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SOME DISTORTION INEQUALITIES ASSOCIATED WITH THE FRACTIONAL DERIVATIVES OF ANALYTIC AND UNIVALENT FUNCTIONS

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ABSTRACT. For the classes $\mathcal{S}$ and $\mathcal{K}$ of (normalized) univalent and convex analytic functions, respectively, a number of authors conjectured interesting extensions of certain known distortion inequalities in terms of a fractional derivative operator. While examining and investigating the validity of these conjectures, many subsequent works considered various generalizations of the distortion inequalities relevant to each of these conjectures. The main object of this paper is to give a direct proof of one of the known facts that these conjectures are false. Several further distortion inequalities involving fractional derivatives are also presented.

Key words and phrases: Distortion inequalities, analytic functions, fractional derivatives, univalent functions, convex functions, hypergeometric function.

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1. INTRODUCTION AND DEFINITIONS

Let $\mathcal{A}$ denote the class of functions $f(z)$ *normalized* by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are *analytic* in the *open* unit disk

$$\mathcal{U} := \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

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Also, let $\mathcal{S}$ and $\mathcal{K}$ denote the subclasses of $\mathcal{A}$ consisting of functions which are, respectively, univalent and convex in $\mathcal{U}$ (see, for details, [4], [5], and [12]).

Geometric Function Theory is the study of the relationship between the *analytic* properties of $f(z)$ and the *geometric* properties of the *image* domain

$$\mathcal{D} = f(\mathcal{U}).$$

An excellent example of this interplay is provided by the following important result which validates a 1916 conjecture of Ludwig Bieberbach (1896-1982):

Theorem 1. de Branges [3]. *If the function* $f(z)$ *given by* (1.1) *is in the class* $\mathcal{S}$, *then*

$$|a_n| \leqq n \qquad (n \in \mathbb{N} \setminus \{1\}\,;\ \mathbb{N} := \{1,2,3,\ldots\}), \tag{1.2}$$

where the equality holds true for all $n \in \mathbb{N} \setminus \{1\}$ *only if* $f(z)$ *is any rotation of the Koebe function*:

$$K(z) := \frac{z}{(1-z)^2} = \sum_{n=1}^{\infty} n z^n \qquad (z \in \mathcal{U}). \tag{1.3}$$

The assertion (1.2) and its *well-known* (rather classical) analogue for the class $\mathcal{K}$ (*cf., e.g.,* [5, p. 117, Theorem 7]) lead us immediately to known distortion inequalities for the nth derivative of functions in the classes $\mathcal{S}$ and $\mathcal{K}$, respectively. Each of the following conjectures, which were made in an attempt to extend these known distortion inequalities for the classes $\mathcal{S}$ and $\mathcal{K}$, involves the *fractional derivative operator* D_z^{λ} *of order* λ, defined by (*cf., e.g.,* [7] and [9])

$$D_z^{\lambda} f(z) := \begin{cases} \dfrac{1}{\Gamma(1-\lambda)} \dfrac{d}{dz} \displaystyle\int_0^z \dfrac{f(\zeta)}{(z-\zeta)^{\lambda}} d\zeta & (0 \leqq \lambda < 1) \\[2ex] \dfrac{d^n}{dz^n} D_z^{\lambda-n} f(z) & (n \leqq \lambda < n+1;\ n \in \mathbb{N}), \end{cases} \tag{1.4}$$

where the function $f(z)$ is analytic in a simply-connected region of the complex z-plane containing the origin, and the multiplicity of $(z-\zeta)^{-\lambda}$ is removed by requiring $\log(z-\zeta)$ to be real when $z - \zeta > 0$.

Conjecture 1. [8, p. 88]. *If the function* $f(z)$ *is in the class* $\mathcal{S}$, *then*

$$\left|D_z^{n+\lambda} f(z)\right| \leqq \frac{(n+\lambda+|z|)\ \Gamma(n+\lambda+1)}{(1-|z|)^{n+\lambda+2}} \tag{1.5}$$
$$(z \in \mathcal{U};\ n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}\,;\ 0 \leqq \lambda < 1),$$

where the equality holds true for the Koebe function $K(z)$ *defined by* (1.3).

Conjecture 2. [10, p. 225]. *If the function* $f(z)$ *is in the class* $\mathcal{K}$, *then*

$$\left|D_z^{n+\lambda} f(z)\right| \leqq \frac{\Gamma(n+\lambda+1)}{(1-|z|)^{n+\lambda+1}} \tag{1.6}$$
$$(z \in \mathcal{U};\ n \in \mathbb{N}_0;\ 0 \leqq \lambda < 1),$$

where the equality holds true for the function $L(z)$ *defined by*

$$L(z) := \frac{z}{1-z} = \sum_{n=1}^{\infty} z^n \qquad (z \in \mathcal{U}). \tag{1.7}$$

For $\lambda = 0$ and $n \in \mathbb{N}_0$, Conjectures 1 and 2 can easily be validated by means of the aforementioned known distortion inequalities. Each of these conjectures has indeed been proven to be false for $0 < \lambda < 1$ and $n \in \mathbb{N}_0$ (see, for details, [1], [2], and [6]; see also a recent work of Srivastava [11], which presents various *further* developments and generalizations relevant to the aforementioned conjectures). Our main objective in this paper is to give a *direct* proof of the fact that Conjecture 1 is not true for $0 < \lambda < 1$ and $n \in \mathbb{N}_0$. We also derive several further distortion inequalities involving fractional derivatives.

In our present investigation, we shall also make use of the *hypergeometric function* defined by

$$F(a,b;c;z) := \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!} \tag{1.8}$$
$$\left(a,b,c \in \mathbb{C};\ c \notin \mathbb{Z}_0^- := \{0,-1,-2,\ldots\}\right),$$

where $(\lambda)_k$ denotes the Pochhammer symbol given, in terms of Gamma functions, by

$$(\lambda)_k := \frac{\Gamma(\lambda+k)}{\Gamma(\lambda)} = \begin{cases} 1 & (k=0) \\ \lambda(\lambda+1)\ldots(\lambda+k-1) & (k \in \mathbb{N}). \end{cases} \tag{1.9}$$

The hypergeometric function is analytic in $\mathcal{U}$ and

$$F(a,b;c;z) = F(b,a;c;z). \tag{1.10}$$

Furthermore, it possesses the following integral representation:

$$F(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\,dt \tag{1.11}$$
$$(\Re(c) > \Re(b) > 0;\ |\arg(1-z)| \leqq \pi - \varepsilon;\ 0 < \varepsilon < \pi).$$

It is easily seen from the definition (1.4) that

$$D_z^\lambda\left\{z^{\mu-1}\right\} = \frac{\Gamma(\mu)}{\Gamma(\mu-\lambda)} z^{\mu-\lambda-1} \qquad (0 \leqq \lambda < 1;\ \mu > 0), \tag{1.12}$$

so that

$$D_z^\lambda\left\{z^{\mu-1}(1-z)^{-\nu}\right\} = \frac{\Gamma(\mu)}{\Gamma(\mu-\lambda)} z^{\mu-\lambda-1} F(\mu,\nu;\mu-\lambda;z) \tag{1.13}$$
$$(0 \leqq \lambda < 1;\ \mu > 0;\ \nu \in \mathbb{R};\ z \in \mathcal{U}).$$

Thus, for the extremal functions $K(z)$ and $L(z)$ defined by (1.3) and (1.7), respectively, by suitably further specializing the fractional derivative formula (1.13) *with* $\mu = 2$, we obtain

$$D_z^\lambda K(z) = \frac{z^{1-\lambda}}{\Gamma(2-\lambda)} F(2,2;2-\lambda;z) \tag{1.14}$$
$$(0 \leqq \lambda < 1;\ z \in \mathcal{U})$$

and (*cf.* [6])

$$D_z^\lambda L(z) = \frac{z^{1-\lambda}}{\Gamma(2-\lambda)} F(2,1;2-\lambda;z) \tag{1.15}$$
$$(0 \leqq \lambda < 1;\ z \in \mathcal{U}).$$

2. MAIN RESULTS RELEVANT TO CONJECTURE 1

We begin by proving

Theorem 2. *Let* $0 < \lambda < 1$. *Then Conjecture* 1 *is not true for* $n \in \mathbb{N}$.

Proof. For $L(z) \in \mathcal{S}$, it follows from (1.15) and the definition (1.8) that

$$D_z^{\lambda} L(z) = z^{-\lambda} \sum_{k=1}^{\infty} \frac{\Gamma(k+1)}{\Gamma(k-\lambda+1)} z^k \tag{2.1}$$
$$(0 < \lambda < 1;\ z \in \mathcal{U} \setminus \{0\}),$$

where $z^{-\lambda}$ is analytic in $\mathcal{U} \setminus \{0\}$ and the multiplicity of $z^{-\lambda}$ is removed by requiring $\log z$ to be real when $z > 0$. Thus, by the definition (1.4), we have

$$\begin{aligned} D_z^{1+\lambda} L(z) &= \left(z^{-\lambda}\right)' \sum_{k=1}^{\infty} \frac{\Gamma(k+1)}{\Gamma(k-\lambda+1)} z^k + z^{-\lambda} \left(\sum_{k=1}^{\infty} \frac{\Gamma(k+1)}{\Gamma(k-\lambda 1)} z^k \right)' \\ &= z^{-1-\lambda} \sum_{k=1}^{\infty} \frac{\Gamma(k+1)}{\Gamma(k-\lambda)} z^k \qquad (0 < \lambda < 1;\ z \in \mathcal{U} \setminus \{0\}). \end{aligned} \tag{2.2}$$

By the principle of mathematical induction, it can be shown by using (2.2) that

$$\begin{aligned} D_z^{n+\lambda} L(z) &= z^{-n-\lambda} \sum_{k=1}^{\infty} \frac{\Gamma(k+1)}{\Gamma(k-\lambda-n+1)} z^k \\ &= \frac{z^{1-n-\lambda}}{\Gamma(2-n-\lambda)} F(2,1;2-n-\lambda;z) \end{aligned} \tag{2.3}$$
$$(0 < \lambda < 1;\ n \in \mathbb{N};\ z \in \mathcal{U} \setminus \{0\}).$$

Upon setting $z = r$ $(0 < r < 1)$ in (2.3), if we let $r \to 0$, it is easily seen that

$$D_z^{n+\lambda} L(z) \Big|_{z=r} \to \infty \qquad (r \to 0;\ 0 < \lambda < 1;\ n \in \mathbb{N}). \tag{2.4}$$

On the other hand, if Conjecture 1 is true, the claimed assertion (1.5) readily yields

$$\left| D_z^{n+\lambda} L(z) \right| \leqq M(n;\lambda) \qquad (|z| \to 0;\ 0 < \lambda < 1;\ n \in \mathbb{N}), \tag{2.5}$$

where $M(n;\lambda)$ is a (*finite*) constant depending only on n and λ. This contradiction with (2.4) evidently completes the proof of Theorem 2. □

Next we prove

Theorem 3. *Let the function* $f(z)$ *be in the class* $\mathcal{S}$. *Then*

$$\left| D_z^{\lambda} f(z) \right| \leqq \frac{r^{1-\lambda}}{\Gamma(1-\lambda)} \int_0^1 \frac{1+rt}{(1-t)^{\lambda} (1-rt)^3} dt \tag{2.6}$$
$$(r = |z|;\ z \in \mathcal{U};\ 0 < \lambda < 1),$$

where the equality holds true for the Koebe function $K(z)$ *given by* (1.3).

Proof. Suppose that the function $f(z) \in \mathcal{S}$ is given by (1.1). Then, by using (1.12) in conjunction with (1.1), we obtain

$$D_z^{\lambda} f(z) = z^{-\lambda} \sum_{k=1}^{\infty} \frac{\Gamma(k+1)}{\Gamma(k-\lambda+1)} a_k z^k \tag{2.7}$$
$$(a_1 := 1;\ 0 < \lambda < 1;\ z \in \mathcal{U}),$$

where the multiplicity of $z^{-\lambda}$ is removed as in Theorem 2.

By applying the assertion (1.2) of Theorem 1 on the right-hand side of (2.7), we have

$$\begin{aligned} \left|D_z^{\lambda} f(z)\right| &\leqq r^{-\lambda} \sum_{k=1}^{\infty} \frac{\Gamma(k+1)}{\Gamma(k-\lambda+1)}\, k r^k \\ &= \frac{r^{1-\lambda}}{\Gamma(2-\lambda)} \sum_{k=0}^{\infty} \frac{(2)_k (1)_k}{(2-\lambda)_k}\, \frac{(k+1) r^k}{k!} \\ &= \frac{r^{1-\lambda}}{\Gamma(2-\lambda)} \left(rF(2,1;2-\lambda;r)\right)' \end{aligned} \tag{2.8}$$

$$\left(r=|z|;\ z\in\mathcal{U};\ 0<\lambda<1\right).$$

Since $0<1<2-\lambda$ $(0<\lambda<1)$, we can make use of the integral representation (1.11), and we thus find that

$$\left(rF(2,1;2-\lambda;r)\right)' = (1-\lambda)\int_0^1 \frac{1+rt}{(1-t)^{\lambda}(1-rt)^3}dt, \tag{2.9}$$

which, when substituted for in (2.8), immediately yields the assertion (2.6) of Theorem 3.

Finally, by taking the Koebe function $K(z)$ for $f(z)$ in (2.6), we can see that the result is sharp. □

Remark 1. Theorem 3 can also be deduced by applying the case $n=0$ of a known result due to Cho *et al.* [2, p. 120, Theorem 3].

Remark 2. By comparing the assertions (2.6) and (1.5) *with* $n=0$, it readily follows that Conjecture 1 is not true also when $n=0$ and $0<\lambda<1$.

3. A DISTORTION INEQUALITY INVOLVING THE HYPERGEOMETRIC FUNCTION

In this section, we prove a distortion inequality involving the hypergeometric function, which is given by

Theorem 4. *Let the function* $f(z)$ *be in the class* $\mathcal{S}$. *Then*

$$\left|D_z^{1+\lambda} f(z)\right| \leqq \frac{r^{-\lambda}}{\Gamma(1-\lambda)}\left(rF(2,1;1-\lambda;r)\right)' \tag{3.1}$$
$$\left(r=|z|;\ z\in\mathcal{U}\setminus\{0\};\ 0<\lambda<1\right),$$

where the equality holds true for the Koebe function $K(z)$ *given by* (1.3).

Proof. For the function $f(z)\in\mathcal{S}$ given by (1.1), it follows from (2.7) and the definition (1.4) that

$$D_z^{1+\lambda} f(z) = z^{-1-\lambda}\sum_{k=1}^{\infty}\frac{\Gamma(k+1)}{\Gamma(k-\lambda)}\, a_k z^k \tag{3.2}$$
$$\left(a_1:=1;\ 0<\lambda<1;\ z\in\mathcal{U}\setminus\{0\}\right),$$

since $z^{-\lambda}$ is analytic in $\mathcal{U}\setminus\{0\}$.

Applying the assertion (1.2) of Theorem 1 once again, we find from (3.2) that

$$\begin{aligned} \left|D_z^{1+\lambda} f(z)\right| &\leqq r^{-1-\lambda} \sum_{k=1}^{\infty} \frac{\Gamma(k+1)}{\Gamma(k-\lambda)}\, k r^k \\ &= \frac{r^{-\lambda}}{\Gamma(1-\lambda)} \sum_{k=0}^{\infty} \frac{(2)_k\,(1)_k}{(1-\lambda)_k}\, \frac{(k+1)\, r^k}{k!} \\ &= \frac{r^{-\lambda}}{\Gamma(1-\lambda)}\, \left(rF(2,1;1-\lambda;r)\right)' \end{aligned} \tag{3.3}$$

$$(r=|z|\,;\; z\in\mathcal{U}\setminus\{0\}\,;\; 0<\lambda<1)\,,$$

which proves the inequality (3.1).

By taking the Koebe function $K(z)$ for $f(z)$ in (3.1), we thus complete our direct proof of Theorem 4. □

Remark 3. The assertion (3.1) of Theorem 4 can also be proven by appealing to the case $n=1$ of the aforementioned known result due to Cho *et al*. [2, p. 120, Theorem 3].

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