

In this section bounds on the n^{th} moment about a point γ are investigated. Define for n a nonnegative integer,

$$M_n(\gamma) := \int_a^b (x-\gamma)^n h(x)\, dx, \quad \gamma \in \mathbb{R}. \quad (3.1)$$

If $\gamma = 0$ then $M_n(0)$ are the moments about the origin while taking $\gamma = M_1(0)$ gives the central moments. Further the expectation of a continuous random variable is given by

$$E(X) = \int_a^b h(x)\, dx, \quad (3.2)$$

where $h(x)$ is the probability density function of the random variable X and so $E(X) = M_1(0)$. Also, the variance of the random variable X, $\sigma^2(X)$ is given by

$$\sigma^2(X) = E\left[(X - E(X))^2\right] = \int_a^b (x - E(X))^2 h(x)\,dx, \tag{3.3}$$

which may be seen to be the second moment about the mean, namely

$$\sigma^2(X) = M_2(M_1(0)).$$

The following corollary is valid.

Corollary 3.1. *Let $f : [a,b] \to \mathbb{R}$ be integrable on $[a,b]$, then*

$$\left| M_n(\gamma) - \frac{B^{n+1} - A^{n+1}}{n+1}\mathcal{M}(f) \right| \tag{3.4}$$
$$\leq \begin{cases} \sup_{t\in[a,b]} |\phi(t)| \cdot \frac{1}{n+1} \bigvee_a^b (f), & \text{for } f \text{ of bounded variation on } [a,b], \\ \dfrac{L}{n+1} \int_a^b |\phi(t)|\,dt, & \text{for } f \ L-\text{Lipschitzian}, \\ \dfrac{1}{n+1} \int_a^b |\phi(t)|\,df(t), & \text{for } f \text{ monotonic nondecreasing.} \end{cases}$$

where $M_n(\gamma)$ is as given by (3.1), $\mathcal{M}(f)$ is the integral mean of f as defined in (1.2),

$$B = b - \gamma, \quad A = a - \gamma$$

and

$$\phi(t) = (t-\gamma)^n - \left[\left(\frac{t-a}{b-a}\right)(b-\gamma)^{n+1} + \left(\frac{b-t}{b-a}\right)(a-\gamma)^{n+1}\right]. \tag{3.5}$$

Proof. From (2.12) taking $g(t) = (t-\gamma)^n$ then using (1.1) and (1.2) gives

$$(b-a)\left|T\left(f,(t-\gamma)^n\right)\right| = \left| M_n(\gamma) - \frac{B^{n+1} - A^{n+1}}{n+1}\mathcal{M}(f) \right|.$$

The right hand side is obtained on noting that for $g(t) = (t-\gamma)^n$, $\phi(t) = -\frac{\psi(t)}{b-a}$. □

Remark 3.2. It should be noted here that Cerone and Dragomir [2] obtained bounds on the left hand expression for $f' \in L_p[a,b]$, $p \geq 1$. They obtained the following Lemmas which will prove useful in procuring expressions for the bounds in (3.4) in a more explicit form.

Lemma 3.3. *Let $\phi(t)$ be as defined by (3.5), then*

$$\phi(t) \begin{cases} < 0 & \begin{cases} n \text{ odd}, & \text{any } \gamma \text{ and } t \in (a,b) \\ n \text{ even} & \begin{cases} \gamma < a, & t \in (a,b) \\ a < \gamma < b, & t \in [c,b) \end{cases} \end{cases} \\ > 0, & n \text{ even} \begin{cases} \gamma > b, & t \in (a,b) \\ a < \gamma < b, & t \in (a,c) \end{cases} \end{cases} \tag{3.6}$$

where $\phi(c) = 0$, $a < c < b$ *and*

$$c \begin{cases} > \gamma, & \gamma < \frac{a+b}{2} \\ = \gamma, & \gamma = \frac{a+b}{2} \\ < \gamma, & \gamma > \frac{a+b}{2}. \end{cases}$$

Lemma 3.4. *For* $\phi(t)$ *as given by (3.5) then*

$$\int_a^b |\phi(t)|\, dt \tag{3.7}$$

$$= \begin{cases} \frac{B-A}{2}\left[B^{n+1} - A^{n+1}\right] - \frac{B^{n+2}-A^{n+2}}{n+2}, & \begin{cases} n \text{ odd and any } \gamma \\ n \text{ even and } \gamma < a \end{cases}; \\ \frac{2C^{n+2}-B^{n+2}-A^{n+2}}{n+2} + \frac{1}{2(b-a)}\left\{\left[(b-a)^2 - 2(c-a)^2\right]B^{n+1} \right. \\ \qquad \left. + \left[2(b-c)^2 - (b-a)^2\right]\right\} A^{n+1}, & n \text{ even and } a < \gamma < b; \\ \frac{B^{n+2}-A^{n+2}}{n+2} - \frac{B-A}{2}\left[B^{n+1} - A^{n+1}\right], & n \text{ even and } \gamma > b, \end{cases}$$

where

$$\begin{cases} B = b - \gamma,\ A = a - \gamma,\ C = c - \gamma, \\ C_1 = \int_a^c C(t)\, dt, \quad C_2 = \int_c^b C(t)\, dt, \\ \text{with} \quad C(t) = \left(\frac{t-a}{b-a}\right) B^{n+1} + \left(\frac{b-t}{b-a}\right) A^{n+1} \end{cases} \tag{3.8}$$

and $\phi(c) = 0$ *with* $a < c < b$*.*

Lemma 3.5. *For* $\phi(t)$ *as defined by (3.5), then*

$$\sup_{t\in[a,b]} \left|\tilde{\phi}(t)\right| = \begin{cases} C(t^*) - \frac{B^{n+1}-A^{n+1}}{(n+1)(B-A)}, & n \text{ odd, } n \text{ even and } \gamma < a; \\ \frac{B^{n+1}-A^{n+1}}{(n+1)(B-A)} - C(t^*) & n \text{ even and } \gamma > b; \\ \frac{m_1+m_2}{2} + \left|\frac{m_1-m_2}{2}\right| & n \text{ even and } a < \gamma < b, \end{cases} \tag{3.9}$$

where

$$(t^* - \gamma)^n = \frac{B^{n+1} - A^{n+1}}{(n+1)(B-A)}, \tag{3.10}$$

$C(t)$ *is as defined in (3.8),* $m_1 = \tilde{\phi}(t_1^*)$*,* $m_2 = -\tilde{\phi}(t_2^*)$ *and* t^**,* t_1^**,* t_2^* *satisfy (3.10) with* $t_1^* < t_2^*$*.*

The following lemma is required to determine the bound in (3.4) when f is monotonic non-decreasing. This was not covered in Cerone and Dragomir [2] since they obtained bounds assuming that f were differentiable.

Lemma 3.6. *The following result holds for* $\phi(t)$ *as defined by (3.5),*

$$\frac{1}{n+1}\int_a^b |\phi(t)|\, df = \begin{cases} \chi_n(a,b), & n \text{ odd or } n \text{ even and } \gamma < a, \\ -\chi_n(a,b), & n \text{ even and } \gamma > b, \\ \chi_n(c,b) - \chi_n(a,c), & n \text{ even and } a < \gamma < b \end{cases} \tag{3.11}$$

and for $f : [a,b] \to \mathbb{R}$*, monotonic nondecreasing*

$$\frac{1}{n+1}\int_a^b |\phi(t)|\,df \tag{3.12}$$

$$\leq \begin{cases} \dfrac{B(B^n-1)-A(A^n-1)}{n+1} f(b), & n \text{ odd or } n \text{ even and } \gamma < a; \\ \dfrac{A(A^n-1)-B(B^n-1)}{n+1} f(b), & n \text{ even and } \gamma > b; \\ \left[B^{n+1}-C^{n+1}-\dfrac{(B^n-A^n)}{b-a}(b-c)\right]\dfrac{f(b)}{n+1} & n \text{ even and} \\ +\left[\dfrac{(B^n-A^n)}{b-a}(c-a)-(C^{n+1}-A^{n+1})\right]\dfrac{f(a)}{n+1}, & a<\gamma<b, \end{cases}$$

where

$$\begin{aligned} \chi_n(a,b) &= \int_a^b \left[(t-\gamma)^n - \frac{(B^n-A^n)}{(n+1)(b-a)}\right] f(t)\,dt, \\ A &= a-\gamma,\quad B=b-\gamma,\quad C=c-\gamma. \end{aligned} \tag{3.13}$$

Proof. Let $\alpha,\beta \in [a,b]$ and

$$\begin{aligned} \chi_n(\alpha,\beta) &= \frac{1}{n+1}\int_\alpha^\beta |\phi(t)|\,df \\ &= \frac{\phi(\alpha) f(\alpha) - \phi(\beta) f(\beta)}{n+1} - \int_\alpha^\beta \left[(t-\gamma)^n - \frac{(B^n-A^n)}{(n+1)(b-a)}\right] f(t)\,dt \end{aligned}$$

and $\chi_n(a,b)$ is as given by (3.13) since $\phi(a)=\phi(b)=0$.

Further, using the results of Lemma 3.3 as represented in (3.6), and, the fact that

$$\frac{1}{n+1}\int_\alpha^\beta |\phi(t)|\,df = \begin{cases} \chi(\alpha,\beta), & \phi(t)<0,\ t\in[\alpha,\beta] \\ -\chi(\alpha,\beta), & \phi(t)>0,\ t\in[\alpha,\beta] \end{cases}$$

gives the results as stated.

We now use the fact that f is monotonic nondecreasing so that from (3.13)

$$\chi_n(a,b) \leq f(b)\int_a^b \left[(t-\gamma)^n - \frac{B^n-A^n}{(n+1)(b-a)}\right] dt.$$

Further,

$$\begin{aligned} \chi_n(c,b) &\leq f(b)\int_c^b \left[(t-\gamma)^n - \frac{B^n-A^n}{(n+1)(b-a)}\right] dt \\ &= f(b)\left[\frac{B^{n+1}-C^{n+1}}{n+1} - \frac{(B^n-A^n)(b-c)}{(n+1)(b-a)}\right] \end{aligned}$$

and

$$\begin{aligned} \chi_n(a,c) &\geq f(a)\int_a^c \left[(t-\gamma)^n - \frac{B^n-A^n}{(n+1)(b-a)}\right] dt \\ &= \left[\frac{C^{n+1}-A^{n+1}}{n+1} - \frac{(B^n-A^n)(c-a)}{(n+1)(b-a)}\right] f(a) \end{aligned}$$

so that the proof of the lemma is now complete. □

The following corollary gives bounds for the expectation.

Corollary 3.7. *Let $f : [a,b] \to \mathbb{R}_+$ be a probability density function associated with a random variable X. Then the expectation $E(X)$ satisfies the inequalities*

$$\left|E(X) - \frac{a+b}{2}\right| \leq \begin{cases} \dfrac{(b-a)^3}{6} \bigvee_a^b (f), & f \text{ of bounded variation,} \\ \left(\dfrac{b-a}{2}\right)^2 \cdot \dfrac{L}{2}, & f\ L-\text{Lipschitzian,} \\ \dfrac{b-a}{2}[a+b-1] f(b), & f \text{ monotonic nondecreasing.} \end{cases} \tag{3.14}$$

Proof. Taking $n = 1$ in Corollary 3.1 and using Lemmas 3.3 – 3.6 gives the results after some straightforward algebra. In particular,

$$\phi(t) = t^2 - (a+b)t + ab = \left(t - \frac{a+b}{2}\right)^2 + \left(\frac{b-a}{2}\right)^2$$

and t^* the one solution of $\phi'(t) = 0$ is $t^* = \frac{a+b}{2}$. □

The following corollary gives bounds for the variance.

We shall assume that $a < \gamma = E[X] < b$.

Corollary 3.8. *Let $f : [a,b] \to \mathbb{R}_+$ be a p.d.f. associated with a random variable X. The variance $\sigma^2(X)$ is such that*

$$\left|\sigma^2(X) - S\right| \leq \begin{cases} [m_1 + m_2 + |m_2 - m_1|] \frac{\bigvee_a^b(f)}{6}, & f \text{ of bounded variation,} \\ \left\{\frac{C^2}{4} - \frac{1}{b-a}\left[(c-a)^3 B^3 - (b-c)^2 A^3\right] \right. \\ \quad \left. + (B^2 + A^2)\left(\frac{b-a}{2}\right)^2 - \frac{(AB)^2}{2}\right\} \cdot \frac{L}{3}, & f \text{ is } L-\text{Lipschitzian,} \\ [B^3 - C^3 - (a+b)(b-c)] \frac{f(b)}{3} \\ \quad + [(a+b)(c-a) - (C^3 - A^3)] \frac{f(a)}{3}, & f \text{ monotonic nondecreasing.} \end{cases} \tag{3.15}$$

where

$$\begin{aligned} S &= \frac{(b - E(X))^3 + (E(X) - a)^3}{3(b-a)}, \\ m_1 &= \phi\left(E(X) - S^{\frac{1}{2}}\right), \quad m_2 = \phi\left(E(X) + S^{\frac{1}{2}}\right), \\ \phi(t) &= (t-\gamma)^3 + \left(\frac{b-t}{b-a}\right)(\gamma - a)^3 - \left(\frac{t-a}{b-a}\right)(b-\gamma)^3, \\ A &= a - \gamma, \quad B = b - \gamma, \quad C = c - \gamma, \quad \phi(c) = 0, \ a < c < b \end{aligned}$$

and $\gamma = E(X)$.

Proof. Taking $n = 2$ in Corollary 3.1 gives from (3.5)

$$\phi(t) = (t-\gamma)^3 + \left(\frac{b-t}{b-a}\right) A^3 - \left(\frac{t-a}{b-a}\right) B^3$$

where $a < \gamma = E(X) < b$.

From Lemma 3.5 and the third inequality in (3.9) with $n = 2$ gives

$$t_1^* = E[X] - S^{\frac{1}{2}}, \quad t_2^* = E[X] + S^{\frac{1}{2}},$$

and hence the first inequality is shown from the first inequality in (3.4).

Now, if f is Lipschitzian, then from the second inequality in (3.4) and since $n = 2$ and $a < \gamma = E(X) < b$, the second identity in (3.7) produces the reported result given in (3.15) after some simplification.

The last inequality is obtained from (3.12) of Lemma 3.6 with $n = 2$ and hence the corollary is proved. □

4. APPROXIMATIONS FOR THE MOMENT GENERATING FUNCTION

Let X be a random variable on $[a, b]$ with probability density function $h(x)$ then the moment generating function $M_X(p)$ is given by

$$M_X(p) = E\left[e^{pX}\right] = \int_a^b e^{px} h(x)\, dx. \tag{4.1}$$

The following lemma will prove useful, in the proof of the subsequent corollary, as it examines the behaviour of the function $\theta(t)$

$$(b-a)\,\theta(t) = tA_p(a,b) - \left[aA_p(t,b) + bA_p(a,t)\right], \tag{4.2}$$

where

$$A_p(a,b) = \frac{e^{bp} - e^{ap}}{p}. \tag{4.3}$$

Lemma 4.1. *Let $\theta(t)$ be as defined by (4.2) and (4.3) then for any $a, b \in \mathbb{R}$, $\theta(t)$ has the following characteristics:*

(i) $\theta(a) = \theta(b) = 0$,

(ii) *$\theta(t)$ is convex for $p < 0$ and concave for $p > 0$,*

(iii) *there is one turning point at $t^* = \frac{1}{p}\ln\left(\frac{A_p(a,b)}{b-a}\right)$ and $a \le t^* \le b$.*

Proof. The result (i) is trivial from (4.2) using standard properties of the definite integral to give $\theta(a) = \theta(b) = 0$.

Now,

$$\theta'(t) = \frac{A_p(a,b)}{b-a} - e^{pt}, \quad \theta''(t) = -pe^{pt} \tag{4.4}$$

giving $\theta''(t) > 0$ for $p < 0$ and $\theta''(t) < 0$ for $p > 0$ and (ii) holds.

Further, from (4.4) $\theta'(t^*) = 0$ where

$$t^* = \frac{1}{p}\ln\left(\frac{A_p(a,b)}{b-a}\right).$$

To show that $a \le t^* \le b$ it suffices to show that

$$\theta'(a)\,\theta'(b) < 0$$

since the exponential is continuous. Here $\theta'(a)$ is the right derivative at a and $\theta'(b)$ is the left derivative at b.

Now,

$$\theta'(a)\,\theta'(b) = \left(\frac{A_p(a,b)}{b-a} - e^{ap}\right)\left(\frac{A_p(a,b)}{b-a} - e^{bp}\right)$$

but

$$\frac{A_p(a,b)}{b-a} = \frac{1}{b-a}\int_a^b e^{pt}dt,$$

the integral mean over $[a,b]$ so that $\theta'(a) > 0$, and $\theta'(b) < 0$ for $p > 0$ and $\theta'(a) < 0$ and $\theta'(b) > 0$ for $p < 0$, giving that there is a point $t^* \in [a,b]$ where $\theta(t^*) = 0$.

Thus the lemma is now completely proved. □

Corollary 4.2. *Let $f : [a,b] \to \mathbb{R}$ be of bounded variation on $[a,b]$ then*

$$(4.5)\quad \left|\int_a^b e^{pt} f(t)\,dt - A_p(a,b)\,\mathcal{M}(f)\right| \le \begin{cases} \left(m(\ln(m)-1) + \dfrac{be^{ap}-ae^{bp}}{b-a}\right)\dfrac{\bigvee_a^b(f)}{|p|}, \\ (b-a)\,m\left[\left(\dfrac{b-a}{2}\right)p-1\right]\dfrac{L}{|p|} \quad \text{for } f\ L-\text{Lipschitzian on } [a,b], \\ \dfrac{p}{|p|}(b-a)\,m\,[f(b)-f(a)], \qquad f \text{ monotonic nondecreasing,} \end{cases}$$

where

$$(4.6)\quad m = \frac{A_p(a,b)}{b-a} = \frac{e^{bp}-e^{ap}}{p(b-a)}.$$

Proof. From (2.12) taking $g(t) = e^{pt}$ and using (1.1) and (1.2) gives

$$(4.7)\quad (b-a)\left|T\left(f,e^{pt}\right)\right| = \left|\int_a^b e^{pt} f(t)\,dt - A_p(a,b)\,\mathcal{M}(f)\right| \le \begin{cases} \sup\limits_{t\in[a,b]}|\theta(t)|\bigvee_a^b(f), & \text{for } f \text{ of bounded variation on } [a,b], \\ L\int_a^b|\theta(t)|\,dt, & \text{for } f\ L-\text{Lipschitzian on } [a,b], \\ \int_a^b|\theta(t)|\,df(t), & f \text{ monotonic nondecreasing on } [a,b], \end{cases}$$

where the bounds are obtained from (2.12) on noting that for $g(t) = e^{pt}$, $\theta(t) = \frac{\psi(t)}{b-a}$ is as given by (4.2) – (4.3).

Now, using the properties of $\theta(t)$ as expounded in Lemma 4.1 will aid in obtaining explicit bounds from (4.7).

Firstly, from (4.2), (4.3) and (4.6)

$$\begin{aligned} \sup_{t\in[a,b]}|\theta(t)| &= |\theta(t^*)| \\ &= \left|t^*m - \left[a\frac{A_p(t^*,b)}{b-a} + b\frac{A_p(a,t^*)}{b-a}\right]\right| \\ &= \left|\frac{m}{p}\ln(m) - \frac{a}{p}\left(\frac{e^{bp}-m}{b-a}\right) - \frac{b}{p}\left(\frac{m-e^{ap}}{b-a}\right)\right| \\ &= \left|\frac{m}{p}(\ln(m)-1) + \frac{be^{ap}-ae^{bp}}{p(b-a)}\right|. \end{aligned}$$

In the above we have used the fact that $m \geq 0$ and that $pt^* = \ln(m)$. Using from Lemma 4.1 the result that $\theta(t)$ is positive or negative for $t \in [a,b]$ depending on whether $p > 0$ or $p < 0$ respectively, the first inequality in (4.5) results.

For the second inequality we have that from (4.2), (4.3) and Lemma 4.1,

$$\begin{aligned}
\int_a^b |\theta(t)|\, dt &= \frac{1}{|p|} \int_a^b \left[pmt - \frac{a\left(e^{bp} - e^{tp}\right) + b\left(e^{tp} - e^{ap}\right)}{b-a} \right] dt \\
&= \frac{1}{|p|} \left[pm\left(\frac{b^2 - a^2}{2}\right) - \left(ae^{bp} - be^{ap}\right) - \int_a^b e^{pt} dt \right] \\
&= \frac{1}{|p|} \left[pm\left(\frac{b^2 - a^2}{2}\right) - \left(ae^{bp} - be^{ap}\right) - (b-a)\, m \right] \\
&= \frac{1}{|p|} \left[(b-a)\, m \left(\frac{a+b}{2} p - 1\right) - \left(ae^{bp} - be^{ap}\right) \right] \\
&= \frac{1}{|p|} \left[\frac{e^{bp} - e^{ap}}{p} \left(\frac{a+b}{2} p - 1\right) - \left(ae^{bp} - be^{ap}\right) \right] \\
&= \frac{1}{|p|} \left(e^{bp} - e^{ap}\right) \left(\frac{b-a}{2} - \frac{1}{p}\right).
\end{aligned}$$

Using (4.6) gives the second result in (4.5) as stated.

For the final inequality in (4.5) we need to determine $\int_a^b |\theta(t)|\, df(t)$ for f monotonic nondecreasing. Now, from (4.2) and (4.3)

$$\begin{aligned}
\int_a^b |\theta(t)|\, df(t) &= \int_a^b \left[mt - \frac{be^{ap} - ae^{bp}}{p(b-a)} - \frac{e^{pt}}{p} \right] df(t) \\
&= \frac{1}{|p|} \int_a^b \left[pmt + \frac{be^{ap} - ae^{bp}}{b-a} - e^{pt} \right] df(t),
\end{aligned}$$

where we have used the fact that $sgn(\theta(t)) = sgn(p)$.

Integration by parts of the Riemann-Stieltjes integral gives

$$\begin{aligned}
&\int_a^b |\theta(t)|\, df(t) \\
&= \frac{1}{|p|} \left\{ \left(pmt + \frac{be^{ap} - ae^{bp}}{b-a} - e^{pt} \right) f(t) \Big]_a^b - p \int_a^b \left[m - e^{pt} \right] f(t)\, dt \right\} \\
&= \frac{p}{|p|} \int_a^b \left(e^{pt} - m \right) f(t)\, dt.
\end{aligned} \tag{4.8}$$

Now,

$$\int_a^b e^{tp} f(t)\, dt \leq f(b) \int_a^b e^{tp} dt = \frac{e^{bp} - e^{ap}}{p} f(b) = (b-a)\, m f(b)$$

and

$$-m \int_a^b f(t)\, dt \leq -m(b-a) f(a)$$

so that combining with (4.8) gives the inequalities for f monotonic nondecreasing. □

Remark 4.3. If f is a probability density function then $\mathcal{M}(f) = \frac{1}{b-a}$ and f is non-negative.

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