Journal of Inequalities in Pure and Applied Mathematics

A PRIORI ESTIMATE FOR A SYSTEM OF DIFFERENTIAL OPERATORS

CHIKH BOUZAR

Département de Mathématiques.
Université d'Oran-Essenia.Algérie.
EMail: bouzarchikh@hotmail.com

volume 2, issue 2, article 17, 2001.

Received 23 October, 2000; accepted 22 January 2001.

Communicated by: S. Saitoh

ISSN (electronic): 1443-5756
040-00

Abstract

We characterize in algebraic terms an inequality in Sobolev spaces for a system of differential operators with constant coefficients.

2000 Mathematics Subject Classification: 35B45.
Key words: Differential operators, a priori estimate

Contents

1. Introduction

We are interested in the following inequality

$$\exists C > 0, \|R(D)u\| \le C \sum_{j=1}^{k} \|P_j(D)u\|, \forall u \in C_0^\infty(\Omega), \tag{1.1}$$

where $S = \{P_j(D); j = 1, .., k\}$, $R(D)$ are linear differential operators of order $\le m$ with constant complex coefficients and $C_0^\infty(\Omega)$ is the space of infinitely differentiable functions with compact supports in a bounded open set Ω of the Euclidian space $\mathbb{R}^n$. By $\|.\|$ we denote the norm of the Hilbert space $L^2(\Omega)$ of square integrable functions.

Each differential operator $P_j(D)$ has a complete symbol $P_j(\xi)$ such that

$$P_j(\xi) = p_j(\xi) + q_j(\xi) + r_j(\xi) + \dots, \tag{1.2}$$

where $p_j(\xi)$, $q_j(\xi)$ and $r_j(\xi)$ are the homogeneous polynomial parts of $P_j(\xi)$ in $\xi \in \mathbb{R}^n$ of orders, respectively, m, $m-1$ and $m-2$.

It is well-known that the system S satisfies the inequality (1.1) for all differential operators $R(D)$ of order $\le m$ if and only if it is elliptic, i.e.

$$\sum_{j=1}^{k} |p_j(\xi)| \ne 0, \forall \xi \in \mathbb{R}^n \setminus 0. \tag{1.3}$$

In this paper we give an necessary and sufficient algebraic condition on the system S such that it satisfies the inequality (1.1) for all differential operators

$R(D)$ of order $\leq m-1$.

The estimate (1.1) has been used in our work [1], without proof, in the study of local estimates for certain classes of pseudodifferential operators.

2. The Results

To prove the main theorem we need some lemmas. The first one gives an algebraic characterization of the inequality (1.1) based on a well-known result of Hörmander [3].

Recall the Hörmander function

$$\widetilde{P}_j(\xi) = \left(\sum_{\alpha} \left| P_j^{(\alpha)}(\xi) \right|^2 \right)^{\frac{1}{2}}, \tag{2.1}$$

where $P_j^{(\alpha)}(\xi) = \frac{\partial^{|\alpha|}}{\partial \xi_1^{\alpha_1} \ldots \partial \xi_n^{\alpha_n}} P_j(\xi)$, (see [3]) .

Lemma 2.1. *The inequality (1.1) holds for every $R(D)$ of order $\leq m-1$ if and only if*

$$\exists C > 0, \quad |\xi|^{m-1} \leq C \sum_{j=1}^{k} \widetilde{P}_j(\xi)\,, \forall \xi \in \mathbb{R}^n. \tag{2.2}$$

Proof. The proof of this lemma follows essentially from the classical one in the case of $k = 1$, and it is based on Hörmander's inequality (see [3, p. 7]). □

The scalar product in the complex Euclidian space C^k of $A = (a_1, .., a_k)$ and $B = (b_1, .., b_k)$ is denoted as usually by $A \cdot B = \sum_{i=1}^{k} a_i \overline{b}_i$, and the norm of C^k by $|\cdot|$.

Let, by definition,

$$|A \wedge B|^2 = \sum_{i<j}^{k} |a_i b_j - b_i a_j|^2 \,. \tag{2.3}$$

The next lemma is a consequence of the classical Lagrange's identity (see [2]).

Lemma 2.2. *Let* $A = (a_1, .., a_k) \in C^k$ *and* $B = (b_1, .., b_k) \in C^k$*, then*

(2.4)
$$|At + B|^2 = \left(|A|\,t + \frac{Re\,(A \cdot B)}{|A|}\right)^2 + \frac{|Im\,(A \cdot B)|^2 + |A \wedge B|^2}{|A|^2}, \forall t \in R.$$

Proof. We have

$$\begin{aligned} |At + B|^2 &= (|A|\,t)^2 + 2tRe\,(A \cdot B) + |B|^2 \\ &= \left(|A|\,t + \frac{Re\,(A \cdot B)}{|A|}\right)^2 + |B|^2 - \left(\frac{Re\,(A \cdot B)}{|A|}\right)^2. \end{aligned}$$

We obtain (2.4) from the next classical Lagrange's identity

$$|A|^2\,|B|^2 = |Re\,(A \cdot B)|^2 + |Im\,(A \cdot B)|^2 + |A \wedge B|^2\,.$$

□

For $\xi \in \mathbb{R}^n$ we define the vector functions

(2.5) $$A(\xi) = (p_1(\xi), .., p_k(\xi)) \text{ and } B(\xi) = (q_1(\xi), .., q_k(\xi)).$$

Let

(2.6) $$\Xi = \left\{\omega \in S^{n-1} : |A(\omega)|^2 = \sum_{j=1}^{k} |p_j(\omega)|^2 \neq 0\right\},$$

where S^{n-1} is the unit sphere of $\mathbb{R}^n$, and

$$F(t,\xi) = |gradA(\xi)|^2 + |A(\xi)t + B(\xi)|^2, \tag{2.7}$$

where $|grad\ A(\xi)|^2 = \sum_{j=1}^{k} |grad\ p_j(\xi)|^2$.

Lemma 2.3. *The inequality (2.2) holds if and only if there exist no sequences of real numbers $t_j \longrightarrow +\infty$ and $\omega_j \in S^{n-1}$ such that*

$$F(t_j,\omega_j) \longrightarrow 0. \tag{2.8}$$

Proof. Let t_j be a sequence of real numbers and ω_j a sequence of S^{n-1}, using the homogeneity of the functions p, q and r, then (2.2) is equivalent to

$$\frac{|t_j\omega_j|^{2(m-1)}}{\sum_{l=1}^{k} \widetilde{P}_l(t_j\omega_j)^2} = \frac{1}{F(t_j,\omega_j) + 2\sum_{l=1}^{k} Re\left(p_l(\omega_j).\overline{r}_l(\omega_j)\right) + \chi(\omega_j).O(\frac{1}{t_j})} \leq C,$$

where χ is a bounded function. Hence it is easy to see Lemma 2.3. □

If $\omega \in \Xi$ we define the function G by

$$G(\omega) = |gradA(\omega)|^2 + \frac{|Im\left(A(\omega)\cdot B(\omega)\right)|^2 + |A(\omega)\wedge B(\omega)|^2}{|A(\omega)|^2}.$$

Theorem 2.4. *The estimate (1.1) holds if and only if*

$$\exists C > 0, G(\omega) \geq C, \forall \omega \in \Xi \tag{2.9}$$

Proof. All positive constants are denoted by C. If (2.9) holds then from (2.4) and (2.7) we have

(2.10)
$$F(t,\omega) = \left(|A(\omega)|\, t + \frac{Re\,(A(\omega).B(\omega))}{|A(\omega)|} \right)^2 + G(\omega) \geq C, \forall \omega \in \Xi, \forall t \geq 0.$$

The vector function A is analytic and the set Ξ is dense in S^{n-1}, therefore by continuity we obtain

$$F(t,\omega) \geq C, \forall t \geq 0, \forall \omega \in S^{n-1}. \tag{2.11}$$

For $\xi \in \mathbb{R}^n$, set $\omega = \frac{\xi}{|\xi|}$ and $t = |\xi|$ in (2.11), as the vector functions A and B are homogeneous, we obtain

$$|A(\xi) + B(\xi)|^2 + |gradA(\xi)|^2 \geq C\, |\xi|^{2(m-1)}, \forall \xi \in \mathbb{R}^n,$$

and then, for $|\xi| \geq C$, we have

$$\sum_{j=1}^{k} \left(|P_j(\xi)|^2 + |gradP_j(\xi)|^2 \right) + O\left(\left(1 + |\xi|^2\right)^{m-2} \right) \geq C\, |\xi|^{2(m-1)}. \tag{2.12}$$

From the last inequality we easily get (2.2) of Lemma 2.1.

Suppose that (2.9) does not hold, then there exists a sequence $\omega_j \in \Xi$ such that $G(\omega_j) \longrightarrow 0$, i.e.

$$|gradA(\omega_j)|^2 \rightarrow 0, \tag{2.13}$$

and

$$|Im\,(A(\omega_j).B(\omega_j))|^2 + |A(\omega_j) \wedge B(\omega_j)|^2 \over |A(\omega_j)|^2 \to 0. \tag{2.14}$$

As S^{n-1} is compact we can suppose that $\omega_j \longrightarrow \omega_0 \in S^{n-1}$. Hence, from (2.14) and (2.4) with $t = 0$, we obtain

$$\frac{Re\,(A(\omega_j).B(\omega_j))}{|A(\omega_j)|} \longrightarrow \pm |B(\omega_0)|\,. \tag{2.15}$$

From (2.13), due to Euler's identity for homogeneous functions,

$$A(\omega_0) = \overrightarrow{0}\,. \tag{2.16}$$

Now if $B(\omega_0) = 0$ then $F(t, \omega_0) \equiv 0$, which contradicts (2.8).

Let $B(\omega_0) \neq 0$, and suppose that

$$\frac{Re\,(A(\omega_j).B(\omega_j))}{|A(\omega_j)|} \longrightarrow -|B(\omega_0)|\,, \tag{2.17}$$

then setting $t_j = \frac{|B(\omega_j)|}{|A(\omega_j)|}$ in (2.10), it is clear that $t_j \longrightarrow +\infty$, so, with $G(\omega_j) \longrightarrow 0$, $F(t_j, \omega_j)$ will converge to 0, which contradicts (2.8).

If

$$\frac{Re\,(A(\omega_j).B(\omega_j))}{|A(\omega_j)|} \longrightarrow +|B(\omega_0)|\,,$$

then changing ω_j to $-\omega_j$ and using the homogeneity of the functions A and B, we obtain the same conclusion. □

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