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LOWER BOUNDS FOR THE INFIMUM OF THE SPECTRUM OF THE SCHRÖDINGER OPERATOR IN $\mathbb{R}^N$ AND THE SOBOLEV INEQUALITIES

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ABSTRACT. This article is concerned with the infimum e_1 of the spectrum of the Schrödinger operator $\tau = -\Delta + q$ in $\mathbb{R}^N$, $N \geq 1$. It is assumed that $q_- = \max(0, -q) \in L^p(\mathbb{R}^N)$, where $p \geq 1$ if $N = 1$, $p > N/2$ if $N \geq 2$. The infimum e_1 is estimated in terms of the L^p-norm of q_- and the infimum $\lambda_{N,\theta}$ of a functional $\Lambda_{N,\theta}(\nu) = \|\nabla v\|_2^\theta \|v\|_2^{1-\theta} \|v\|_r^{-1}$, with ν element of the Sobolev space $H^1(\mathbb{R}^N)$, where $\theta = N/(2p)$ and $r = 2N/(N - 2\theta)$. The result is optimal. The constant $\lambda_{N,\theta}$ is known explicitly for $N = 1$; for $N \geq 2$, it is estimated by the optimal constant $C_{N,s}$ in the Sobolev inequality, where $s = 2\theta = N/p$. A combination of these results gives an explicit lower bound for the infimum e_1 of the spectrum. The results improve and generalize those of Thirring [*A Course in Mathematical Physics III. Quantum Mechanics of Atoms and Molecules*, Springer, New York 1981] and Rosen [*Phys. Rev. Lett.*, **49** (1982), 1885-1887] who considered the special case $N = 3$. The infimum $\lambda_{N,\theta}$ of the functional $\Lambda_{N,\theta}$ is calculated numerically (for $N = 2, 3, 4, 5$, and 10) and compared with the lower bounds as found in this article. Also, the results are compared with these by Nasibov [*Soviet. Math. Dokl.*, **40** (1990), 110-115].

Key words and phrases: Optimal lower bound, infimum spectrum Schrődinger operator, Sobolev inequality.

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1. RESULTS

In this article we study the Schrödinger operator $\tau = -\Delta + q$ on $\mathbb{R}^N$. The real-valued potential q is such that $q = q_+ + q_-$, where

$$q_+ = \max(0, q) \in L^2_{loc}(\mathbb{R}^N), \tag{1.1}$$

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(1.2) $$q_- = \max(0, -q) \in L^p(\mathbb{R}^N), \quad N = 1: \quad 1 \le p < \infty, \quad N \ge 2: \quad N/2 < p < \infty.$$

Associated with q is the closed hermitian form h,

(1.3) $$h(u, v) = (\nabla u, \overline{\nabla v}) + \int_{\mathbb{R}^N} q u \bar{v} dx, \qquad u, v \in Q(h),$$

(1.4) $$Q(h) = H^1(\mathbb{R}^N) \cap \{u \mid u \in L^2(\mathbb{R}^N), \quad q_+^{1/2} \in L^2(\mathbb{R}^N)\}.$$

As will be shown in the course of the proof of Theorem 1.1, h is semibounded below if the condition (1.2) is satisfied. Hence, we can define a unique self-adjoint operator H, such that $Q(h)$ is its quadratic form (see [22, Theorem VIII.15] or [26, Theorem 2.5.19]).

We remark that τ restricted to $C_0^\infty(\mathbb{R}^N)$ is essentially self-adjoint for the following values of p :

(1.5) $$\begin{array}{ll} p \ge 2 & \text{if } N = 1, 2, 3; \\ p > 2 & \text{if } N = 4; \\ p \ge N/2 & \text{if } N \ge 5; \end{array}$$

see [21, Corollary, p. 199, with $V_1 = q_+$, $c = d = 0$, $V_2 = q_-$]. For $N = 1, 2, 3$ condition (1.5) imposes a restriction on the values of p allowed in (1.2). Furthermore, $\mathcal{D}(H) = H_0^2(\mathbb{R}^N) = H^2(\mathbb{R}^N)$ if $q_+ \in L^\infty(\mathbb{R}^N)$, $p > N/2$, $N \ge 4$; see [6, pp. 123, 246 (vi)].

It is our purpose to give a lower bound for the infimum of the spectrum of H by estimating the Rayleigh quotient $e_1 = \inf_{u \in \mathcal{D}(H)} h(u, u)/\|u\|_2^2$. Since q_+ enlarges e_1, it suffices to consider the Rayleigh quotient for the case $q_+ = 0$.

Let $\Lambda_{N,\theta}$ be the following functional on $H^1(\mathbb{R}^N)$:

(1.6) $$\Lambda_{N,\theta}(v) = \frac{\|\nabla v\|_2^\theta \|v\|_2^{1-\theta}}{\|v\|_r}, \quad r = 2N/(N - 2\theta), \quad v \in H^1(\mathbb{R}^N),$$

where

$$0 < \theta \le 1/2 \text{ if } N = 1, \qquad \text{and} \qquad 0 < \theta < 1 \text{ if } N \ge 2.$$

Let $\lambda_{N,\theta}$ be its infimum

(1.7) $$\lambda_{N,\theta} = \inf \left\{\Lambda_{N,\theta}(v) | v \in H^1(\mathbb{R}^N),\ v \ne 0\right\}.$$

It is possible to include the cases $\theta = 0$, with $\lambda_{N,0} = \Lambda_{N,0}(v) = 1$, and $\theta = 1$, provided $N \ge 2$; see below. The functional $\Lambda_{N,\theta}(v)$ is invariant for dilations in the argument of v and for scaling of v.

We recall the following imbeddings

(1.8) $$H^1(\mathbb{R}^1) \hookrightarrow C^{0,\lambda}(\overline{\mathbb{R}^1}),\ 0 < \lambda \le 1/2,$$

(1.9) $$H^1(\mathbb{R}^2) \hookrightarrow L^s(\mathbb{R}^2),\ 2 \le s < \infty,$$

(1.10) $$H^1(\mathbb{R}^N) \hookrightarrow L^s(\mathbb{R}^N),\ 2 \le s \le 2N/(N-2),\ N \ge 3;$$

see [1, pp. 97, 98]. Here, $C^{0,\lambda}(\overline{\mathbb{R}^1})$ is the space of bounded, uniformly continuous functions v on $\mathbb{R}^1$ with

$$\sup_{x,y \in \mathbb{R}^1,\, x \ne y} |v(x) - v(y)|/|x - y|^\lambda < \infty.$$

Hence, $u \in H^1(\mathbb{R}^1)$ implies $u \in L^2(\mathbb{R}^1) \cap L^\infty(\mathbb{R}^1)$ and, therefore, $u \in L^s(\mathbb{R}^1)$, $2 \le s \le \infty$. Thus, (1.8), (1.9), and (1.10) imply that there exist positive constants K such that

$$[\|\nabla v\|_2^2 + \|v\|_2^2]^{1/2}/\|v\|_s \ge K, \qquad \begin{array}{r} 2 \le s \le \infty \text{ if } N = 1, \\ 2 \le s < \infty \text{ if } N = 2, \\ 2 \le s \le 2N/(N-2) \text{ if } N \ge 3. \end{array} \tag{1.11}$$

Returning to the functional $\Lambda_{N,\theta}$, we make for $0 < \theta < 1$ $(0 < \theta \le 1/2$ if $N = 1)$ a dilation $x = \epsilon y$, $x, y \in \mathbb{R}^N$, $w(y) = v(x)$, such that

$$\|\nabla w\|_2^2/\|w\|_2^2 = \theta/(1-\theta).$$

The inequality

$$ab \le a^P/P + b^Q/Q,\ a, b \ge 0,\ 1 < P < \infty,\ 1/P + 1/Q = 1, \tag{1.12}$$

with equality if and only if $a^P = b^Q$, applied to $\Lambda^2_{N,\theta}(w)$ gives ($P = 1/\theta$, $Q = 1/(1-\theta)$, $a = \eta\|\nabla w\|_2^{2\theta}$, $b = \|w\|_2^{2\theta}/\eta$)

$$\Lambda^2_{N,\theta}(w) \le \frac{\theta\eta^{1/\theta}\|\nabla w\|_2^2 + (1-\theta)\eta^{-1/(1-\theta)}\|w\|_2^2}{\|w\|_r^2}, \tag{1.13}$$

for some number $\eta > 0$. Equality holds if and only if

$$\eta^{1/\theta}\|\nabla w\|_2^2 = \eta^{-1/(1-\theta)}\|w\|_2^2, \quad \text{i.e. } \eta^{-1/(\theta(1-\theta))} = \theta/(1-\theta).$$

In this case,

$$\Lambda^2_{N,\theta}(w) = \theta^\theta(1-\theta)^{1-\theta}\frac{\|\nabla w\|_2^2 + \|w\|_2^2}{\|w\|_r^2}. \tag{1.14}$$

Since it is possible to perform this dilation for any $v \in H^1(\mathbb{R}^N)$, and since $\theta^\theta(1-\theta)^{1-\theta} > 0$ we conclude that $\lambda_{N,\theta} > 0$ for $0 < \theta < 1$. The case $N = 1$, $\theta = 1/2$ (in that case r becomes undefined) is covered by the value $s = \infty$ in (1.11). The cases $\theta = 1$, $N \ge 2$ are covered by a special form of the Sobolev inequality

$$\|\nabla w\|_s \ge C_{N,s}\|w\|_t,\ t = sN/(N-s),\ 1 \le s < N,\ w \in H^{1,s}(\mathbb{R}^N), \tag{1.15}$$

where $C_{N,s}$ are the optimal constants and

$$H^{1,s}(\mathbb{R}^N) = \text{completion of } \{w \mid w \in C^1(\mathbb{R}^N), \|u\|_{1,s}^s = \|u\|_s^s + \|\nabla u\|_s^s < \infty\} \\ \text{with respect to the norm } \|\cdot\|_{1,s}. \tag{1.16}$$

If we take $s = 2$ we have $\lambda_{N,1} = C_{N,2}$, $N \ge 3$. Since $H^1(\mathbb{R}^2) \not\hookrightarrow L^\infty(\mathbb{R}^2)$, it follows that $\lambda_{2,1} = C_{2,2} = 0$, $i.e.$ $K = 0$ in (1.11). The numbers $C_{N,s}$ are known explicitly by the work of [2] and [25], see also [14]

$$C_{N,s} = N^{1/s}\left(\frac{N-s}{s-1}\right)^{(s-1)/s}\left[N\omega_N B\left(\frac{N}{s}, N+1-\frac{N}{s}\right)\right]^{1/N},\ 1 < s < N, \tag{1.17}$$

$$C_{N,1} = N\omega_N^{1/N},\ N \ge 2, \tag{1.18}$$

where ω_N is the volume of the unit ball in $\mathbb{R}^N$:

$$\omega_N = \pi^{N/2}/\Gamma(1 + N/2), \tag{1.19}$$

$$B(a,b) = \Gamma(a)\Gamma(b)/\Gamma(a+b),\ a, b > 0, \tag{1.20}$$

and there is equality in (1.15) for functions of the form

$$w_{N,s}(x_1, ..., x_N) = \left\{a + b|x|^{s/(s-1)}\right\}^{1-N/s},\ a, b > 0,\ 1 < s < N. \tag{1.21}$$

Note that $w_{N,s} \notin L^s(\mathbb{R}^N)$ if $s \geq N^{1/2}$. For $s = 1$ there are no functions such that there is equality, but by taking an approximating sequence $\{w^i\} \in H^{1,1}(\mathbb{R}^N)$ of the characteristic function of the unit ball, the bound $C_{N,1}$ can be approximated arbitrarily close. See further Lemma 2.1 for more information about $\Lambda_{N,\theta}$ and the explicit form for $\lambda_{1,\theta}$.

In Theorem 1.1 we give the lowest possible point of the spectrum of this Schrödinger equation for all q_- satisfying (1.2). Let us define the number $l(N,\theta)$, where $\theta = N/(2p)$, as follows

$$l(N,\theta) = \inf_{q_- \in L^p(\mathbb{R}^N)} \inf_{u \in H^1(\mathbb{R}^N)} \frac{\|\nabla u\|_2^2 + \int_{\mathbb{R}^N} q\|u\|_2^2 dx}{\|u\|_2^2} \|q_-\|_p^{-1/(1-\theta)}. \tag{1.22}$$

Theorem 1.1. *Let $q_- \in L^p(\mathbb{R}^N)$, $1 \leq p < \infty$ if $N = 1$, $N/2 < p < \infty$ if $N \geq 2$ (i.e. (1.2)). Then*

$$l(N,\theta) = -(1-\theta)\theta^{\theta/(1-\theta)}\lambda_{N,\theta}^{-2/(1-\theta)}, \quad \begin{array}{l} 0 < \theta < 1/2 \text{ if } N = 1, \\ 0 < \theta < 1 \text{ if } N \geq 2, \end{array} \tag{1.23}$$

and explicitly for $N = 1$

$$l(1,\theta) = -\left\{(2\theta)^{2\theta}(1-2\theta)^{1-2\theta}\left[B\left(\frac{1}{2},\frac{1}{2\theta}\right)\right]^{-2\theta}\right\}^{1/(1-\theta)}, 0 < \theta < 1/2,$$

$$= -\left\{p^{-p}(p-1)^{p-1}\left[B\left(\frac{1}{2},p\right)\right]^{-1}\right\}^{2/(2p-1)}, 1 < p < \infty, \tag{1.24}$$

$$l(1,1/2) = -1/4. \tag{1.25}$$

Remark 1.2. Of course, for any application of this method to find a lower bound for e_1 (the smallest eigenvalue) one can take the following infimum over the allowed set Θ of θ-values (depending on q_-).

$$e_1 \geq -\inf_{\theta \in \Theta}(1-\theta)\theta^{\theta/(1-\theta)}\lambda_{N,\theta}^{-2/(1-\theta)}\|q_-\|_{N/(2\theta)}^{1/(1-\theta)}. \tag{1.26}$$

Remark 1.3. Note that we do not include $\theta = 1$ in the allowed θ-range, although for $N \geq 2$ $\lambda_{N,1}$ is defined. It turns out that the method of the proof does not work in this case; it gives however a criterion such that $\sigma_d(H) = \emptyset$ ($i.e.$ there are no isolated eigenvalues), see the Remark 2.4 after the proof of Theorem 1.1.

Remark 1.4. It is possible to allow the case $p = \infty$, $i.e.$ $\theta = 0$, then $l(N,0) = -1$. If $q = -\|q_-\|_\infty$ this bound is achieved arbitrarily close by a sequence of functions $\{u^i\} \in H^1(\mathbb{R}^N)$, where each u^i is a smooth approximation of the characteristic function of the i-ball in $\mathbb{R}^N$, because then the quotient

$$\|\nabla u^i\|_2^2/\|u^i\|_2^2 \to N\omega_N i^{-1},\ i \to \infty, \quad \text{and} \quad \frac{\int_{\mathbb{R}^N} q|u^i|^2\,dx}{\|u^i\|_2^2}\|q_-\|_\infty^{-1} = -1.$$

Remark 1.5. Already Lieb and Thirring [15] characterize the infimum of the spectrum with a number $-(L^1_{\gamma,N})^{1/\gamma}$ (in their notation, $\gamma = p - N/2$), with $\gamma > \max(0, 1-N/2)$, and $\gamma = 1/2$, $N = 1$. Therefore,

$$(L^1_{\gamma,N})^{1/\gamma}\big|_{\gamma=(1-\theta)N/(2\theta)} = (1-\theta)\theta^{\theta/(1-\theta)}\lambda_{N,\theta}^{-2/(1-\theta)}. \tag{1.27}$$

They give $L^1_{\gamma,1}$ for $\gamma > 1/2$ explicitly. Here, we also include the case $N = 1$, $\gamma = 1/2$ (i.e. $\theta = 1/2$, $p = 1$). However, the main reason of this article is to show how one can give an explicit estimate for e_1 by sharp estimates of the numbers $\lambda_{N,\theta}$, $N \geq 2$, in terms of the numbers $C_{N,s}$ for some $s = s(\theta)$, see Theorems 1.7 and 1.8. For a survey for other integral inequalities results related to the infimum of the spectrum see [9] and [16].

Remark 1.6. The results for the ordinary differential case ($N = 1, \Omega = \mathbb{R}$) are related to those for $\Omega = \mathbb{R}^+$ with either a Dirichlet or a Neumann boundary condition at $x = 0$ (respectively the operators T_0 and $T_{\pi/2}$ in the work of [8], [27] and [10]). In those cases there holds $1 \le p \le \infty$

$$\inf_{q_- \in L^p(\mathbb{R}^+)} \inf_{u \in \mathcal{D}(T_0)} \frac{\|u'\|_2^2 + \int_0^\infty q|u|_2^2 dx}{\|u\|_2^2} \|q_-\|_p^{-2p/(2p-1)} = l(1, 1/(2p)), \tag{1.28}$$

$$\inf_{q_- \in L^p(\mathbb{R}^+)} \inf_{u \in \mathcal{D}(T_{\pi/2})} \frac{\|u'\|_2^2 + \int_0^\infty q|u|_2^2 dx}{\|u\|_2^2} \|q_-\|_p^{-2p/(2p-1)} = 2^{2/(2p-1)} l(1, 1/(2p)). \tag{1.29}$$

See for related work [3].

Theorem 1.7. *The following inequalities hold for $N \ge 2$*

$$\text{i) } \lambda_{N,\theta} > (\lambda_{N,\theta'})^\alpha (\lambda_{N,\theta''})^{1-\alpha},\ 0 < \alpha < 1,\ \theta = \alpha\theta' + (1-\alpha)\theta'',\ \theta' \ne \theta'', \tag{1.30}$$

$$\text{ii) } \lambda_{N,\theta} > (\theta C_{N,2\theta})^\theta, \quad 1/2 \le \theta < 1, \tag{1.31}$$

$$\begin{aligned} \text{iii) } \lambda_{N,\theta} &> (\theta_N C_{N,2\theta_N})^\theta, \quad 0 < \theta \le \theta_N, \\ \lambda_{N,\theta} &> (\theta C_{N,2\theta})^\theta, \quad \theta_N \le \theta < 1, \end{aligned} \tag{1.32}$$

$$\text{iv) } \lambda_{N,\theta} > (C_{N,2})^\theta, \quad 0 < \theta < 1, \tag{1.33}$$

where $C_{N,s}$ is given by (1.17) and (1.18) and $\theta_N = \theta(N) \in (1/2, 1)$ is the unique maximum of $\theta C_{N,2\theta}$, $1/2 \le \theta \le 1$. θ_N is given by $\theta_N = N/(2p_N)$ where p_N is the solution of $M(N,p) = 0$, with

$$M(N,p) = \log\left(\frac{N-p}{p-1}\right) + \frac{N-p}{p(p-1)} + \psi(p) - \psi(N+1-p), \tag{1.34}$$

$$\psi(x) = \frac{d}{dx}(\log(\Gamma(x)) = \left(\frac{d}{dx}\Gamma(x)\right)/\Gamma(x), \quad x > 0. \tag{1.35}$$

It is now easy to combine both theorems in

Theorem 1.8. *Under the conditions of Theorem 1.1 there holds*

$$l(N,\theta) > \begin{cases} -(1-\theta)\theta^{\theta/(1-\theta)}(\theta_N C_{N,2\theta_N})^{-2\theta/(1-\theta)}, & 0 < \theta \le \theta_N, \\ -(1-\theta)\theta^{-\theta/(1-\theta)}(C_{N,2\theta})^{-2\theta/(1-\theta)}, & \theta_N \le \theta < 1, \end{cases} \tag{1.36}$$

and also (generally less than optimal)

$$l(N,\theta) > -(1-\theta)\theta^{\theta/(1-\theta)}(\theta' C_{N,2\theta'})^{-2\theta/(1-\theta)}, \quad 0 < \theta < 1, \quad \textit{for any } \theta' \ge \theta,\ 1/2 \le \theta' \le 1. \tag{1.37}$$

Proof. Equation (1.36) follows from (1.23) and (1.32); (1.37) follows from (1.23), (1.30) (with $\theta'' = 0$) and (1.31). □

Remark 1.9. For $N = 3$, $\theta' = 1$ the result (1.37) reads explicitly

$$l(3,\theta) > -(1-\theta)\theta^{\theta/(1-\theta)}[3^{1/2}2^{-2/3}\pi^{2/3}]^{-2\theta/(1-\theta)}, \quad 0 < \theta < 1, \tag{1.38}$$

and this is the same result as [23, (14)].

Remark 1.10. [26, (3.5.30), and private communication by H. Grosse] gives the following result for $N = 3$

$$l(3, 3/(2p)) > -((p-1)/p)^2 (4\pi)^{-2/(2p-3)} \left[\Gamma\left(\frac{2p-3}{p-1}\right)\right]^{(2p-2)/(2p-3)}, \quad 3/2 < p < \infty, \tag{1.39}$$

or in terms of θ,

$$l(3,\theta) > -(1-2\theta/3)^2(4\pi)^{-2\theta/(3-3\theta)}\left[\Gamma\left(\frac{6-6\theta}{3-2\theta}\right)\right]^{(3-2\theta)/(3-3\theta)}, \; 0<\theta<1. \tag{1.40}$$

It can be proved that (1.38) is better than (1.40) for all $0<\theta<1$. For $\theta=0$ the right-hand sides of both (1.38) and (1.40) give the correct value $l(3,0)=-1$.

Remark 1.11. To show the superiority of (1.37) with $\theta'<1$ against (1.37) with $\theta'=1$, $i.e.$ (1.38), we evaluate the bound for $l(3,3/4)$ of (1.37) with $\theta=\theta'=3/4$. We find

$$l(3,3/4) > -2^2 3^{-7}\pi^{-2} \simeq -1.85_{10^{-4}}, \tag{1.41}$$

while (1.38) gives

$$l(3,3/4) > -2^{-4}\pi^{-4} \simeq -6.42_{10^{-4}},$$

and (1.40) gives

$$l(3,3/4) > -2^{-6}\pi^{-2} \simeq -15.83_{10^{-4}}.$$

Based on our numerical calculations (see Section 3) we find $l(3,3/4)=-1.750180_{10^{-4}}$. So the estimate (1.41) comes close to the actual value of $l(3,3/4)$.

Remark 1.12. The results in Theorems 1.1, 1.7, and 1.8 were announced in [28] and [7, p. 337].

Remark 1.13. In the interesting paper [20] Nasibov has given a lower bound (in his notation $1/\overline{k_0}$) for $\lambda_{N,\theta}$:

$$\lambda_{N,\theta} = \frac{1}{k_0} > \frac{1}{\overline{k_0}}, \tag{1.42}$$

with

$$\overline{k_0} = \frac{1}{\sqrt{\theta^\theta(1-\theta)^{1-\theta}}}\left(N\omega_N B\left(\frac{N}{2},\frac{N(1-\theta)}{2\theta}\right)\right)^{\theta/N} k_B\left(\frac{2N}{N+2\theta}\right), \tag{1.43}$$

$$k_B(p) = \left[\left(\frac{p}{2\pi}\right)^{1/p}\left(\frac{p'}{2\pi}\right)^{-1/p'}\right]^{N/2}, \qquad \frac{1}{p}+\frac{1}{p'}=1. \tag{1.44}$$

And, even better

$$\lambda_{N,\theta} = \frac{1}{k_0} > \frac{1}{\overline{\overline{k_0}}}, \qquad \text{with} \quad \frac{1}{\overline{\overline{k_0}}} > \frac{1}{\overline{k_0}}, \quad \text{for} \quad \theta > N/4, \tag{1.45}$$

with

$$\overline{\overline{k_0}} = \left\{\frac{1}{\theta^\theta(1-\theta)^{1-\theta}} k_B\left(\frac{N}{N-2\theta}\right) k_B^2\left(\frac{2N}{N+2\theta}\right)\|G(|x|)\|_{\frac{N}{N-2\theta}}\right\}^{1/2}, \tag{1.46}$$

$$G(|x|) = K_{\frac{N-2}{2}}(|x|)|x|^{-(N-2)/2}, \tag{1.47}$$

with K_α the modified Bessel function of the second kind and order α. The inequality (1.45) is only relevant for $N=2$, $1/2\le\theta\le1$, and $N=3$, $3/4\le\theta\le1$, since $\overline{k_0}<\overline{\overline{k_0}}$, for $N=2$, $0<\theta<1/2$, and $N=3$, $0<\theta<3/4$, and $\overline{k_0}=\overline{\overline{k_0}}$, for $N=2$, $\theta=1/2$, and $N=3$, $\theta=3/4$.

The reader is advised to consult also the original paper (*Dokl. Akad. Nauk SSSR* 307, No. 3, 538-542 (1989)) of [20] since there are a number of misprints in the translated version. In Section 3 this lower bound will be compared with (1.32). The function G reads

$$N = 2, \qquad G(|x|) = K_0(|x|),$$
$$N = 3, \qquad G(|x|) = K_{\frac{1}{2}}(|x|)|x|^{-1/2} = \sqrt{\frac{\pi}{2}}\exp(-|x|)/|x|,$$

so, one has to calculate the integrals in (1.46)

$$N = 2 : \|G(|x|)\|_{\frac{1}{1-\theta}} = \left[\int_0^\infty K_0^{1/(1-\theta)}(r)\, 2\pi r\, dr\right]^{1-\theta}, \tag{1.48}$$

$$N = 3 : \|G(|x|)\|_{\frac{3}{3-2\theta}} = \sqrt{\frac{\pi}{2}}\left[\int_0^\infty r^{(3-4\theta)/(3-2\theta)} \exp\left(-\frac{3r}{3-2\theta}\right) 4\pi\, dr\right]^{(3-2\theta)/3}. \tag{1.49}$$

For $N = 3$ the integral in (1.49) can be evaluated explicitly, while for $N = 2$, i.e. (1.48), that is only possible for $\theta = 1/2$:

$$N = 2 : \|G(|x|)\|_2 = \left[2\pi \int_0^\infty K_0^2(r)\, r\, dr\right]^{1/2}$$
$$= \left(2\pi \left[\frac{r^2}{2}\left(K_0^2(r) - K_1^2(r)\right)\right]\Big|_0^\infty\right)^{1/2} = \sqrt{\pi},$$

$$N = 3 : \|G(|x|)\|_{\frac{3}{3-2\theta}} = \sqrt{\frac{\pi}{2}}\,(4\pi)^{(3-2\theta)/3}\left(\frac{3-2\theta}{3}\right)^{2-2\theta}\left[\Gamma\left(\frac{6-6\theta}{3-2\theta}\right)\right]^{(3-2\theta)/3}.$$

2. PROOFS

Firstly, we give more information on $\Lambda_{N,\theta}$ in a lemma.

Lemma 2.1. *The value* $\lambda_{N,\theta} = \inf_{v\in H^1(\mathbb{R}^N),\, v\neq 0} \Lambda_{N,\theta}(v)$ *for the functional* $\Lambda_{N,\theta}(v)$ *defined in (6) is attained by radial symmetric monotonely decreasing positive functions* $v_{N,\theta}(|x|)$ *which satisfy, except for* $\theta = 1/2$, $N = 1$, *the following ordinary differential equation for* $0 < \theta < 1/2$ *if* $N = 1$, *and* $0 < \theta < 1$ *if* $N \geq 2$,

$$\begin{gathered} -\frac{d^2}{dr^2}v - \frac{(N-1)}{r}\frac{d}{dr}v - v|v|^{(N+2\theta)/(N-2\theta)-1} + v = 0,\ r = |x| > 0, \\ \frac{d}{dr}v(0) = 0,\ \lim_{r\to\infty} v(r) = 0, \end{gathered} \tag{2.1}$$

and the value $\lambda_{N,\theta}$ *is then given by*

$$\lambda_{N,\theta} = \theta^{\theta/2}(1-\theta)^{(N(1-\theta)-2\theta)/(2N)}\left[N\omega_N \int_0^\infty v_{N,\theta}^2(r) r^{N-1} dr\right]^{\theta/N} \quad \textit{for } 0 < \theta < 1,\ N \geq 2. \tag{2.2}$$

For $N = 1$ we have explicitly for $x \geq 0$

(2.3) $$v_{1,\theta}(x) = v_{1,\theta}(-x),\ 0 < \theta \leq 1/2,$$

$$v_{1,\theta}(x) = \left\{(1-2\theta)^{1/2}\cosh\left(\frac{2\theta}{1-2\theta}x\right)\right\}^{-(1-2\theta)/(2\theta)},\ 0 < \theta < 1/2,$$

(2.4) $$v_{1,1/2}(x) = e^{-x},$$

$$\lambda_{1,\theta} = 2^{-\theta}\theta^{-\theta/2}(1-\theta)^{(1-\theta)/2}(1-2\theta)^{-(1-2\theta)/2}\left\{B\left(\frac{1}{2},\frac{1}{2\theta}\right)\right\}^{\theta},\ 0 < \theta < 1/2,$$

$$\lambda_{1,N/(2p)} = 2^{-1/2}\left\{(2p-1)^{(2p-1)/2}(p-1)^{-(p-1)}B\left(\frac{1}{2},p\right)\right\}^{1/(2p)},\ 1 < p < \infty,$$

(2.5) $$\lambda_{1,1/2} = 1.$$

Proof. The case $N = 1$ was treated by [19] and the case $N \geq 2$ was given by [29] who used a rearrangement and an inequality due to Strauss to prove the compactness of the imbedding of radial symmetric functions $u \in H^1(\mathbb{R}^N)$ into $L^s(\mathbb{R}^N)$, $2 < s < \infty$ if $N = 2$, and $2 < s < 2N/(N-2)$ if $N \geq 3$ (see also (1.9), (1.10)). The Euler equation connected with the infimum of $\Lambda_{N,\theta}$ becomes

(2.6) $$-\theta\|\nabla u\|_2^{-2}\Delta u + (1-\theta)\|u\|_2^{-2}u - \|u\|_r^{-r}|u|^{r-2}u = 0,\ r = 2N/(N-2\theta),$$

which can be scaled into the form (2.1) with $\lambda_{N,\theta}$ given by (2.2). The following relations between $\lambda_{N,\theta}$ and the following norms of $\bar{v}_{N,\theta}(x_1,...,x_N) = v_{N,\theta}(|x|)$ hold (*cf.* [24, p. 151], where the factor "$(n-2)$" has to be skipped in the last line on that page)

(2.7) $$\|\bar{v}_{N,\theta}\|_2^2 = L(1-\theta),\ \|\nabla\bar{v}_{N,\theta}\|_2^2 = L\theta,\ \|\bar{v}_{N,\theta}\|_r^r = L,$$

(2.8) $$L = \theta^{-N/2}(1-\theta)^{-N(1-\theta)/(2\theta)}\lambda_{N,\theta}^{N/\theta}.$$

Since (2.1) is nonlinear the value of $v(0)$ has to be chosen properly to satisfy $\lim_{r\to\infty} v(r) = 0$. □

Remark 2.2. We note that the existence of solutions of (2.1) has been proved by many authors: it is just the range $0 < \theta < 1$, see [17]. The uniqueness for the full θ-range has been proved by Kwong, see [11], after preliminary work by [17], and [18]. A proof based on geometrical arguments has been given by [5]. See for related work also [12].

Remark 2.3. Numerical information for $\lambda_{N,\theta}$ for $N = 2, 3$ can be obtained from [15, Appendix], where curves for $L^1_{\gamma,N}$ (see (1.27)) are given $(0 \leq \gamma \leq 2.8,\ N = 2, 3)$. By (1.27) we have

(2.9) $$\lambda_{N,\theta} = \theta^{\theta/2}(1-\theta)^{(1-\theta)/2}(L^1_{\gamma,N})^{-\theta/N},\ \gamma = N(1-\theta)/(2\theta).$$

Comparison with (2.8) learns that $L^1_{\gamma,N} = 1/L$. Besides, the following two values for $\lambda_{N,\theta}$ are known based on numerical calculations

(2.10) $$\lambda_{2,1/2}^{-1} \simeq \left(\frac{1}{\pi(1.86225\cdots)}\right) \simeq 0.642988, \text{ ([29], after (I.5))}$$

$$\to \lambda_{2,1/2} \simeq 1.55524,$$

(2.11) $$\lambda_{2,2/3}^{3} \simeq 4.5981, \text{ ([13], p. 185)}$$

$$\to \lambda_{2,2/3} \simeq 1.66287.$$

Proof of Theorem 1.1. We estimate $h(u,u)$, see (1.3), as follows. All integrals are over $\mathbb{R}^N$.

$$h(u,u) = \|\nabla u\|_2^2 + \int q|u|^2 dx \tag{2.12}$$

$$\geq \|\nabla u\|_2^2 - \int q_-|u|^2 dx$$

$$\geq \|\nabla u\|_2^2 - \|q_-\|_p \|u\|_r^2 \quad [r = 2p/(p-1) = 2N/(N-2\theta)] \tag{2.13}$$

$$\geq \|\nabla u\|_2^2 - \|q_-\|_p \lambda_{N,\theta}^{-2} \|\nabla u\|_2^{2\theta} \|u\|_2^{2(1-\theta)}. \tag{2.14}$$

Apply now (1.12) with

$$P = 1/\theta,\ a = \theta^{-\theta}\|\nabla u\|_2^{2\theta},$$

and

$$ab = \|q_-\|_p \lambda_{N,\theta}^{-2} \|\nabla u\|_2^{2\theta} \|u\|_2^{2(1-\theta)}.$$

Then

$$b = \lambda_{N,\theta}^{-2}\theta^{\theta}\|q_-\|_p \|u\|_2^{2(1-\theta)},$$

and finally we find

$$h(u,u) = -b^Q/Q = -(1-\theta)\theta^{\theta/(1-\theta)}\lambda_{N,\theta}^{-2/(1-\theta)}\|q_-\|_p^{1/(1-\theta)}\|u\|_2^2, \tag{2.15}$$

which is the bound of Theorem 1.1. To prove the optimality part we observe that in such a case we need

$$q = q_- \qquad \text{by (2.12),} \tag{2.16}$$

$$q_- = (const)|u|^{2/(p-1)} \qquad \text{by (2.13),} \tag{2.17}$$

$$u(x_1,\dots,x_N) = (const) v_{N,\theta}(|x|) \qquad \text{by (2.14),} \tag{2.18}$$

$$a^P = b^Q, \qquad \text{by (2.15).} \tag{2.19}$$

that is

$$\theta^{-1}\|\nabla u\|_2^2 = \lambda_{N,\theta}^{-2/(1-\theta)}\theta^{\theta/(1-\theta)}\|q_-\|_p^{1/(1-\theta)}\|u\|_2^2.$$

If one takes

$$u(x_1,\dots,x_N) = v_{N,\theta}(|x|), \tag{2.20}$$

and

$$q(x_1,\dots,x_N) = -q_-(x_1,\dots,x_N) = -\left[v_{N,\theta}(|x|)\right]^{2/(p-1)}, \tag{2.21}$$

then (2.1) becomes $-\Delta u + qu = -u$; this means that the Schrödinger equation and the Euler equation for $\Lambda_{N,\theta}$ are the same if $e_1 = -1$. This is true because for these scalings the lower bound becomes:

$$\begin{aligned} &-(1-\theta)\theta^{\theta/(1-\theta)}\lambda_{N,\theta}^{-2/(1-\theta)}\|q_-\|_p^{1/(1-\theta)} \\ &\quad = -(1-\theta)\theta^{\theta/(1-\theta)}\lambda_{N,\theta}^{-2/(1-\theta)}\left[\|\bar{v}_{N,\theta}\|_r^r\right]^{2\theta/(N(1-\theta))} && \text{by (2.21),} \\ &\quad = -1 && \text{by (2.7), (2.8).} \end{aligned}$$

Finally, (2.19) is implied also by (2.7) and (2.8). It means that the infimum in (1.22) over $q_- \in L^p(\mathbb{R}^N)$ is actually attained. In addition to (2.7) there holds that for q as chosen as in (2.21)

$$\|q_-\|_p^p = L. \tag{2.22}$$

Only the case $\theta = 1/2$, $N = 1$ deserves special attention since $\frac{d}{dx} v_{1,1/2}(x)$ is not continuous at $x = 0$. We take the following sequences (see [27])

$$q_j(x) = -(j+1)[\cosh(jx)]^{-2},\ \|q_j\|_1 = 1 + 1/j, \tag{2.23}$$

$$u_j(x) = [\cosh(jx)]^{-1/j}, \tag{2.24}$$

then u_j, q_j satisfy

$$-\frac{d^2}{dx^2} u_j + q_j u_j = -u_j,$$

so

$$\frac{\|u_j'\|_2^2 + \int_{-\infty}^{\infty} q|u_j|_2^2 dx}{\|u_j\|_2^2} \|q_j\|_1^{-2} = -(1+1/j)^2/4 > -1/4 = l(1, 1/2). \tag{2.25}$$

For these sequences, $j \to \infty$, the bound can be approached arbitrarily close. □

Remark 2.4. As one can observe the proof does not work for $\theta = 1$, $i.e.$ $p = N/2$, however, in that case we can estimate $(N \geq 3)$

$$\begin{aligned} h(u,u) &= \|\nabla u\|_2^2 + \int q|u|^2 dx \\ &\geq \|\nabla u\|_2^2 - \int q_-|u|^2 dx \\ &\geq \|\nabla u\|_2^2 - \|q_-\|_{N/2} \|u\|_{2N/(N-2)}^2 \\ &\geq \|\nabla u\|_2^2 \left(1 - \|q_-\|_{N/2} \lambda_{N,1}^{-2}\right). \end{aligned}$$

So, if

$$\|q_-\|_{N/2} < \lambda_{N,1}^2 = C_{N,2}^2 = \pi N(N-2)[\Gamma(N/2)/\Gamma(N)]^{2/N},\ N \geq 3, \tag{2.26}$$

it follows that $\sigma_d(H) = \emptyset$, $i.e.$ there are no isolated eigenvalues. This is a well-known result, see [15, (4.24)].

Proof of Theorem 1.7. i) By the Hölder inequality we have

$$\|v\|_r < \|v\|_{r'}^{\alpha} \|v\|_{r''}^{1-\alpha},\ 0 < \alpha < 1,\ 1/r = \alpha/r' + (1-\alpha)/r'',\ r' \neq r'', \tag{2.27}$$

which inequality is strict, since $r' \neq r''$. Therefore, by the conditions specified under i)

$$\begin{aligned} \Lambda_{N,\theta}(v) &= \frac{\|\nabla v\|_2^{\theta} \|v\|_2^{1-\theta}}{\|v\|_r} \\ &> \left(\frac{\|\nabla v\|_2^{\theta'} \|v\|_2^{1-\theta'}}{\|v\|_{r'}}\right)^{\alpha} \left(\frac{\|\nabla v\|_2^{\theta''} \|v\|_2^{1-\theta''}}{\|v\|_{r''}}\right)^{1-\alpha} \\ &= \Lambda_{N,\theta'}^{\alpha}(v) \Lambda_{N,\theta''}^{1-\alpha}(v), \end{aligned} \tag{2.28}$$

and we find (1.30), which is also strict, since both infima are attained.

ii) This result is given by [13, (1.5)], by making the transformation $w = v^{1/\theta}$ for $v > 0$ in (1.15) as follows

$$\begin{aligned}
C_{N,s} &\le \frac{\|\nabla w\|_s}{\|w\|_t} = \frac{\|\nabla v^{1/\theta}\|_s}{\|v^{1/\theta}\|_t} = \frac{1/\theta \|v^{(1-\theta)/\theta}\nabla v\|_s}{\|v^{1/\theta}\|_t} && [t = sN/(N-s)] \\
&= \frac{1}{\theta}\frac{\left(\int (\nabla v)^s\, v^{s(1-\theta)/\theta} dx\right)^{1/s}}{\left(\int v^{t/\theta} dx\right)^{1/t}} && \text{[apply Hölder inequality, } 1/P + 1/Q = 1] \\
&\le \frac{1}{\theta}\frac{\left(\int (\nabla v)^{sP}\, dx\right)^{1/(sP)} \left(\int v^{Qs(1-\theta)/\theta} dx\right)^{1/(sQ)}}{\left(\int v^{t/\theta} dx\right)^{1/t}} && \text{[take } P = 2/s,\ Q = 2/(2-s)] \\
&= \frac{1}{\theta}\frac{\left(\int (\nabla v)^2\, dx\right)^{1/2} \left(\int v^{Qs(1-\theta)/\theta} dx\right)^{(2-s)/(2s)}}{\left(\int v^{t/\theta} dx\right)^{1/t}} && \text{[take } s = 2\theta \text{, and } r = t/\theta = 2N/(N-2\theta)] \\
&= \frac{1}{\theta}\frac{\|\nabla v\|_2 \|v\|_2^{(1-\theta)/\theta}}{\|v\|_r^{1/\theta}} = \frac{1}{\theta}\left(\Lambda_{N,\theta}(v)\right)^{1/\theta},
\end{aligned}$$

for the choice $s = 2\theta$. We have to restrict θ to the interval $1/2 \le \theta \le 1$ to give the right-hand side of (31) a meaning. Again, the inequality is strict since $w = v^{\theta}_{N,\theta}$ does not equal a function $w_{N,s}$ (see (1.21)), with $s = 2\theta$.

iii) Combining i) with $\theta'' = 0$ and ii) one finds

$$\Lambda_{N,\theta} > (\theta' C_{N,2\theta'})^{\theta},\ 0 < \theta < 1,\ \theta \le \theta',\ 1/2 \le \theta' < 1. \tag{2.29}$$

This motivates the determination of the maximum of $\theta C_{N,2\theta} = (N/(2p))C_{N,N/p}$ on $1/2 \le \theta < 1$. There holds by (1.17), (1.18)

$$\begin{aligned}
\frac{N}{2p}C_{N,N/p} &= \frac{N^2}{2p}\left(\frac{p-1}{N-p}\right)^{(N-p)/N} \left[N\omega_N B(p, N+1-p)\right]^{1/N}, \quad 1 < p < N, \\
\tfrac{1}{2}C_{N,1} &= (N/2)\omega_N^{1/N}, \quad p = N, \quad \theta = 1/2.
\end{aligned} \tag{2.30}$$

The maximum of (2.30) is found by putting the logaritmic derivative of (2.30) with respect to p equal to zero, which is equation (1.34). It can be proven that (1.34) has a unique solution p_N, $1 < p_N < N$, because $\frac{d}{dp}M(N,p) \le 0$. For this last inequality we use the fact that $\psi'(z) < 1/z + 1/(2z^2) + 3/(4z^3)$. So, with $\theta_N = N/(2p_N)$ and for $0 < \theta \le \theta_N$, there holds $\Lambda_{N,\theta} > (\theta_N C_{N,2\theta_N})^{\theta}$, and for the remaining interval $\theta_N \le \theta < 1$, $\lambda_{N,\theta} > (\theta C_{N,2\theta})^{\theta}$.

iv) Since $\lim_{p\to N} M(N,p) = -\infty$, it follows that $\theta C_{N,2\theta} > C_{N,2}$ for θ in a neighbourhood of $\theta = 1$. So (1.33) follows from (2.29). □

Remark 2.5. Application of Theorem 1.7 i) with $\theta'' = 0$, $\alpha = \theta/\theta'$, gives

$$\lambda^2_{N,\theta} \ge \lambda^{2\theta/\theta'}_{N,\theta'}, \quad \theta' > \theta. \tag{2.31}$$

[15, (2.21)] give the inequality

$$L^1_{\gamma,N} \le L^1_{\gamma-1,N}(\gamma/(\gamma + N/2)),\ \gamma > 2 - N/2. \tag{2.32}$$

By (1.27) this is equivalent with

$$\lambda^2_{N,\theta} \ge \lambda^{2\theta/\theta'}_{N,\theta'} F(\theta,\theta'),\ \theta = N/(2p), \quad \theta' = N/(2(p-1)), \tag{2.33}$$

with

$$F(\theta,\theta') = [(1-\theta)/(1-\theta')]^{\theta(1-\theta')/\theta'}\,(\theta/\theta')^{\theta}.$$

For $\theta' > \theta$ it will be proved that $F(\theta,\theta') < 1$, which means that i) of Theorem 1.7 (equation (2.31)) is better than (2.32). $F(\theta,\theta') < 1$ is equivalent with

$$[\theta(1-\theta')/(\theta'(1-\theta))]^{\theta'} < (1-\theta')/(1-\theta), \tag{2.34}$$

and (2.34) is true by the inequality $(1-a)^b < 1-ab$, $0 < a < 1$, $b < 1$, where $a = (\theta'-\theta)/(\theta'(1-\theta))$, $b = \theta'$.

Remark 2.6. To show the merits Theorem 1.7 of ii) we compare two known values for $\lambda_{N,\theta}$, see (2.10), (2.11), by the estimate (1.31)

$$\lambda_{2,1/2} \simeq 1.55524 > 1.33134\cdots = \pi^{1/4} = (1/2\,C_{2,1})^{1/2}, \tag{2.35}$$

$$\lambda_{2,2/3} \simeq 1.66287 > 1.63696\cdots = (2\pi/3)^{2/3} = (2/3\,C_{2,4/3})^{2/3}. \tag{2.36}$$

Note that in the work of Levine [13, p. 183, third line] the lower bound (2.36) is not calculated correctly. The lower bound C_1 for his variable C (which is $\lambda^3_{2,2/3}$) should be $C_1 = 4\pi^2/9 \simeq 4.38649$, in stead of $C_1 = 2\pi^{3/2}/9 \simeq 1.237$ ([13, p. 183, eighth line]). This corrected value for C_1 is a much better lower bound, since numerically we found $C = \lambda^3_{2,2/3} \simeq 1.66287^3 \simeq 4.5981$. See also Section 3 and Table 1.

Remark 2.7. Approximate solutions p_N of (1.34) for $N = 2$, 3 and $N \to \infty$ are

$$p_2 \simeq 1.647,\ \theta_2 \simeq 0.6070, \tag{2.37}$$

$$p_3 \simeq 2.304,\ \theta_3 \simeq 0.6509, \tag{2.38}$$

$$p_N = 2N/3 + 5/18 + O(1/N),\ \theta_N = 3/4 - 5/(16N) + O(1/N^2),\ N \to \infty. \tag{2.39}$$

The knowledge of (2.37) allows us to improve (2.35) as follows

$$\lambda_{2,1/2} \simeq 1.55524 > 1.46436\cdots = (1/1.647\,C_{2,1.2140})^{1/2}. \tag{2.40}$$

3. NUMERICAL EXPERIMENTS

In order to assess the quality of the estimates (1.31), (1.32), (1.36) and (1.37) we have calculated the numbers $\lambda_{N,\theta}$ for $N = 2,3$ and $\theta = 0.1 + (i-1)0.005$, $i = 1,2,3,\cdots,180$, and for $N = 4,5,10$, and $\theta = 0.0125 + (i-1)0.025$, $i = 1,2,3,\cdots,40$. For $N = 2$ we had to exclude $\theta \geq 0.945$ due to numerical overflow. The method to find $\lambda_{N,\theta}$ consists of a shooting technique to find that value $v(0) = v_0$ such that $v(r)$ is a positive solution of (2.1) with $\lim_{r\to\infty} v(r) = 0$. Therefore, we transformed the interval $r \in (0,\infty)$ into $s = r/(1+r) \in (0,1)$. The transformed differential equation becomes, with $v(r) = u(s)$, $0 < s < 1$,

$$(1-s)^4\frac{d^2}{ds^2}u + \left\{\left(\frac{(N-1)}{s} - 2\right)(1-s)^3\right\}\frac{d}{ds}u - u|u|^{(N+2\theta)/(N-2\theta)-1} - u = 0,$$

$$u(0) = v_0, \qquad \frac{d}{ds}u(0) = 0. \tag{3.1}$$

We solved the transformed differential equation (3.1) by means of a numerical integration method (Runge-Kutta of the fourth order) with a self-adapting stepsize routine such that a prescribed maximal relative error (ε_{rel}) in each component ($u(s), \frac{d}{ds}u(s)$) has been satisfied. We made the choice $\varepsilon_{rel} = 10^{-15}$. For every value of v_0 the numerical integrator will find some point $s = s(v_0) \in (0,1)$ where either $u(s) < 0$, or $\frac{d}{ds}u(s) > 0$. At that point s the integration will be stopped. This integrator is coupled to a numerical zero-finding routine (see [4]),

N	θ	p	s	ρ	$\lambda_{N,\theta}$ numerical	$\lambda_{N,\theta}$ lower bnd.	Comment
2	$1/3$	3	$1/2$	1	$1.379427(6)$		numerical, this work
						1.28953	see (1.32), this work
						N.A.	see (1.31), this work
						1.37026	see (1.42), Nasibov
						1.35157	see (1.45), Nasibov
2	$1/2$	2	1	2	1.55524		numerical (2.10), based on Weinstein [29]
					$1.555239(5)$		numerical, this work
						1.46436	see (1.32), this work
						1.33134	see (1.31), this work
						1.51739	see (1.42), Nasibov
						1.51739	see (1.45), Nasibov
2	$2/3$	$3/2$	2	4	1.66287		numerical (2.11), based on Levine [13]
					$1.663066(0)$		numerical, this work
						1.63696	see (1.32), this work
						1.63696	see (1.31), this work
						1.55436	see (1.42), Nasibov
						1.61962	see (1.45), Nasibov
3	$3/4$	2	1	2	$2.2258(9)$		numerical, this work
						2.21005	see (1.32), this work
						2.21005	see (1.31), this work
						2.05668	see (1.42), Nasibov
						2.05668	see (1.45), Nasibov

Table 1: Comparison of some cases for $\lambda_{N,\theta}$; $p = N/(2\theta)$; $s = 2\theta/(N-2\theta)$ (notation Weinstein); $\rho = 4\theta/(N-2\theta)$ (notation Nasibov).

which can also be applied for finding a discontinuity. The function f for which such a discontinuity has to been found is specified by if $u(s(v_0)) < 0$, $f(v_0) = -(1-s(v_0))$ else (that means thus $\frac{d}{ds}u(s(v_0)) > 0$) $f(v_0) = (1-s(v_0))$. The sought value v_0 has been found if this numerical routine has come up with two values v_0 and v_0^1 such that $|v_0 - v_0^1| < r_p|v_0| + a_p$, (with $r_p = a_p = 10^{-15}$ relative and absolute precisions, respectively) and $|f(v_0)| \leq |f(v_0^1)|$, while $sign(f(v_0) = -sign(f(v_0^1))$. During the integration processes the norms in (2.7) will be calculated. As a check upon this procedure the following expressions

$$\|\bar{v}_{N,\theta}\|_2^2/(1-\theta), \quad \|\nabla\bar{v}_{N,\theta}\|_2^2/\theta, \quad \|\bar{v}_{N,\theta}\|_r^r, \tag{3.2}$$

are compared. They should be all equal, see (2.7). In the Table 1 the value for $\lambda_{N,\theta}$ are given with one digit less than the number of equal digits in this comparison; between brackets the next digit is given.

The results of the calculations are shown in the Figures 1, 3, 5, 6, 7. For $N = 2, 3$ part of the θ-range has been enlarged to show better the approximations and the infimum of the functional, see Figures 2, 4. (All figures appear in Appendix A at the end of this paper.)

In Fig. 13 the value $v(0)$ of the minimizer $v(r)$ of the functional $\Lambda_{N,\theta}$ as function of θ for $N = 2, 3, 4, 5, 10$ has been shown. Note the logarithmic ordinate axis for $v(0)$.

4. DISCUSSION

In this article the infimum of the spectrum of the Schrödinger operator $\tau = -\Delta + q$ in $\mathbb{R}^N$ has been expressed in the infimum $\lambda_{N,\theta}$ of the functional $\Lambda_{N,\theta}$, and known estimates for $\lambda_{N,\theta}$ have been optimized and applied to supply estimates of the infimum of the spectrum. Moreover, numerical experiments have been done to calculate $\lambda_{N,\theta}$ as function of θ for $N = 2, 3, 4, 5,$ and 10. These results have been used to compare the estimates found in this article with these found by Nasibov [20].

Except for $N = 2$, in general, the estimate of Nasibov is better for the lower half of the θ-interval, while the estimate in this article is better for the upper half. For $N = 2$ there is an interval (θ_-,θ_+) (with $\theta_- \in (0.615, 0.620)$, and $\theta_+ \in (0.745, 0.750)$) where the bound in this article is better, while the opposite is true outside that interval, see Fig. 8. For $0 < \theta \leq \theta_0$ (where $\theta_0 \in (0.55, 0.65)$ is depending on the value of N, $N = 3, 4, 5, 10$), the lower bound by Nasibov is better, but the bounds are of the same order of magnitude and very close to the actual value of $\lambda_{N,\theta}$; for $\theta_0 < \theta < 1$, the bound of Nasibov is worse, see Figs. 9, 10, 11, and 12.

The ratio of the estimate in this article with $\lambda_{N,\theta}$, for $\theta \to 1$, $N \geq 3$, approaches the value 1, since $\lambda_{N,1} = C_{N,2}$, $N \geq 3$ (see just after (1.16) and the Figs. 9, 10, 11, and 12).

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APPENDIX A. FIGURES

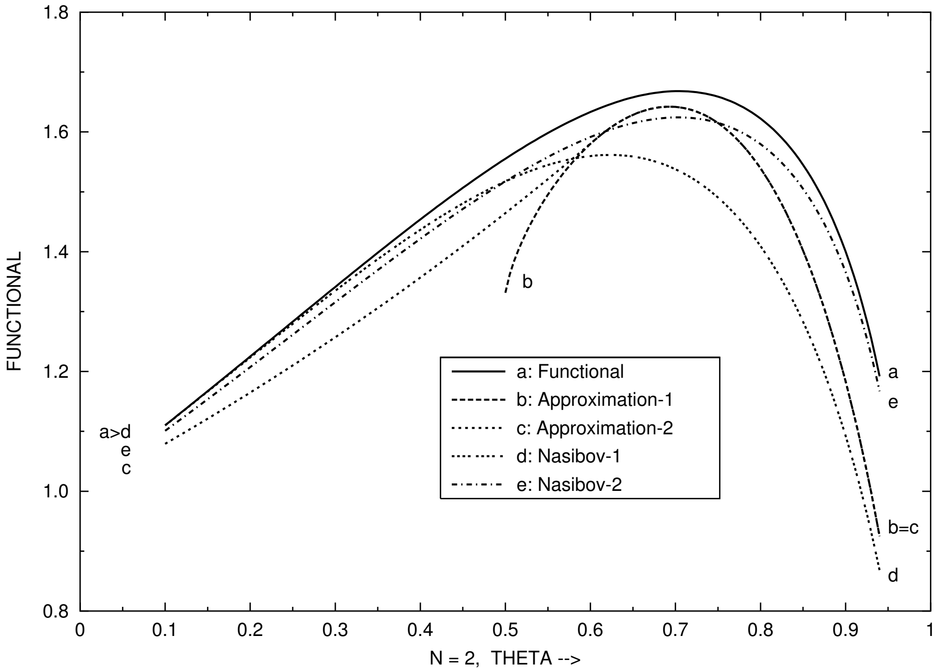

Figure 1: $N = 2$: $\lambda_{2,\theta}$ with four approximations; Approximation-1 corresponds with Theorem 1.7-(ii), Approximation-2 with Theorem 1.7-(iii), Nasibov-1 with (1.43), Nasibov-2 with (1.46).

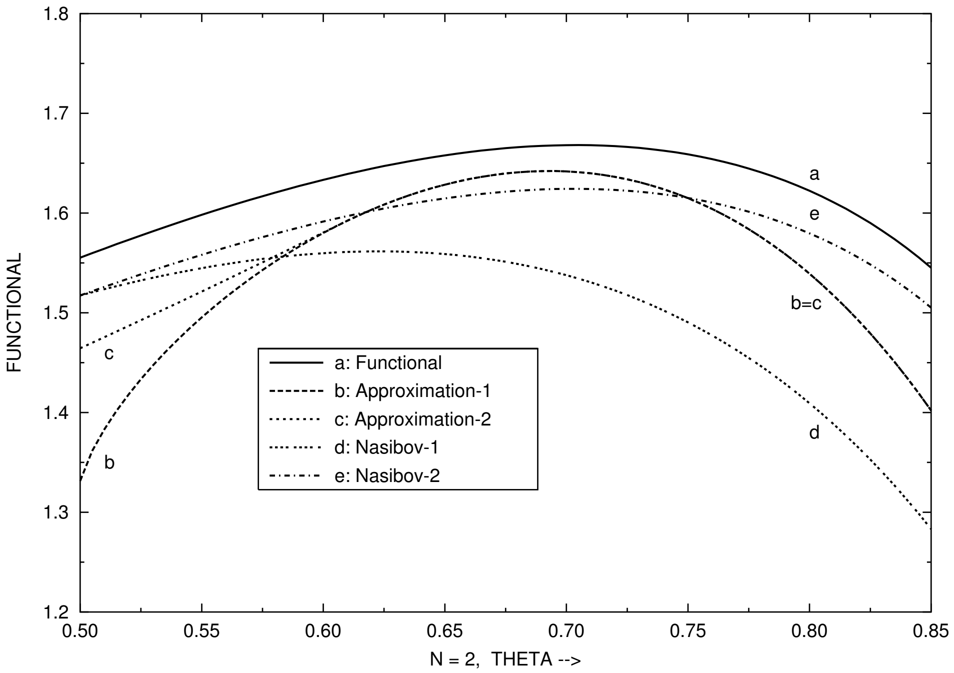

Figure 2: $N = 2$: $\lambda_{2,\theta}$ with four approximations; Approximation-1 corresponds with Theorem 1.7-(ii), Approximation-2 with Theorem 1.7-(iii), Nasibov-1 with (1.43), Nasibov-2 with (1.46).

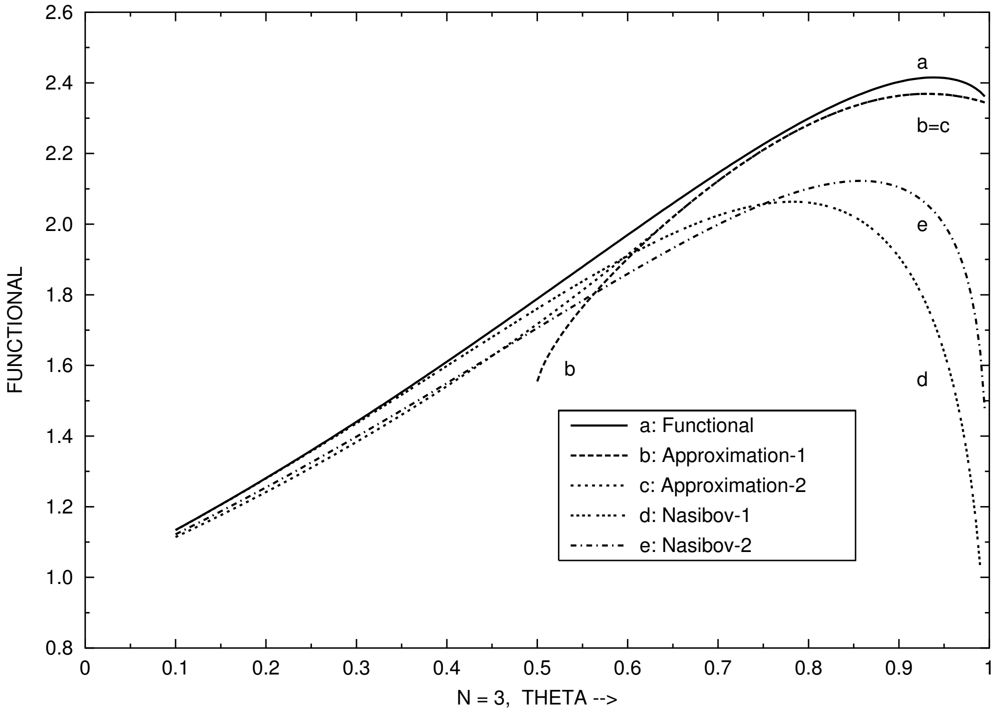

Figure 3: $N = 3$: $\lambda_{3,\theta}$ with four approximations; Approximation-1 corresponds with Theorem 1.7-(ii), Approximation-2 with Theorem 1.7-(iii), Nasibov-1 with (1.43), Nasibov-2 with (1.46).

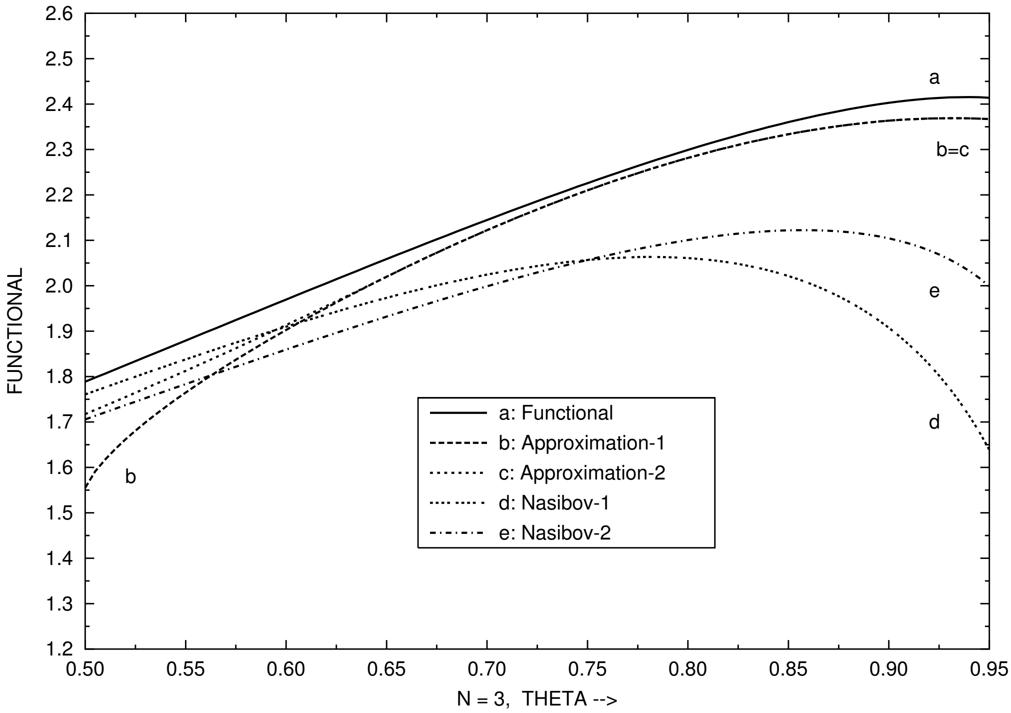

Figure 4: $N = 3$: $\lambda_{3,\theta}$ with four approximations; Approximation-1 corresponds with Theorem 1.7-(ii), Approximation-2 with Theorem 1.7-(iii), Nasibov-1 with (1.43), Nasibov-2 with (1.46).

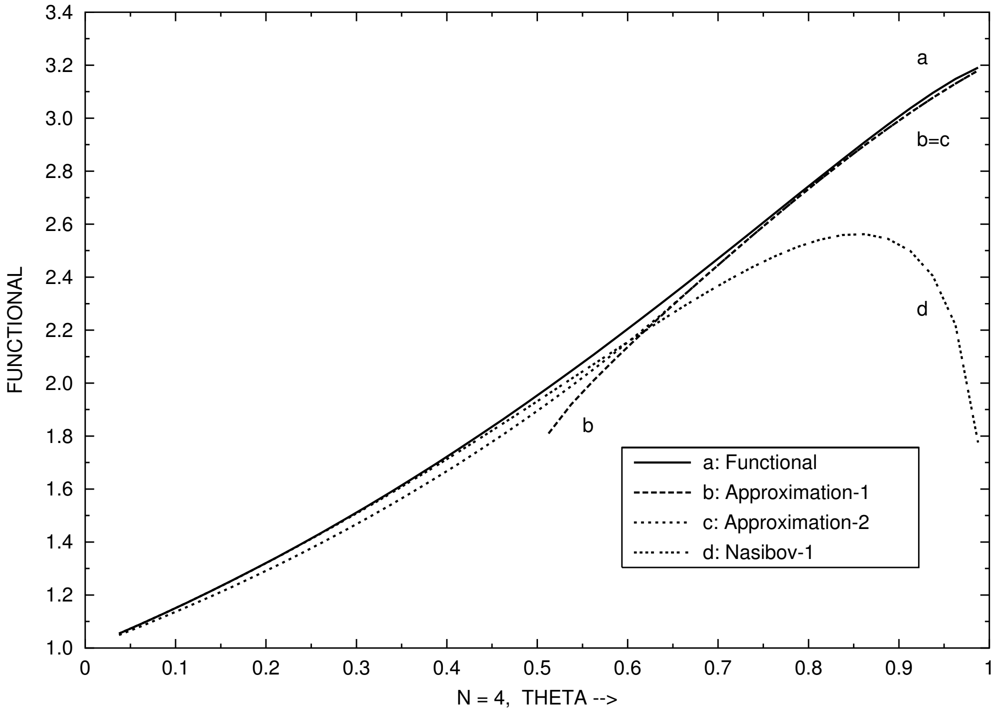

Figure 5: $N = 4$: $\lambda_{4,\theta}$ with three approximations; Approximation-1 corresponds with Theorem 1.7-(ii), Approximation-2 with Theorem 1.7-(iii), Nasibov-1 with (1.43).

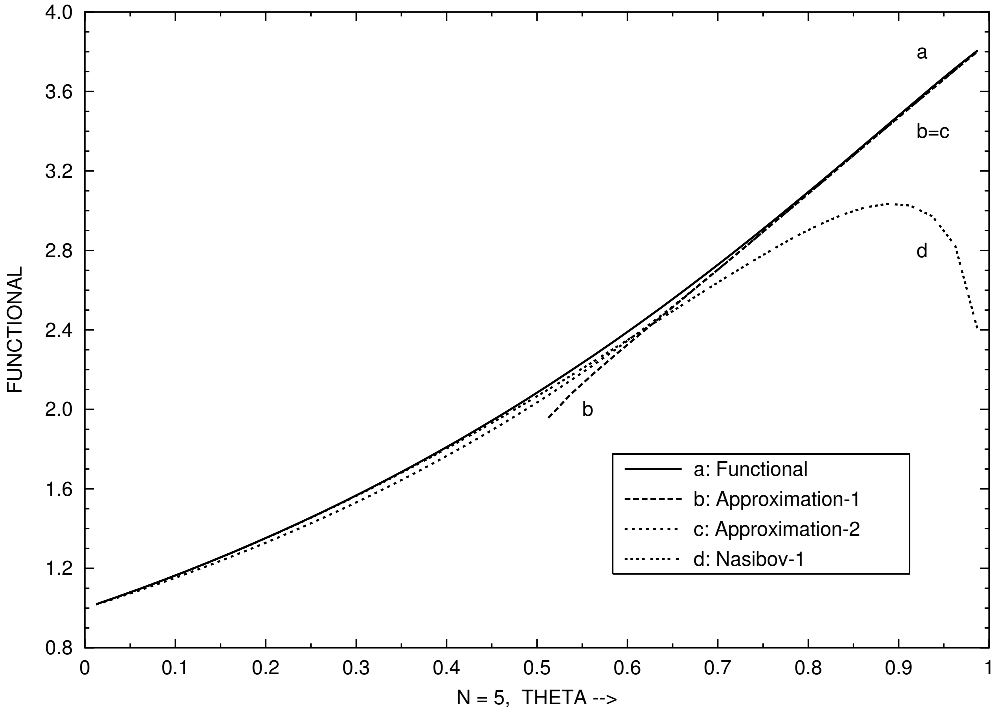

Figure 6: $N = 5$: $\lambda_{5,\theta}$ with three approximations; Approximation-1 corresponds with Theorem 1.7-(ii), Approximation-2 with Theorem 1.7-(iii), Nasibov-1 with (1.43).

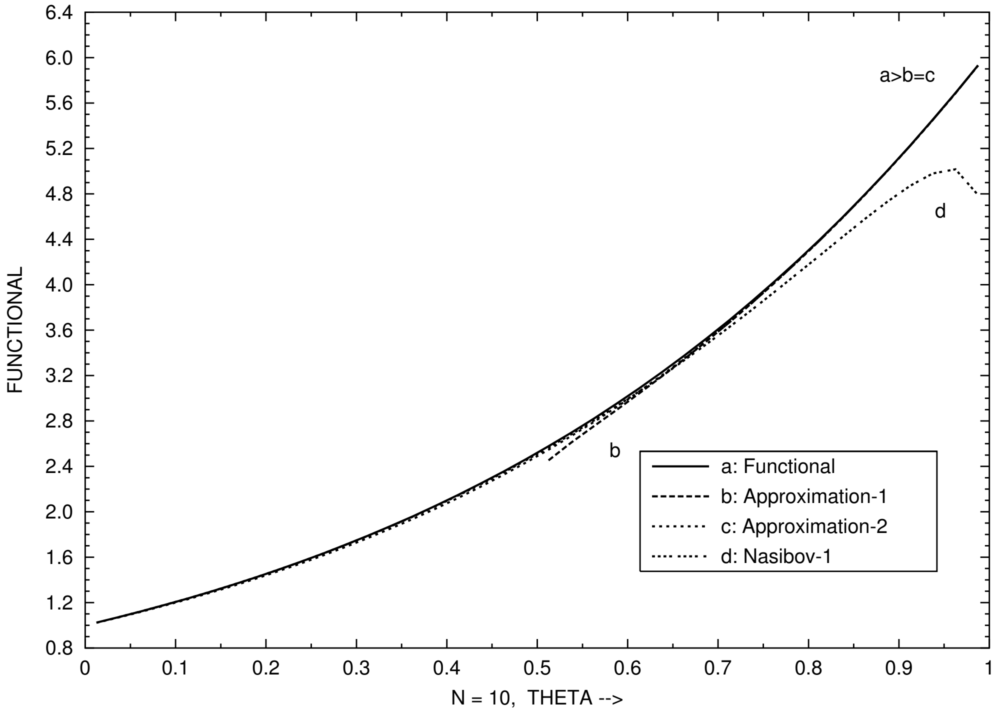

Figure 7: $N = 10$: $\lambda_{10,\theta}$ with three approximations; Approximation-1 corresponds with Theorem 1.7-(ii), Approximation-2 with Theorem 1.7-(iii), Nasibov-1 with (1.43).

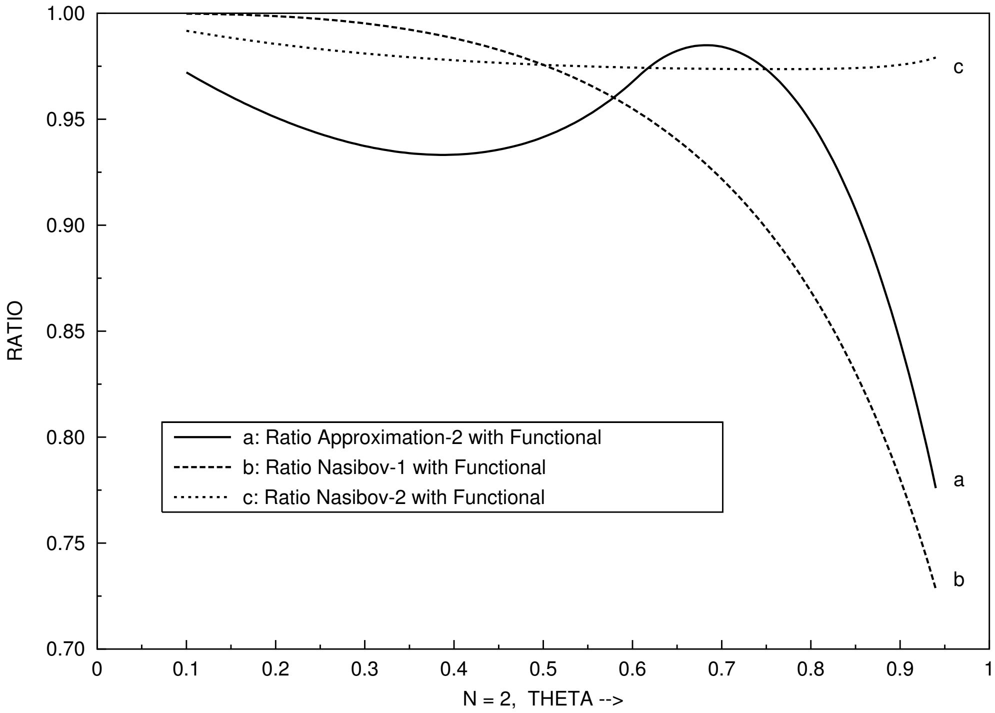

Figure 8: $N = 2$: Ratio of three approximations with $\lambda_{2,\theta}$: Approximation-2 (Theorem 1.7-(iii)), Nasibov-1 (1.43), and Nasibov-2 (1.46).

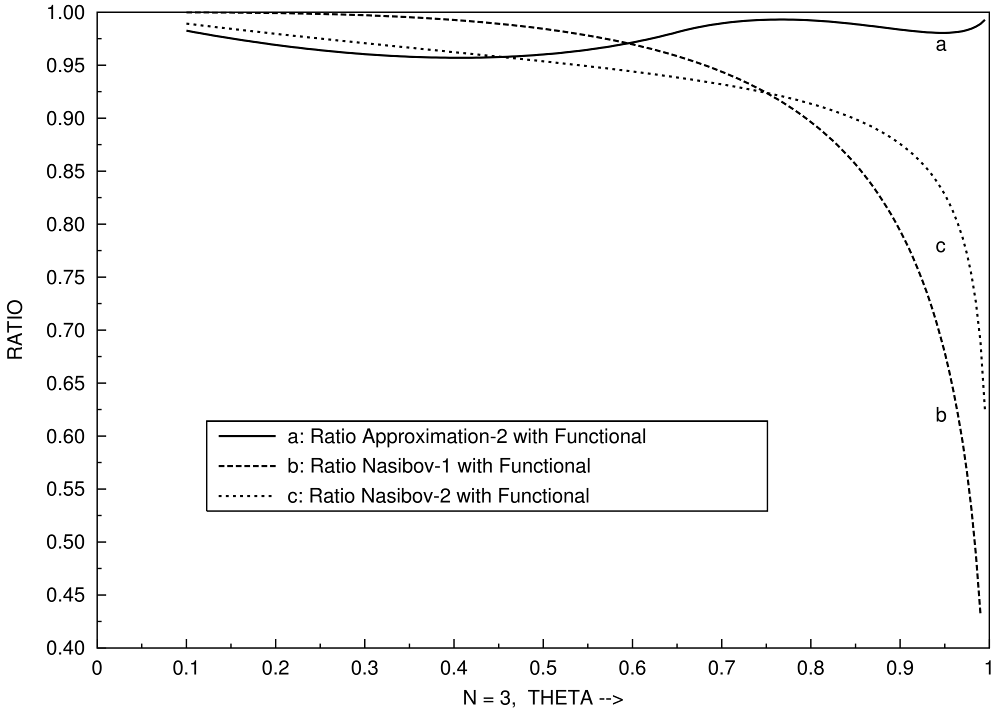

Figure 9: $N = 3$: Ratio of three approximations with $\lambda_{3,\theta}$: Approximation-2 (Theorem 1.7-(iii)), Nasibov-1 (1.43), and Nasibov-2 (1.46).

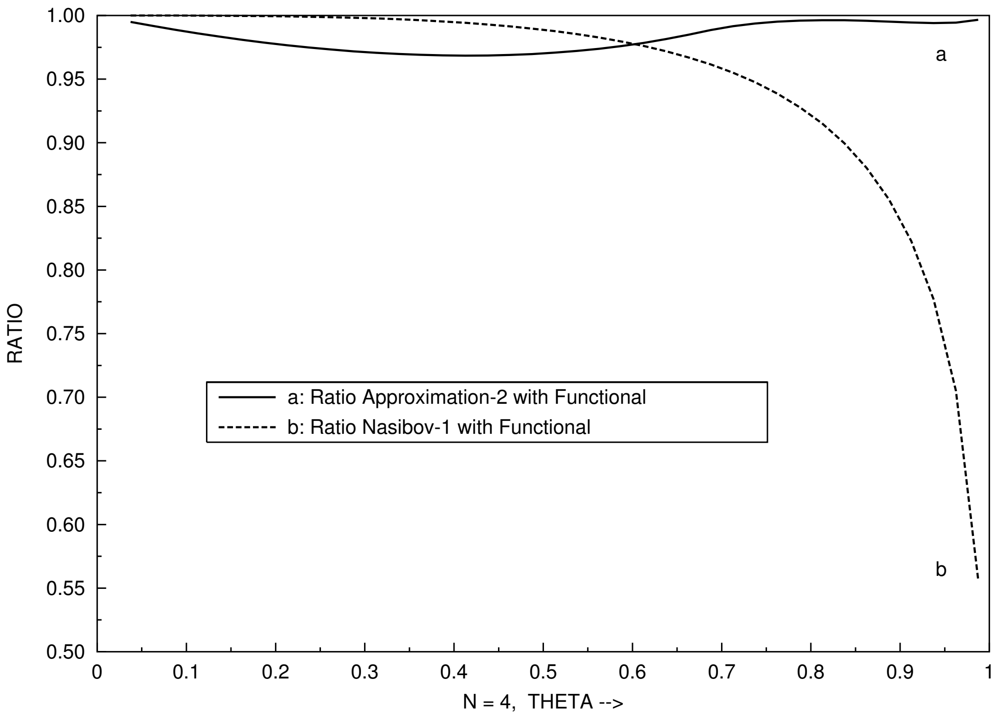

Figure 10: $N = 4$: Ratio of two approximations with $\lambda_{4,\theta}$: Approximation-2 (Theorem 1.7-(iii)) and Nasibov-1 (1.43).

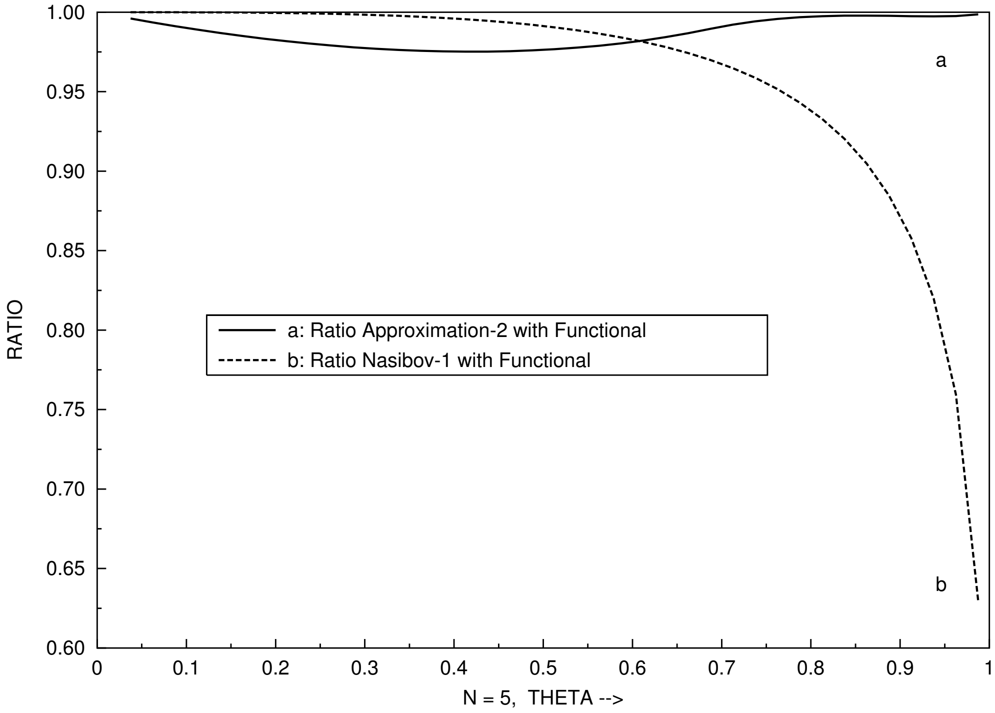

Figure 11: $N = 5$: Ratio of two approximations with $\lambda_{5,\theta}$: Approximation-2 (Theorem 1.7-(iii)) and Nasibov-1 (1.43).

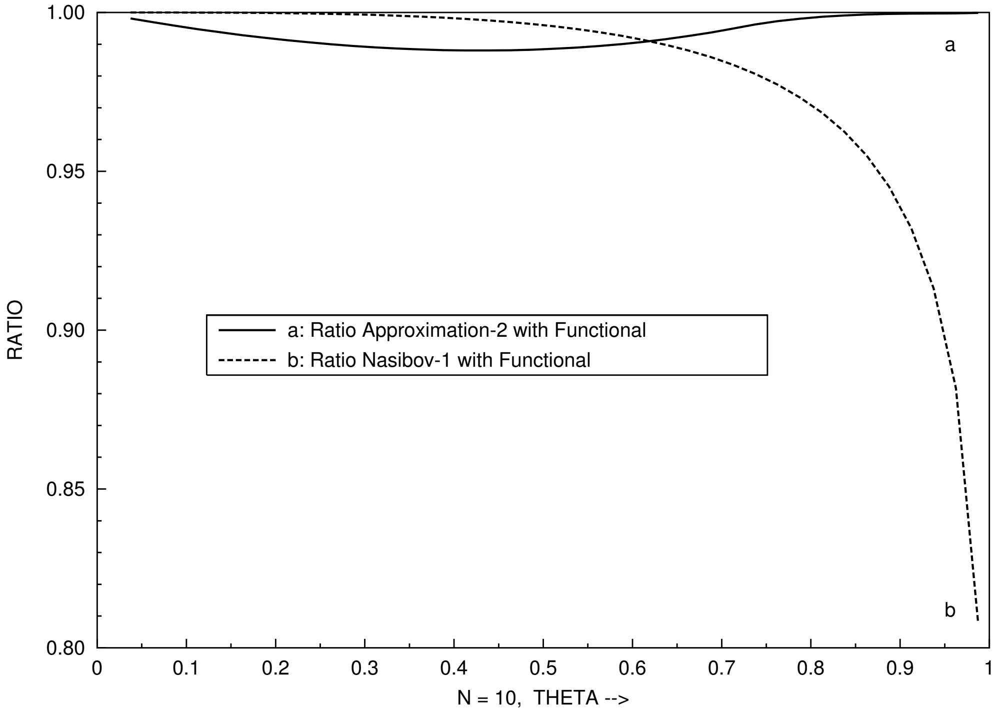

Figure 12: $N = 10$: Ratio of two approximations with $\lambda_{10,\theta}$: Approximation-2 (Theorem 1.7-(iii)) and Nasibov-1 (1.43).

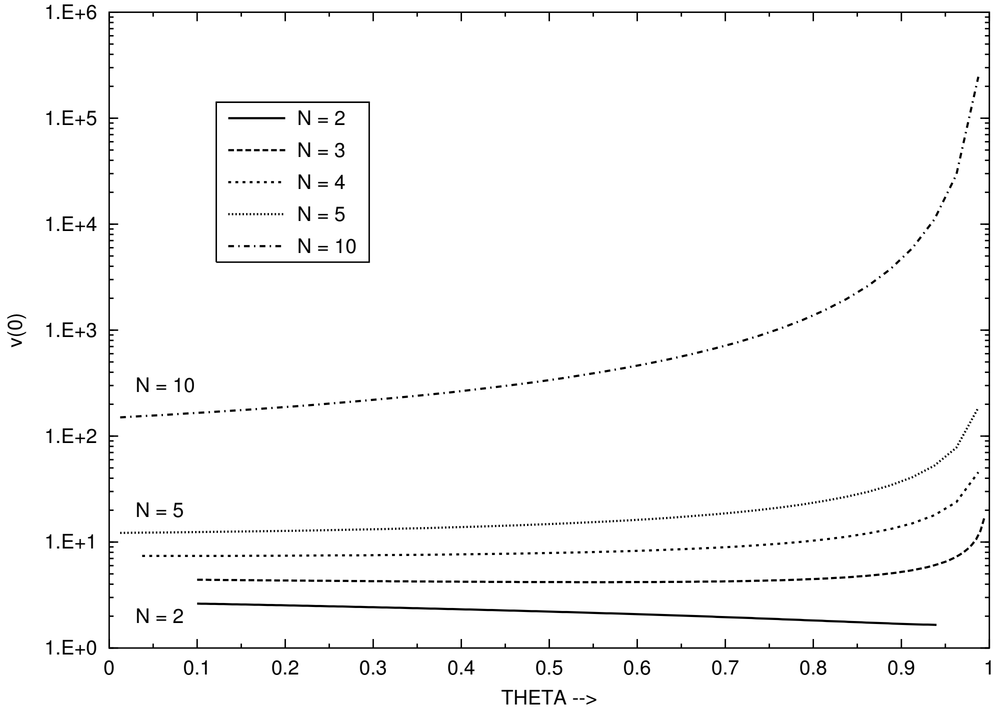

Figure 13: The value $v(0)$ of the minimizer $v(r)$ of the functional $\Lambda_{N,\theta}$ as function of θ for $N = 2, 3, 4, 5, 10$.