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A PRIORI ESTIMATE FOR A SYSTEM OF DIFFERENTIAL OPERATORS

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[Abstract](#)

[Contents](#)



[Home Page](#)

[Go Back](#)

[Close](#)

[Quit](#)

Abstract

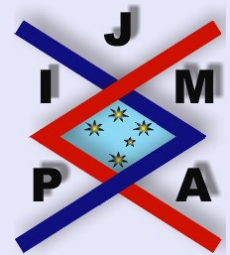
We characterize in algebraic terms an inequality in Sobolev spaces for a system of differential operators with constant coefficients.

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Key words: Differential operators, a priori estimate

Contents

1	Introduction	3
2	The Results	5
	References	



A Priori Estimate for a System of Differential Operators

Chikh Bouzar

Title Page

Contents



Go Back

Close

Quit

Page 2 of 10

1. Introduction

We are interested in the following inequality

$$(1.1) \quad \exists C > 0, \|R(D)u\| \leq C \sum_{j=1}^k \|P_j(D)u\|, \forall u \in C_0^\infty(\Omega),$$

where $S = \{P_j(D); j = 1, \dots, k\}$, $R(D)$ are linear differential operators of order $\leq m$ with constant complex coefficients and $C_0^\infty(\Omega)$ is the space of infinitely differentiable functions with compact supports in a bounded open set Ω of the Euclidian space \mathbb{R}^n . By $\|\cdot\|$ we denote the norm of the Hilbert space $L^2(\Omega)$ of square integrable functions.

Each differential operator $P_j(D)$ has a complete symbol $P_j(\xi)$ such that

$$(1.2) \quad P_j(\xi) = p_j(\xi) + q_j(\xi) + r_j(\xi) + \dots,$$

where $p_j(\xi)$, $q_j(\xi)$ and $r_j(\xi)$ are the homogeneous polynomial parts of $P_j(\xi)$ in $\xi \in \mathbb{R}^n$ of orders, respectively, m , $m - 1$ and $m - 2$.

It is well-known that the system S satisfies the inequality (1.1) for all differential operators $R(D)$ of order $\leq m$ if and only if it is elliptic, i.e.

$$(1.3) \quad \sum_{j=1}^k |p_j(\xi)| \neq 0, \forall \xi \in \mathbb{R}^n \setminus 0.$$

In this paper we give an necessary and sufficient algebraic condition on the system S such that it satisfies the inequality (1.1) for all differential operators



A Priori Estimate for a System
of Differential Operators

Chikh Bouzar

Title Page

Contents



Go Back

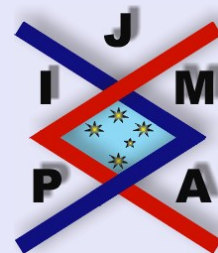
Close

Quit

Page 3 of 10

$R(D)$ of order $\leq m - 1$.

The estimate (1.1) has been used in our work [1], without proof, in the study of local estimates for certain classes of pseudodifferential operators.



**A Priori Estimate for a System
of Differential Operators**

Chikh Bouzar

Title Page

Contents



Go Back

Close

Quit

Page 4 of 10

2. The Results

To prove the main theorem we need some lemmas. The first one gives an algebraic characterization of the inequality (1.1) based on a well-known result of Hörmander [3].

Recall the Hörmander function

$$(2.1) \quad \tilde{P}_j(\xi) = \left(\sum_{\alpha} \left| P_j^{(\alpha)}(\xi) \right|^2 \right)^{\frac{1}{2}},$$

where $P_j^{(\alpha)}(\xi) = \frac{\partial^{|\alpha|}}{\partial \xi_1^{\alpha_1} \dots \partial \xi_n^{\alpha_n}} P_j(\xi)$, (see [3]).

Lemma 2.1. *The inequality (1.1) holds for every $R(D)$ of order $\leq m - 1$ if and only if*

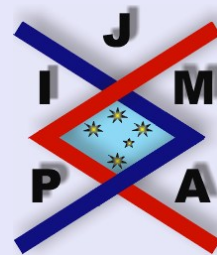
$$(2.2) \quad \exists C > 0, \quad |\xi|^{m-1} \leq C \sum_{j=1}^k \tilde{P}_j(\xi), \quad \forall \xi \in \mathbb{R}^n.$$

Proof. The proof of this lemma follows essentially from the classical one in the case of $k = 1$, and it is based on Hörmander's inequality (see [3, p. 7]). \square

The scalar product in the complex Euclidian space C^k of $A = (a_1, \dots, a_k)$ and $B = (b_1, \dots, b_k)$ is denoted as usually by $A \cdot B = \sum_{i=1}^k a_i \bar{b}_i$, and the norm of C^k by $|\cdot|$.

Let, by definition,

$$(2.3) \quad |A \wedge B|^2 = \sum_{i < j}^k |a_i b_j - b_i a_j|^2.$$



A Priori Estimate for a System
of Differential Operators

Chikh Bouzar

Title Page

Contents



Go Back

Close

Quit

Page 5 of 10

The next lemma is a consequence of the classical Lagrange's identity (see [2]).

Lemma 2.2. Let $A = (a_1, \dots, a_k) \in C^k$ and $B = (b_1, \dots, b_k) \in C^k$, then

$$(2.4) \quad |At + B|^2 = \left(|A|t + \frac{\operatorname{Re}(A \cdot B)}{|A|} \right)^2 + \frac{|\operatorname{Im}(A \cdot B)|^2 + |A \wedge B|^2}{|A|^2}, \forall t \in \mathbb{R}.$$

Proof. We have

$$\begin{aligned} |At + B|^2 &= (|A|t)^2 + 2t\operatorname{Re}(A \cdot B) + |B|^2 \\ &= \left(|A|t + \frac{\operatorname{Re}(A \cdot B)}{|A|} \right)^2 + |B|^2 - \left(\frac{\operatorname{Re}(A \cdot B)}{|A|} \right)^2. \end{aligned}$$

We obtain (2.4) from the next classical Lagrange's identity

$$|A|^2 |B|^2 = |\operatorname{Re}(A \cdot B)|^2 + |\operatorname{Im}(A \cdot B)|^2 + |A \wedge B|^2.$$

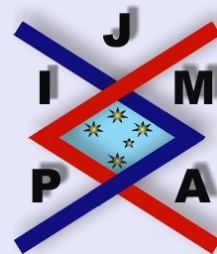
□

For $\xi \in \mathbb{R}^n$ we define the vector functions

$$(2.5) \quad A(\xi) = (p_1(\xi), \dots, p_k(\xi)) \text{ and } B(\xi) = (q_1(\xi), \dots, q_k(\xi)).$$

Let

$$(2.6) \quad \Xi = \left\{ \omega \in S^{n-1} : |A(\omega)|^2 = \sum_{j=1}^k |p_j(\omega)|^2 \neq 0 \right\},$$



A Priori Estimate for a System
of Differential Operators

Chikh Bouzar

Title Page

Contents



Go Back

Close

Quit

Page 6 of 10

where S^{n-1} is the unit sphere of \mathbb{R}^n , and

$$(2.7) \quad F(t, \xi) = |\text{grad}A(\xi)|^2 + |A(\xi)t + B(\xi)|^2,$$

where $|\text{grad} A(\xi)|^2 = \sum_{j=1}^k |\text{grad} p_j(\xi)|^2$.

Lemma 2.3. *The inequality (2.2) holds if and only if there exist no sequences of real numbers $t_j \rightarrow +\infty$ and $\omega_j \in S^{n-1}$ such that*

$$(2.8) \quad F(t_j, \omega_j) \rightarrow 0.$$

Proof. Let t_j be a sequence of real numbers and ω_j a sequence of S^{n-1} , using the homogeneity of the functions p, q and r , then (2.2) is equivalent to

$$\frac{|t_j \omega_j|^{2(m-1)}}{\sum_{l=1}^k \tilde{P}_l(t_j \omega_j)^2} = \frac{1}{F(t_j, \omega_j) + 2 \sum_{l=1}^k \text{Re}(p_l(\omega_j) \cdot \bar{r}_l(\omega_j)) + \chi(\omega_j) \cdot O(\frac{1}{t_j})} \leq C,$$

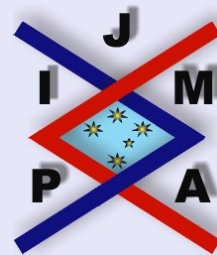
where χ is a bounded function. Hence it is easy to see Lemma 2.3. □

If $\omega \in \Xi$ we define the function G by

$$G(\omega) = |\text{grad}A(\omega)|^2 + \frac{|\text{Im}(A(\omega) \cdot B(\omega))|^2 + |A(\omega) \wedge B(\omega)|^2}{|A(\omega)|^2}.$$

Theorem 2.4. *The estimate (1.1) holds if and only if*

$$(2.9) \quad \exists C > 0, G(\omega) \geq C, \forall \omega \in \Xi$$



A Priori Estimate for a System of Differential Operators

Chikh Bouzar

Title Page

Contents



Go Back

Close

Quit

Page 7 of 10

Proof. All positive constants are denoted by C . If (2.9) holds then from (2.4) and (2.7) we have

$$(2.10) \quad F(t, \omega) = \left(|A(\omega)|t + \frac{\operatorname{Re}(A(\omega).B(\omega))}{|A(\omega)|} \right)^2 + G(\omega) \geq C, \forall \omega \in \Xi, \forall t \geq 0.$$

The vector function A is analytic and the set Ξ is dense in S^{n-1} , therefore by continuity we obtain

$$(2.11) \quad F(t, \omega) \geq C, \forall t \geq 0, \forall \omega \in S^{n-1}.$$

For $\xi \in \mathbb{R}^n$, set $\omega = \frac{\xi}{|\xi|}$ and $t = |\xi|$ in (2.11), as the vector functions A and B are homogeneous, we obtain

$$|A(\xi) + B(\xi)|^2 + |\operatorname{grad}A(\xi)|^2 \geq C |\xi|^{2(m-1)}, \forall \xi \in \mathbb{R}^n,$$

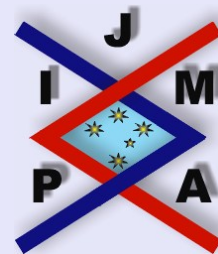
and then, for $|\xi| \geq C$, we have

$$(2.12) \quad \sum_{j=1}^k (|P_j(\xi)|^2 + |\operatorname{grad}P_j(\xi)|^2) + O\left((1 + |\xi|^2)^{m-2}\right) \geq C |\xi|^{2(m-1)}.$$

From the last inequality we easily get (2.2) of Lemma 2.1.

Suppose that (2.9) does not hold, then there exists a sequence $\omega_j \in \Xi$ such that $G(\omega_j) \rightarrow 0$, i.e.

$$(2.13) \quad |\operatorname{grad}A(\omega_j)|^2 \rightarrow 0,$$



A Priori Estimate for a System of Differential Operators

Chikh Bouzar

Title Page

Contents



Go Back

Close

Quit

Page 8 of 10

and

$$(2.14) \quad \frac{|Im(A(\omega_j).B(\omega_j))|^2 + |A(\omega_j) \wedge B(\omega_j)|^2}{|A(\omega_j)|^2} \rightarrow 0.$$

As S^{n-1} is compact we can suppose that $\omega_j \rightarrow \omega_0 \in S^{n-1}$. Hence, from (2.14) and (2.4) with $t = 0$, we obtain

$$(2.15) \quad \frac{Re(A(\omega_j).B(\omega_j))}{|A(\omega_j)|} \rightarrow \pm |B(\omega_0)|.$$

From (2.13), due to Euler's identity for homogeneous functions,

$$(2.16) \quad A(\omega_0) = \vec{0}.$$

Now if $B(\omega_0) = 0$ then $F(t, \omega_0) \equiv 0$, which contradicts (2.8).

Let $B(\omega_0) \neq 0$, and suppose that

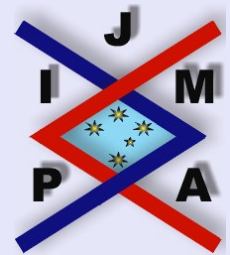
$$(2.17) \quad \frac{Re(A(\omega_j).B(\omega_j))}{|A(\omega_j)|} \rightarrow -|B(\omega_0)|,$$

then setting $t_j = \frac{|B(\omega_j)|}{|A(\omega_j)|}$ in (2.10), it is clear that $t_j \rightarrow +\infty$, so, with $G(\omega_j) \rightarrow 0$, $F(t_j, \omega_j)$ will converge to 0, which contradicts (2.8).

If

$$\frac{Re(A(\omega_j).B(\omega_j))}{|A(\omega_j)|} \rightarrow +|B(\omega_0)|,$$

then changing ω_j to $-\omega_j$ and using the homogeneity of the functions A and B , we obtain the same conclusion. \square



A Priori Estimate for a System of Differential Operators

Chikh Bouzar

Title Page

Contents



Go Back

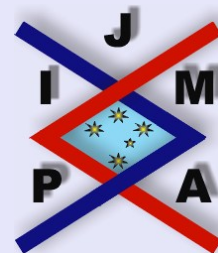
Close

Quit

Page 9 of 10

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A Priori Estimate for a System of Differential Operators

Chikh Bouzar

Title Page

Contents



Go Back

Close

Quit

Page 10 of 10