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COERCIVENESS INEQUALITY FOR NONLOCAL BOUNDARY VALUE PROBLEMS FOR SECOND ORDER ABSTRACT ELLIPTIC DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper we give conditions which assure the coercive solvability of an abstract differential equation of elliptic type with an operator in the boundary conditions, and the completeness of generalized eigenfunctions. We apply the abstract result to show that a non regular boundary value problem for a second order partial differential equation of an elliptic type in a cylindrical domain is coercive solvable.

Key words and phrases: Ellipticity, Abstract Differential Equation, Coerciveness, Non Regular Elliptic Problems.

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1. INTRODUCTION

Many works are devoted to the study of hyperbolic or parabolic abstract equations [16, 18, 9]. In [16, 20] regular boundary value problems for elliptic abstract equations are considered. A few works are concerned with non regular problems.

In this paper we establish conditions guaranteeing that non local boundary value problem for elliptic abstract differential equation of the second order in an interval is coercive solvable in the Hilbert space $L_2(0,1;H)$. A coercive estimates, when the problem is regular, was proved in [1, 3]. The considered problem is not regular, since the boundary conditions are non local, similar problems have been considered in [4, 5, 7, 22]. Moreover, we prove the completeness of root functions. The completeness of root functions for regular boundary value problems are proved in [1, 6, 10, 14] and in the book [22].

The obtained results are then applied to study a non local boundary value problem for the Laplace equation in a cylinder.

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2. NOTATIONS AND DEFINITIONS

Let H be a Hilbert space, A a linear closed operator in H and $D(A)$ its domain. We denote by $L(H)$ the space of bounded linear operators acting on H, with the usual operator norm, and by $L_p(0,1;H)$ the space of strongly measurable functions $x \to u(x) : [0,1] \to H$, whose $p^{th}-$power are summable, with the norm

$$\|u\|_{0,p}^p = \|u\|_{L_p(0,1;H)}^p = \int_0^1 \|u(x)\|_H^p \, dx < \infty, \;\; p \in [1, \infty] .$$

Now, introduce the $L_p(0,1;H)$ vector-valued Sobolev spaces $W_p^2(0,1;H_1,H)$, where H_1, H are Hilbert spaces such that $H_1 \subset H$ with continuous embedding

$$W_p^2(0,1;H_1,H) = \{u : u'' \in L_p(0,1;H) \text{ and } u \in L_p(0,1;H_1)\}$$

and

$$\|u\|_{W_p^2(0,1;H_1,H)} = \|u\|_{L_p(0,1;H_1)} + \|u''\|_{L_p(0,1;H)} < \infty.$$

Let E_0, E_1 be two Banach spaces, which are continuously injected in the Banach space E, the pair $\{E_0, E_1\}$ is said to be an interpolation couple. Consider the Banach space

$$E_0 + E_1 = \{u : u \in E, \; \exists u_j \in E_j, \; j = 0,1, \text{ with } u = u_0 + u_1\} ,$$

$$\|u\|_{E_0+E_1} = \inf_{u=u_0+u_1; u_j \in E_j} \left(\|u_0\|_{E_0} + \|u_1\|_{E_1} \right) ,$$

and the functional

$$K(t,u) = \inf_{u=u_0+u_1; u_j \in E_j} \left(\|u_0\|_{E_0} + t \|u_1\|_{E_1} \right) .$$

The interpolation space for the couple $\{E_0, E_1\}$ is defined, by the K-method, as follows

$$(E_0, E_1)_{\theta,p} = \left\{ u : u \in E_0 + E_1, \|u\|_{\theta,p} = \left(\int_0^\infty t^{-1-\theta p} K^p(t,u) dt \right)^{\frac{1}{p}} < \infty \right\} ,$$

$0 < \theta < 1, \; 1 \le p \le \infty.$

$$(E_0, E_1)_{\theta,\infty} = \left\{ u : u \in E_0 + E_1, \|u\|_{\theta,\infty} = \sup_{t \in (0,\infty)} t^{-\theta} K(t,u) dt < \infty \right\} ,$$

$0 < \theta < 1$. Let A be a closed operator in H. $H(A)$ is the domain of A provided with the Hilbertian graph norm

$$\|u\|_{H(A)}^2 = \|Au\|^2 + \|u\|^2 , u \in D(A).$$

If $-A$ is the infinitesimal generator of the semigroup $\exp(-xA)$ which is analytic for $x > 0$, decreasing at infinity and strongly continuous for $x \ge 0$, then the following holds [19, p. 96]:

$$(H, H(A^n))_{\theta,p} = \left\{ u : u \in H, \;\; \|u\|_{\theta,p}^p < \infty \right\} ,$$

where

$$\|u\|_{\theta,p}^p = \int_0^\infty t^{-n(1-\theta)p-1} \|A^n \exp(-tA)u\|^p \, dt + \|u\|^p ,$$

$0 < \theta < 1$, $n \in \mathbb{N}$, $1 \le p < \infty$ and $\|u\|_{\theta,p}$ its norm.

Let H and H_1 be Hilbert spaces such that the imbedding $H_1 \subset H$ is continuous and $\overline{H_1} = H$. Then $(H, H_1)_{\theta,2}$, is a Hilbert space, we denote it by $(H, H_1)_\theta$. It is known that $(H, H_1)_\theta = H(S^\theta)$, where S is a self-adjoint positive-definite operator in H [17].

Let $Ff(\sigma) = (2\pi)^{-\frac{1}{2}} \int\limits_{-\infty}^{+\infty} \exp(i\sigma x) f(x) dx$ be the Fourier transform of the function f.

Lemma 2.1. [22, p. 300] *Let A be a self adjoint and positive definite operator in H. Then*

(1) $\exists\omega > 0$, $\left\|A^{\alpha}\exp\left[-x\left(A+\lambda I\right)^{\frac{1}{2}}\right]\right\| \leq C\exp\left[-\omega x\left|\lambda\right|^{\frac{1}{2}}\right]$ *for all* $\alpha \in \mathbb{R}$, $x \geq x_0 > 0$, $|\arg\lambda| \leq \varphi < \pi$, *where C does not depend on x and λ.*

(2) $\int_0^1 \left\|(A+\lambda I)^{\alpha}\exp\left[-x\left(A+\lambda I\right)^{\frac{1}{2}}\right]u\right\| \leq C\left(\left\|A^{\alpha-\frac{1}{4}}u\right\|^2 + |\lambda|^{2\alpha-\frac{1}{2}}\|u\|^2\right)$ *for all* $\alpha \geq \frac{1}{4}$, $|\arg\lambda| \leq \varphi < \pi$, *and* $u \in D\left(A^{\alpha-\frac{1}{4}}\right)$, *where C does not depend on u and λ.*

(3) $\left\|A^{\alpha}\left(A+\lambda I\right)^{-\beta}\right\| \leq C\left(1+|\lambda|\right)^{\alpha-\beta}$, *for all* $0 \leq \alpha \leq \beta$, $|\arg\lambda| \leq \varphi < \pi$, *where C does not depend on λ.*

3. Solvability of the Principal Problem

3.1. **Homogeneous Problem.** Consider in the Hilbert space H the boundary value problem for the second order abstract differential equation

$$L(D)u = -u''(x) + Au(x) + A(x)u(x) = f(x) \qquad x \in (0,1), \tag{3.1}$$

$$\begin{aligned} L_1 u &= \delta u(0) &&= f_1, \\ L_2 u &= u'(1) + Bu(0) &&= f_2, \end{aligned} \tag{3.2}$$

A, $A(x)$, B are linear operators and δ is a complex number.

Looking to the principal part of the problem $(3.1), (3.2)$ with a parameter

$$L(D)u = -u''(x) + (A+\lambda I)u(x) = 0 \qquad x \in (0,1), \tag{3.3}$$

$$\begin{aligned} L_1 u &= \delta u(0) &&= f_1, \\ L_2 u &= u'(1) + Bu(0) &&= f_2. \end{aligned} \tag{3.4}$$

Theorem 3.1. *Assume that the following conditions are satisfied*

(1) *A is a self-adjoint and positive-definite operator in H.*
(2) $\delta \neq 0$.
(3) *B is continuous from $H(A^{1/2})$ in $H(A)$ and from H in $H(A^{1/2})$.*

Then the problem (3.3), (3.4) *for* $f_1 \in (H, H(A))_{\frac{1}{4}}$, $f_2 \in (H, H(A))_{\frac{3}{4}}$ *and for λ such that* $|\arg\lambda| \leq \phi < \pi$, $|\lambda| \longrightarrow \infty$, *has a unique solution in the space* $W_2^2(0,1;H(A),H)$, *and for the solution of the problem* (3.3), (3.4) *the following coercive estimate holds*

$$\|u''\|_{L_2(0,1;H)} + \|Au\|_{L_2(0,1;H)} + |\lambda|\,\|u\|_{L_2(0,1;H)} \leq C\left(\left\|A^{\frac{3}{4}}f_1\right\|_H + |\lambda|^{\frac{3}{4}}\|f_1\|_H + \left\|A^{\frac{1}{4}}f_2\right\|_H + |\lambda|^{\frac{1}{4}}\|f_2\|_H\right), \tag{3.5}$$

where C does not depend on λ.

Proof. The solution u, belonging to $W_2^2(0,1;H(A),H)$, of the equation (3.3) is in the form

$$u(x) = e^{-xA_\lambda^{\frac{1}{2}}}g_1 + e^{-(1-x)A_\lambda^{\frac{1}{2}}}g_2 \tag{3.6}$$

with $A_\lambda = A + \lambda I$ and $g_1, g_2 \in (H, H(A))_{\frac{3}{4}}$.

Indeed, let $u \in W_2^2(0,1;H(A),H)$ be a solution of (3.3). Then we have

$$\left(D - A_\lambda^{\frac{1}{2}}\right)\left(D + A_\lambda^{\frac{1}{2}}\right)u(x) = 0.$$

Note by

$$v(x)=\left(D+A_\lambda^{\frac{1}{2}}\right)u(x).$$

From [22, p. 168] $v\in W_2^1(0,1;H(A^{\frac{1}{2}}),H)$ and

$$\left(D-A_\lambda^{\frac{1}{2}}\right)v(x)=0. \tag{3.7}$$

So

$$v(x)=e^{-(1-x)A_\lambda^{\frac{1}{2}}}v(1), \tag{3.8}$$

where, according to [19, p. 44],

$$v(1)\in\left(H(A^{\frac{1}{2}}),H\right)_{\frac{1}{2}}=\left(H,H(A^{\frac{1}{2}})\right)_{\frac{1}{2}}.$$

From (3.7), (3.8) we have

$$\begin{aligned}u(x)&=e^{-xA_\lambda^{\frac{1}{2}}}u(0)+\int_0^x e^{-(x-y)A_\lambda^{\frac{1}{2}}}e^{-(1-y)A_\lambda^{\frac{1}{2}}}v(1)dy\\&=e^{-xA_\lambda^{\frac{1}{2}}}u(0)+\frac{1}{2}A_\lambda^{-\frac{1}{2}}\left\{e^{-(1-x)A_\lambda^{\frac{1}{2}}}-e^{-xA_\lambda^{\frac{1}{2}}}e^{-A_\lambda^{\frac{1}{2}}}\right\}v(1),\end{aligned}$$

where $u(0)\in(H(A),H)_{\frac{1}{2}}$ [19, p. 44]. Now,

$$A^{\frac{1}{2}}:(H,H(A))_{\frac{3}{4}}\to(H,H(A))_{\frac{1}{4}}=\left(H,H(A^{\frac{1}{2}})\right)_{\frac{1}{2}}$$

is an isomorphism. Consequently the last inequality is in the form (3.6).

Let us show the reverse, i.e. the function u in the form (3.6) with g_1 and g_2 in $(H,H(A))_{\frac{3}{4}}$, belongs to $W_2^2(0,1;H(A),H)$. From interpolation spaces properties see [15], [19, p. 96] and the expression (3.6) of the function u we have

$$\begin{aligned}&\|u\|_{W_2^2(0,1;H(A),H)}\\&\quad\le\left(\left\|AA_\lambda^{-1}\right\|+1\right)\\&\qquad\times\left\{\left(\int_0^1\left\|A_\lambda e^{-xA_\lambda^{\frac{1}{2}}}g_1\right\|^2dx\right)^{\frac{1}{2}}+\left(\int_0^1\left\|A_\lambda e^{-(1-x)A_\lambda^{\frac{1}{2}}}g_2\right\|^2dx\right)^{\frac{1}{2}}\right\}\\&\quad\le C\left(\|g_1\|_{(H,H(A_\lambda))_{\frac{3}{4}}}+\|g_2\|_{(H,H(A_\lambda))_{\frac{3}{4}}}\right)\\&\quad\le C(\lambda)\left(\|g_1\|_{(H,H(A))_{\frac{3}{4}}}+\|g_2\|_{(H,H(A))_{\frac{3}{4}}}\right).\end{aligned} \tag{3.9}$$

The function u satisfies the boundary conditions (3.4) if

$$\left\{\begin{array}{lcl}\delta g_1+\delta e^{-A_\lambda^{\frac{1}{2}}}g_2&=&f_1,\\-A_\lambda^{\frac{1}{2}}e^{-A_\lambda^{\frac{1}{2}}}g_1+Bg_1+A_\lambda^{\frac{1}{2}}g_2+Be^{-A_\lambda^{\frac{1}{2}}}g_2&=&f_2,\end{array}\right.$$

which we can write in matrix form as:

$$\left[\begin{pmatrix}\delta I&0\\B&A_\lambda^{\frac{1}{2}}\end{pmatrix}+\begin{pmatrix}0&\delta e^{-A_\lambda^{\frac{1}{2}}}\\-A_\lambda^{\frac{1}{2}}e^{-A_\lambda^{\frac{1}{2}}}&Be^{-A_\lambda^{\frac{1}{2}}}\end{pmatrix}\right]\begin{pmatrix}g_1\\g_2\end{pmatrix}=\begin{pmatrix}f_1\\f_2\end{pmatrix}. \tag{3.10}$$

The first matrix of operators is invertible, its inverse is

$$\begin{pmatrix} \frac{1}{\delta}I & 0 \\ -\frac{1}{\delta}A_\lambda^{-\frac{1}{2}}B & A_\lambda^{-\frac{1}{2}} \end{pmatrix}. \tag{3.11}$$

Multiplying the two members of (3.10) by the matrix inverse (3.11), we get the following system:

$$\begin{cases} g_1 + e^{-A_\lambda^{\frac{1}{2}}} g_2 &= \frac{1}{\delta} f_1 \\ e^{-A_\lambda^{\frac{1}{2}}} g_1 + g_2 &= -\frac{1}{\delta} A_\lambda^{-\frac{1}{2}} B f_1 + A_\lambda^{-\frac{1}{2}} f_2 \end{cases}$$

we can solve it by Cramer's method, because the coefficients of the linear system are bounded linear operators. The determinant is given by $I+e^{-2A_\lambda^{\frac{1}{2}}}$ which is invertible as a little perturbation of unity, in fact $\left\| e^{-2A_\lambda^{\frac{1}{2}}} \right\| \leq q < 1$.

Hence the solution is written as

$$\begin{cases} g_1 &= \frac{1}{\delta} f_1 + R_{11}(\lambda) f_1 + R_{12}(\lambda) f_2, \\ g_2 &= -\frac{1}{\delta}\left(I + T(\lambda)\right) A_\lambda^{-\frac{1}{2}} B f_1 + \left(I + T(\lambda)\right) A_\lambda^{-\frac{1}{2}} f_2 + R_{21}(\lambda) f_1, \end{cases} \tag{3.12}$$

where $R_{ij}(\lambda)$ are given by

$$\begin{cases} R_{11}(\lambda) &= -\frac{1}{\delta}\left(I + T(\lambda)\right) e^{-2A_\lambda^{\frac{1}{2}}} + \frac{1}{\delta}\ \left(I + T(\lambda)\right) A_\lambda^{-\frac{1}{2}} e^{-A_\lambda^{\frac{1}{2}}} B, \\ R_{12}(\lambda) &= -\left(I + T(\lambda)\right) A_\lambda^{-\frac{1}{2}} e^{-A_\lambda^{\frac{1}{2}}}, \\ R_{21}(\lambda) &= \frac{1}{\delta}\ \left(I + T(\lambda)\right) e^{-A_\lambda^{\frac{1}{2}}}, \end{cases}$$

and satisfy $\|R_{ij}(\lambda)\| \to 0$ when $|\lambda| \to \infty$. $(I + T(\lambda))$ is the inverse of $I + e^{-2A_\lambda^{\frac{1}{2}}}$ obtained from the corresponding Neumann series.

Finally the solution u is given by

$$\begin{aligned} u(x) = e^{-xA_\lambda^{\frac{1}{2}}} &\left(\frac{1}{\delta} f_1 + R_{11}(\lambda) f_1 + R_{12}(\lambda) f_2 \right) \\ &+ e^{-(1-x)A_\lambda^{\frac{1}{2}}} \left(-\frac{1}{\delta}\left(I + T(\lambda)\right) A_\lambda^{-\frac{1}{2}} B f_1 + \left(I + T(\lambda)\right) A_\lambda^{-\frac{1}{2}} f_2 + R_{21}(\lambda) f_1 \right). \end{aligned}$$

From the assumptions of Theorem 3.1 and the properties of interpolation spaces, the following applications are continuous,

$$\begin{aligned} \left(I + T(\lambda)\right) A_\lambda^{-\frac{1}{2}} B : &\quad (H, H(A))_{\frac{3}{4}} \longmapsto (H, H(A))_{\frac{3}{4}}\ , \\ \left(I + T(\lambda)\right) A_\lambda^{-\frac{1}{2}} : &\quad (H, H(A))_{\frac{1}{4}} \longmapsto (H, H(A))_{\frac{3}{4}}\ . \end{aligned}$$

Then we have the estimates

$$\left\| \left(I + T(\lambda)\right) A_\lambda^{-\frac{1}{2}} B f_1 \right\|_{(H,H(A))_{\frac{3}{4}}} \leq C \left\| f_1 \right\|_{(H,H(A))_{\frac{3}{4}}}$$

and

$$\left\| \left(I + T(\lambda)\right) A_\lambda^{-\frac{1}{2}} f_2 \right\|_{(H,H(A))_{\frac{3}{4}}} \leq C \left\| f_2 \right\|_{(H,H(A))_{\frac{1}{4}}}.$$

Set $u(x) = u_1(x) + u_2(x) + u_3(x)$, where

$$u_1(x) = \frac{1}{\delta} e^{-xA_\lambda^{\frac{1}{2}}} f_1,$$
$$u_2(x) = -\frac{1}{\delta} e^{-(1-x)A_\lambda^{\frac{1}{2}}} \left((I + T(\lambda))\, A_\lambda^{-\frac{1}{2}} B f_1 \right),$$
$$u_2(x) = e^{-(1-x)A_\lambda^{\frac{1}{2}}} \left((I + T(\lambda))\, A_\lambda^{-\frac{1}{2}} f_2 \right).$$

Then

$$\|u_1\|_{W_2^2(0,1;H(A),H)} = \frac{1}{|\delta|} \left\| A_\lambda e^{-xA_\lambda^{\frac{1}{2}}} f_1 \right\|_{L_2(0,1;H)} + \frac{1}{|\delta|} \left\| A e^{-xA_\lambda^{\frac{1}{2}}} f_1 \right\|_{L_2(0,1;H)}.$$

However, from Lemma 2.1, we have

$$\left\| A_\lambda e^{-xA_\lambda^{\frac{1}{2}}} f_1 \right\|_{L_2(0,1;H)} \leq C \left(\left\| A^{\frac{3}{4}} f_1 \right\|_H + |\lambda|^{\frac{3}{4}} \|f_1\|_H \right).$$

Similarly we obtain bounds for u_2 and u_3. □

3.2. **Non Homogeneous Problem.** Consider, now, the principal problem for the non homogeneous equation with a parameter

$$L_0(\lambda, D)u = -u''(x) + A_\lambda u(x) = f(x) \qquad x \in (0,1), \tag{3.13}$$

$$\begin{aligned} L_{10}u &= \delta u(0) &&= f_1, \\ L_{20}u &= u'(1) + Bu(0) &&= f_2. \end{aligned} \tag{3.14}$$

We have the result.

Theorem 3.2. *Suppose the following conditions satisfied*

(1) *A is a self-adjoint and positive-definite operator in H.*
(2) *B is continuous from $H(A^{1/2})$ in $H(A)$ and from H in $H(A^{1/2})$.*
(3) $\delta \neq 0$.

Then the problem (3.13), (3.14), for f, f_1 and f_2 in $L_2(0,1;H)$, $(H, H(A))_{\frac{3}{4}}$ and $(H, H(A))_{\frac{1}{4}}$ respectively, and for λ such that $|\arg \lambda| \leq \phi < \pi$, $|\lambda| \longrightarrow \infty$, has a unique solution belonging to the space $W_p^2(0,1;H(A),H)$, for $p \in (1,\infty)$, and the following coercive estimate holds

$$\begin{aligned} &\|u''\|_{L_2(0,1;H)} + \|Au\|_{L_2(0,1;H)} + |\lambda| \, \|u\|_{L_2(0,1;H)} \\ &\qquad \leq C \left(\|f\|_{L_2(0,1;H)} + \left\| A^{\frac{3}{4}} f_1 \right\|_H + |\lambda|^{\frac{3}{4}} \|f_1\|_H + \left\| A^{\frac{1}{4}} f_2 \right\|_H + |\lambda|^{\frac{1}{4}} \|f_2\|_H \right), \end{aligned} \tag{3.15}$$

where C does not depend on λ.

Proof. In Theorem 3.1, we proved the uniqueness. Let us now show that the solution of the problem (3.13), (3.14) belonging to $W_p^2(0,1;H(A),H)$ can be written in the form $u(x) = u_1(x) + u_2(x)$, $u_1(x)$ is the restriction to $[0,1]$ of $\widetilde{u}_1(x)$, where $\widetilde{u}_1(x)$ is the solution of the equation

$$L_0(\lambda, D)\widetilde{u}_1(x) = \widetilde{f}(x), \qquad x \in \mathbb{R}, \tag{3.16}$$

with $\widetilde{f}(x) = f(x)$ if $x \in [0,1]$ and $\widetilde{f}(x) = 0$ otherwise. $u_2(x)$ is the solution of the problem

$$L_0(\lambda, D)u_2 = 0,\ L_{10}u_2 = f_1 - L_{10}u_1,\ L_{20}u_2 = f_2 - L_{20}u_1. \tag{3.17}$$

The solution of the equation (3.16) is given by the formula

$$\widehat{u}_1(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\mu x} L_0(\lambda, i\mu)^{-1} \widehat{\widetilde{f}}(\mu) d\mu \tag{3.18}$$

where $\widehat{\widetilde{f}}$ is the Fourier transform of the function $\widetilde{f}(x)$, $L_0(\lambda, s)$ is the characteristic pencil of the equation (3.16) i.e. $L_0(\lambda, s) = -s^2 I + A + \lambda I$.
From (3.18) and Plancherel equality it follows that:

$$\begin{aligned} |\lambda| \|u_1\|_{L_2(0,1;H)} &+ \|u_1''\|_{L_2(0,1;H)} + \|Au_1\|_{L_2(0,1;H)} \\ &\leq |\lambda| \|\widehat{u}_1\|_{L_2(\mathbb{R};H(A))} + \|\widehat{u}_1''\|_{L_2(\mathbb{R};H)} + \|A\widehat{u}_1\|_{L_2(\mathbb{R};H)} \\ &\leq |\lambda| \left\| F^{-1} L_0(\lambda, i\mu)^{-1} F\widetilde{f}(\mu) \right\|_{L_2(\mathbb{R};H)} + \left\| F^{-1} (i\mu)^2 L_0(\lambda, i\mu)^{-1} F\widetilde{f}(\mu) \right\|_{L_2(\mathbb{R};H)} \\ &\quad + \left\| F^{-1} A L_0(\lambda, i\mu)^{-1} F\widetilde{f}(\mu) \right\|_{L_2(\mathbb{R};H)}, \end{aligned} \tag{3.19}$$

where F is the Fourier transform.

From condition (1) of Theorem 3.2, for $|\arg \lambda| \leq \varphi < \pi$ and $|\lambda|$ sufficiently large, we have

$$\left\| L_0(\lambda, i\mu)^{-1} \right\| = \left\| (A + \lambda I + \mu^2 I)^{-1} \right\| \leq C(1 + \left|\lambda + \mu^2\right|)^{-1} \leq C |\mu|^{-2} \tag{3.20}$$

$$\left\| A L_0(\lambda, i\mu)^{-1} \right\| = \left\| A(A + \lambda I + \mu^2 I)^{-1} \right\| \leq C \tag{3.21}$$

$$|\lambda| \left\| L_0(\lambda, i\mu)^{-1} \right\| = |\lambda| \left\| (A + \lambda I + \mu^2 I)^{-1} \right\| \leq C |\lambda| (1 + \left|\lambda + \mu^2\right|)^{-1} \leq C. \tag{3.22}$$

Then it follows that

$$|\lambda| \|u_1\|_{L_2(0,1;H)} + \|u_1''\|_{L_2(0,1;H)} + \|Au_1\|_{L_2(0,1;H)} \leq C \|f\|_{L_2(0,1;H)}.$$

Since $u_1 \in W_2^2(0,1; H(A), H)$ and from [19, p. 44] we have

$$u_1'(0) \in (H(A), H)_{\frac{3}{4}} = (H, H(A))_{\frac{1}{4}},$$

$$u_1(0) \in (H, H(A))_{\frac{3}{4}}.$$

Therefore $L_{10} u_1 \in (H, H(A))_{\frac{3}{4}}$ and $L_{20} u_1 \in (H, H(A))_{\frac{1}{4}}$.

From Theorem 3.1, the problem (3.17), when $|\arg \lambda| \leq \phi < \pi$, $|\lambda| \to \infty$, has a solution $u_2(x)$ which is in $W_2^2(0,1; H(A), H)$. Now, we have to find bounds for the following terms

$$\left\| A^{\frac{3}{4}} L_{10} u_1 \right\|_H = \left\| A^{\frac{3}{4}} u_1(0) \right\|_H,$$

$$|\lambda|^{\frac{3}{4}} \|L_{10} u_1\|_H = |\lambda|^{\frac{3}{4}} \|u_1(0)\|_H,$$

$$\left\| A^{\frac{1}{4}} L_{20} u_1 \right\|_H \leq \left\| A^{\frac{1}{4}} u_1'(1) \right\|_H + \left\| A^{\frac{1}{4}} B u_1(0) \right\|_H$$

and

$$|\lambda|^{\frac{1}{4}} \|L_{20} u_1\|_H \leq |\lambda|^{\frac{1}{4}} \|u_1'(1)\|_H + |\lambda|^{\frac{1}{4}} \|Bu_1(0)\|_H.$$

For example, we have

$$\begin{aligned} |\lambda|^{\frac{3}{4}} \|u_1(0)\|_H &\leq C \left(\|u_1\|_{W_2^2(0,1;H(A),H)} + |\lambda| \|u_1\|_{L_2(0,1;H)} \right) \\ &\leq C \|f\|_{L_2(0,1;H)}. \end{aligned}$$

Similarly, we get the other bounds and by the same way the coerciveness estimate. □

4. SOLVABILITY OF THE GENERAL PROBLEM

Consider, now, the general problem with a parameter

$$(4.1)\qquad L_0(\lambda, D)u \;=\; \lambda u(x) - u''(x) + Au(x) + A(x)u(x) \;=\; f(x) \qquad x \in (0,1),$$

$$(4.2)\qquad \begin{aligned} L_{10}u &= \delta u(0) &&= f_1, \\ L_{20}u &= u'(1) + Bu(0) &&= f_2. \end{aligned}$$

We have the result.

Theorem 4.1. *Suppose the following conditions satisfied*

1. A *is a self-adjoint and positive-definite operator in* H.
2. *The imbedding* $H(A) \subset H$ *is compact.*
3. B *is continuous from* $H(A^{1/2})$ *in* $H(A)$ *and from* H *in* $H(A^{1/2})$.
4. $\delta \neq 0$.
5. $\|A(.)u\|_{L_2(0,1;H)} \leq \epsilon \|Au\|_{L_2(0,1;H)} + C(\epsilon) \|u\|_{L_2(0,1;H)}$.

Then the problem (4.1), (4.2), *for* f, f_1 *and* f_2 *in* $L_2(0,1;H)$, $(H, H(A))_{\frac{3}{4}}$ *and* $(H, H(A))_{\frac{1}{4}}$ *respectively, and for* λ *such that* $|\arg\lambda| \leq \phi < \pi,\ |\lambda| \longrightarrow \infty$, *has a unique solution belonging to the space* $W_p^2(0,1;H(A),H)$, *for* $p \in (1,\infty)$, *and the following coercive estimate holds*

$$(4.3)\qquad \begin{aligned} &\|u''\|_{L_2(0,1;H)} + \|Au\|_{L_2(0,1;H)} + |\lambda| \|u\|_{L_2(0,1;H)} \\ &\quad \leq C\left(\|f\|_{L_2(0,1;H)} + \left\|A^{\frac{3}{4}} f_1\right\|_H + |\lambda|^{\frac{3}{4}} \|f_1\|_H + \left\|A^{\frac{1}{4}} f_2\right\|_H + |\lambda|^{\frac{1}{4}} \|f_2\|_H \right). \end{aligned}$$

where C *does not depend on* u, f, $f_{1,}$ $f_{2,}$ *and* λ.

Proof. Let u be a solution of (4.1), (4.2) belonging to $W_2^2(0,1;H(A),H)$. Then u is a solution of the problem

$$(P_0)\begin{cases} L_0(\lambda, D)u = f(x) - A(x)u(x) & x \in (0,1), \\ L_{10}u = \delta u(0) = f_1 & \\ L_{20}u = u'(1) + Bu(0) = f_2. & \end{cases}$$

From Theorem 3.2, we get the estimate

$$\begin{aligned} &\|u''\|_{L_2(0,1;H)} + \|Au\|_{L_2(0,1;H)} + |\lambda| \|u\|_{L_2(0,1;H)} \\ &\quad \leq C\left(\|f - A(.)u\|_{L_2(0,1;H)} + \left\|A^{\frac{3}{4}} f_1\right\|_H + |\lambda|^{\frac{3}{4}} \|f_1\|_H + \left\|A^{\frac{1}{4}} f_2\right\|_H + |\lambda|^{\frac{1}{4}} \|f_2\|_H \right). \end{aligned}$$

Using condition (5) of Theorem 3.2, we get

$$\begin{aligned} &\|u''\|_{L_2(0,1;H)} + (1 - C\epsilon) \|Au\|_{L_2(0,1;H)} + (|\lambda| - C.C(\epsilon)) \|u\|_{L_2(0,1;H)} \\ &\quad \leq C\left(\|f\|_{L_2(0,1;H)} + \left\|A^{\frac{3}{4}} f_1\right\|_H + |\lambda|^{\frac{3}{4}} \|f_1\|_H + \left\|A^{\frac{1}{4}} f_2\right\|_H + |\lambda|^{\frac{1}{4}} \|f_2\|_H \right). \end{aligned}$$

Choosing ϵ such that $C \cdot \epsilon < 1$, the coerciveness estimates follows easily. ☐

5. COMPLETENESS OF ROOT FUNCTIONS

Let us define the operator $\mathfrak{L}$ by

$$\mathfrak{L}u \equiv -u'' + Au,$$
$$D(\mathfrak{L}) = W_2^2(0,1; H(A), H, L_k u = 0, k = 0,1).$$

Lemma 5.1. *Suppose that $s_j(I, H(A), H) \simeq Cj^{-q}$ then*

$$s_j\left(\mathcal{I}, W_2^2(0,1; H(A), H), L_2(0,1; H)\right) \simeq Cj^{-\frac{1}{\frac{1}{2}+q}}.$$

I (resp. $\mathcal{I}$) is the imbedding of $H(A)$ in H (resp. of $W_2^2(0,1; H(A), H)$ in $L_2(0,1; H)$) and $s_j(I, H(A), H)$ are the s-numbers of the operator I from $H(A)$ to H.

Proof. Let S_1 be the operator defined in $L_2(0,1)$ such that $S_1 = S_1^* \geq \gamma^2 I$, $D(S_1) = H(S_1) = W_2^2(0,1)$. From [17], we know that if $H_1 \subset H$ and H_1 is dense in H then there exists an operator S_1 such that $S_1 = S_1^*$ and $D(S_1) = H$. Otherwise, let S_2 be the operator defined by $S_2 = S_2^* \geq \gamma^2 I$, $D(S_2) = H(A)$. If we define the operator S on $L_2(0,1) \otimes H = L_2(0,1; H)$ by $S = S_1 \otimes I_2 + I_1 \otimes S_2$, where I_1 (resp. I_2) is the identity operator in $L_2(0,1)$ (resp. H). We have

$$s_j\left(S_1^{-1}, L_2(0,1), L_2(0,1)\right) \simeq s_j(I, H(S_1), L_2(0,1)) \simeq Cj^{-2},$$
$$s_j\left(S_2^{-1}, H, H\right) \simeq s_j(I, H(A), H) \simeq Cj^{-q}.$$

Hence, from [11], we obtain $s_j(S^{-1}, L_2(0,1; H), L_2(0,1; H)) \simeq Cj^{-\frac{1}{\frac{1}{2}+q}}$. □

Theorem 5.2. *Let the conditions of Theorem 3.2 hold along with $A^{-1} \in \sigma_q(H)$, $q > 0$. Then, the system of root functions of the operator $\mathcal{L}$ is complete in $L_2(0,1; H)$.*

Proof. From Theorem 4.1, we have $\|R(\lambda, \mathcal{L})\| \leq C|\lambda|^{-1}$ for $|\arg\lambda| \leq \varphi < \pi$ and $|\lambda|$ sufficiently large. Using Lemma 5.1, we have $R(\lambda, \mathcal{L}) \in \sigma_p(L_2(0,1; H))$ for $p > \frac{1}{2} + \frac{1}{q}$, so, for the operator $\mathcal{L}$, all the conditions of [8, Theorem 126, 2.3, p. 50], are fulfilled. This achieves the proof of the theorem. □

Theorem 5.3. *Suppose that the conditions of Theorem 5.2 are satisfied as well as the condition $D(A(x)) \subset D(A)$ and $\forall\epsilon > 0$, $\|A(\cdot)u\|_H \leq \epsilon\|Au\|_H + C(\epsilon)\|u\|_H$. $u \in D(A)$. Let $\mathcal{A}$ be the operator defined by $(\mathcal{A}u)(x) = A(x)u(x)$, $D(\mathcal{A}) = L_2(0,1; H)$. Then the system of root functions of $\mathcal{L} + \mathcal{A}$ is complete in $L_2(0,1; H)$.*

Proof. It is clear that

$$\|\mathcal{A}u\|_{L_2(0,1;H)} \leq \epsilon\|Au\|_{L_2(0,1;H)} + C(\epsilon)\|u\|_{L_2(0,1;H)}.$$

Since by Theorem 4.1, we have

$$\|Au\|_{L_2(0,1;H)} \leq C\|f\|_{L_2(0,1;H)} = C\|(\mathcal{L} - \lambda I)u\|_{L_2(0,1;H)},$$

hence

$$\|\mathcal{A}u\|_{L_2(0,1;H)} \leq \epsilon\|(\mathcal{L} - \lambda I)u\|_{L_2(0,1;H)} + C(\epsilon)\|u\|_{L_2(0,1;H)}$$

and so, for $|\lambda|$ sufficiently large and $|\arg\lambda| \leq \varphi < \pi$,

$$R(\lambda, \mathcal{L} + \mathcal{A}) \in \sigma_p(L_2(0,1; H)),$$

and from Theorem 4.1 we have

$$\|R(\lambda, \mathcal{L} + \mathcal{A})\| \leq C|\lambda|^{-1}$$

for $|\lambda|$ sufficiently large and $|\arg\lambda| \leq \varphi < \pi$. Then the system of root functions is complete in $L_2(0,1; H)$. □

6. APPLICATION

Let us consider, in the cylindrical domain $\Omega = [0,1] \times G$ the non local boundary value problem for the Laplace equation with a parameter

$$
(P)\begin{cases} L(\lambda)u = & \lambda u(x,y) - \Delta u(x,y) + b(x,y)u(x,y) = f(x,y), & (x,y) \in \Omega; \\ L_1 u = & \delta u(0,y) = f_1(y), & y \in G; \\ L_2 u = & \frac{\partial}{\partial x}u(1,y) + Bu(0,y) = f_2(y), & y \in G; \\ Pu = & u(x,y') = 0, & (x,y') \in \Gamma, \end{cases} \tag{6.1}
$$

where $\Gamma = [0,1] \times \partial G$ and ∂G is the boundary of G.

A number λ_0 is called an eigenvalue of (P) if the problem

$$
(P')\begin{cases} L(\lambda_0)u = 0, & (x,y) \in \Omega; \\ L_1 u = 0, & y \in G; \\ L_2 u = 0, & y \in G; \\ Pu = 0, & (x,y') \in \Gamma, \end{cases} \tag{6.2}
$$

has a non trivial solution that belongs to $W_2^2(\Omega)$. The non trivial solution u_0 of (P') that belongs to $W_2^2(\Omega)$ is called the eigenfunction of (P) corresponding to the eigenvalue $\lambda_{0.}$ Solutions u_k of

$$
(P'')\begin{cases} L(\lambda_0)u_k + u_{k-1} = 0, & (x,y) \in \Omega; \\ L_1 u_k = 0, & y \in G; \\ L_2 u_k = 0, & y \in G; \\ Pu_k = 0, & (x,y') \in \Gamma, \end{cases} \tag{6.3}
$$

belonging to $W_2^2(\Omega)$ are associated functions of the $k-th$ rank to the eigenvalue u_0 of (P). Eigenfunctions and associated functions of (P) are gathered under the general name of root functions of (P).

Theorem 6.1. *Let $b(x,y) \in W_\infty^{0,1}(\Omega)$, $\delta \neq 0$, $\partial G \in C^2$ then*

(1) (P), *for $f \in W_2^{0,1}(\Omega, Pu=0)$, $f_k \in W_2^{-\frac{m_k}{2}+\frac{3}{4}}(G, Pu=0)$ and for λ such that $|\lambda|$ sufficiently large and $|\arg\lambda| \leq \varphi < \pi$, has a unique solution that belongs to the space $W_2^2(\Omega)$, and for this solution we have the coercive estimate*

$$
|\lambda| \|u''\|_{L_2(\Omega)} + \|u\|_{W_2^2(\Omega)} \leq C\left(\|f\|_{L_2(\Omega)} + \sum_{k=1}^{2} \|f_k\|_{W_2^{-\frac{m_k}{2}+\frac{3}{4}}(G,Pu=0)} + \sum_{k=1}^{2} |\lambda|^{-\frac{m_k}{2}+\frac{3}{4}} \|f_k\|_H \right),
$$

where C does not depend on u and λ.

(2) *Root functions of (P) are complete in $L_2(\Omega)$.*

Proof. Consider in $H = L_2(\Omega)$ the operators A and $A(x)$ defined by

$$
Au = -\Delta u(y) + \lambda_0 u(y), \qquad D(A) = W_2^2(G, Pu=0)
$$

$$
A(x)\,u = b(x,y)u(y) - \lambda_0 u(y), \qquad D(A(x)) = W_2^1(G, Pu=0, m=0).
$$

Then the problem (P) can be written in the form

$$\begin{cases} \lambda u(x) - u''(x) + Au(x) + A(x)u(x) & = & 0 \quad x \in (0,1), \\ \delta u(0) & = & f_1 \\ u'(1) + Bu(0) & = & f_2. \end{cases}$$

We have the compact imbedding $W_2^2(\Omega) \subset L_2(\Omega)$. On the other hand

$$s_j\left(I, W_2^2(\Omega), L_2(\Omega)\right) \simeq j^{-\frac{2}{r+1}}.$$

By virtue of Lemma 3.1 in [21, p. 60] we have

$$s_j\left(I, H(A), L_2(\Omega)\right) \simeq s_j\left(A^{-1}, L_2(\Omega), L_2(\Omega)\right).$$

Since $H(A) \subset W_2^2(\Omega)$, then it follows that $A^{-1} \in \sigma_p\left(L_2(\Omega), L_2(\Omega)\right)$, then $\|R(\lambda, A)\| \leq C\,|\lambda|^{-1}$ for $|\arg\lambda| \leq \varphi < \pi$ and $|\lambda|$ sufficiently large. Hence, all conditions of Theorem 5.3 has been checked. □

REFERENCES

[1] S. AGMON, On the eigenfunctions and eigenvalues of general boundary value problems, *Comm. Pure Appl. Math.* **15**(1962), 119–147.

[2] S. AGMON AND L. NIRENBERG, Properties of solutions of ordinary differential equation in Banach spaces, *Comm. Pure Appl. Math.*, **16** (1963), 121–239.

[3] M.S. AGRANOVIĆ AND M.L.VISIK, Elliptic problems with a parameter and parabolic problems of general type, *Russian Math. Surveys*, **19** (1964), 53–161.

[4] A. AIBECHE, Coerciveness estimates for a class of elliptic problems, *Diff. Equ. Dynam. Sys.*, **4** (1993), 341–351.

[5] A. AIBECHE, Completeness of generalized eigenvectors for a class of elliptic problems, *Math. Results*, **31** (1998), 1–8.

[6] F.E. BROWDER, On the spectral theory of elliptic differential operators I, *Math. Ann.*, **142** (1961), 22–130.

[7] M. DENECHE, Abstract differential equation with a spectral parameter in the boundary conditions, *Math. Results*, **35** (1999), 217–227.

[8] N. DUNFORD AND J.T. SCHWARTZ, *Linear Operators Vol.II*, Interscience, 1963.

[9] H. O. FATTORINI, *Second Order Linear Differential Equations in Banach Spaces*, North-Holland, Amsterdam, 1985.

[10] G. GEYMONAT AND P. GRISVARD, Alcuni risultati di teoria spettrale, *Rend. Sem. Mat. Univ. Padova*, **38** (1967), 121–173.

[11] V.I. GORBACHUK AND M.I. GORBACHUK, *Boundary Value Problems for Operator Differential Equations*, Kluwer Academic Publishers, 1991.

[12] P. GRISVARD, Caractérisation de quelques espaces d'interpolation, *Arch. Rat. Mech. Anal.*, **25** (1967), 40–63.

[13] T. KATO, *Perturbation Theory for Linear Operators*, Springer-Verlag, New-York, 1966.

[14] M.V. KELDYSH, On the eigenvalues and eigenfunctions of certain class of non self-adjoint equations, *Dokl. Akad. Nauk. SSSR*, **77** (1951), 11–14.

[15] H. KOMATSU, Fractional powers of operators II. Interpolation spaces, *Pacif. J. Math.*, **21** (1967), 89–111.

[16] S.G. KREIN, *Linear Differential Equations in Banach Spaces*, Providence, 1971.

[17] J.L. LIONS AND E. MAGENES, *Problèmes aux Limites non Homogènes et Applications, Vol. I*, Dunod, 1968.

[18] H. TANABE, *Equations of Evolution*, Pitman, London, 1979.

[19] H. TRIEBEL, *Interpolation Theory, Functions spaces, Differential Operators*, North Holland, Amsterdam, 1978.

[20] S.Y. YAKUBOV, *Linear Differential Equations and Applications* (in Russian), Baku, elm, 1985.

[21] S.Y. YAKUBOV, *Completeness of Root Functions of Regular Differential Operators*, Longman, Scientific and Technical, New-York, 1994.

[22] S.Y. YAKUBOV AND Y.Y. YAKUBOV, *Differential-Operator Equations. Ordinary and Partial Differential Equations*, CRC Press, New York, 1999.