Journal of Inequalities in Pure and Applied Mathematics

http://jipam.vu.edu.au/

Volume 6, Issue 2, Article 50, 2005

INTEGRAL MEANS FOR STARLIKE AND CONVEX FUNCTIONS WITH NEGATIVE COEFFICIENTS

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Received 17 February, 2005; accepted 06 April, 2005
Communicated by N.E. Cho

Memorial Paper for Professor Nicolae N. Pascu

ABSTRACT. Let $\mathcal{T}$ be the class of functions $f(z)$ with negative coefficients which are analytic and univalent in the open unit disk $\mathbb{U}$ with $f(0) = 0$ and $f'(0) = 1$. The classes $\mathcal{T}^*$ and $\mathcal{C}$ are defined as the subclasses of $\mathcal{T}$ which are starlike and convex in $\mathbb{U}$, respectively. In view of the interesting results for integral means given by H. Silverman (*Houston J. Math.* **23**(1977)), some generalization theorems are discussed in this paper.

Key words and phrases: Univalent, Starlike, Convex, Integral mean.

2000 *Mathematics Subject Classification.* Primary 30C45.

ISSN (electronic): 1443-5756

041-05

1. INTRODUCTION

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

that are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of all univalent functions $f(z)$ in $\mathbb{U}$. A function $f(z) \in \mathcal{A}$ is said to be starlike with respect to the origin in $\mathbb{U}$ if it satisfies

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0 \qquad (z \in \mathbb{U}). \tag{1.2}$$

We denote by $\mathcal{S}^*$ the subclass of $\mathcal{S}$ consisting of all starlike functions $f(z)$ with respect to the origin in $\mathbb{U}$. Further, a function $f(z) \in \mathcal{A}$ is said to be convex in $\mathbb{U}$ if it satisfies

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0 \qquad (z \in \mathbb{U}). \tag{1.3}$$

We also denote by $\mathcal{K}$ the subclass of $\mathcal{S}$ consisting of $f(z)$ which are convex in $\mathbb{U}$. By the above definitions, we know that $f(z) \in \mathcal{K}$ if and only if $zf'(z) \in \mathcal{S}^*$, and that $\mathcal{K} \subset \mathcal{S}^* \subset \mathcal{S} \subset \mathcal{A}$.

The class $\mathcal{T}$ is defined as the subclass of $\mathcal{S}$ consisting of all functions $f(z)$ which are given by

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \qquad (a_n \geq 0). \tag{1.4}$$

Further, we denote by $\mathcal{T}^* = \mathcal{S}^* \cap \mathcal{T}$ and $\mathcal{C} = \mathcal{K} \cap \mathcal{T}$. It is well-known by Silverman [6] that

Remark 1.1. A function $f(z) \in \mathcal{T}^*$ if and only if

$$\sum_{n=2}^{\infty} n a_n \leq 1. \tag{1.5}$$

A function $f(z) \in \mathcal{C}$ if and only if

$$\sum_{n=2}^{\infty} n^2 a_n \leq 1. \tag{1.6}$$

For $f(z) \in \mathcal{A}$ and $g(z) \in \mathcal{A}$, $f(z)$ is said to be subordinate to $g(z)$ in $\mathbb{U}$ if there exists an analytic function $\omega(z)$ in $\mathbb{U}$ such that $\omega(0) = 0$, $|\omega(z)| < 1$ $(z \in \mathbb{U})$, and $f(z) = g(\omega(z))$. We denote this subordination by

$$f(z) \prec g(z). \quad \text{(cf. Duren [1]).} \tag{1.7}$$

For subordinations, Littlewood [2] has given the following integral mean.

Theorem A. *If $f(z)$ and $g(z)$ are analytic in $\mathbb{U}$ with $f(z) \prec g(z)$, then, for $\lambda > 0$ and $|z| = r$ $(0 < r < 1)$,*

$$\int_0^{2\pi} |f(re^{i\theta})|^{\lambda} d\theta \leq \int_0^{2\pi} |g(re^{i\theta})|^{\lambda} d\theta. \tag{1.8}$$

Furthermore, Silverman [6] has shown that

Remark 1.2. $f_1(z) = z$ and $f_n(z) = z - \frac{z^n}{n}$ $(n \geq 2)$ are extreme points of the class $\mathcal{T}^*$(or $\mathcal{T}$). $f_1(z) = z$ and $f_n(z) = z - \frac{z^n}{n^2}$ $(n \geq 2)$ are extreme points of the class $\mathcal{C}$.

Applying Theorem A with extreme points of $\mathcal{T}$, Silverman [7] has proved the following results.

Theorem B. *Suppose that $f(z) \in \mathcal{T}^*$, $\lambda > 0$ and $f_2(z) = z - \frac{z^2}{2}$. Then, for $z = re^{i\theta}$ $(0 < r < 1)$,*

$$\int_0^{2\pi} |f(z)|^\lambda d\theta \leq \int_0^{2\pi} |f_2(z)|^\lambda d\theta. \tag{1.9}$$

Theorem C. *If $f(z) \in \mathcal{T}^*$, $\lambda > 0$, and $f_2(z) = z - \frac{z^2}{2}$, then, for $z = re^{i\theta}$ $(0 < r < 1)$,*

$$\int_0^{2\pi} |f'(z)|^\lambda d\theta \leq \int_0^{2\pi} |f_2'(z)|^\lambda d\theta. \tag{1.10}$$

In the present paper, we consider the generalization properties for Theorem B and Theorem C with $f(z) \in \mathcal{T}^*$ and $f(z) \in \mathcal{C}$.

Remark 1.3. More recently, applying Theorem A by Littlewood [2], Sekine, Tsurumi and Srivastava [4]; and Sekine, Tsurumi, Owa and Srivastava [5] have discussed some interesting properties of integral means inequalities for fractional derivatives of some general subclasses of analytic functions $f(z)$ in the open unit disk $\mathbb{U}$. Further, Owa and Sekine [3] have considered the integral means with some coefficient inequalities for certain analytic functions $f(z)$ in $\mathbb{U}$.

2. GENERALIZATION PROPERTIES

Our first result for the generalization properties is contained in

Theorem 2.1. *Let $f(z) \in \mathcal{T}^*$, $\lambda > 0$, and $f_k(z) = z - \frac{z^k}{k}$ $(k \geq 2)$. If $f(z)$ satisfies*

$$\sum_{j=0}^{k-3} \frac{j+1}{k}(a_{2k+j-1} + a_{k+j+1} - a_{k-j-1}) \geq 0 \tag{2.1}$$

for $k \geq 3$, and if there exists an analytic function $\omega(z)$ in $\mathbb{U}$ given by

$$(\omega(z))^{k-1} = k\left(\sum_{n=2}^{\infty} a_n z^{n-1}\right),$$

then, for $z = re^{i\theta}$ $(0 < r < 1)$,

$$\int_0^{2\pi} |f(z)|^\lambda d\theta \leq \int_0^{2\pi} |f_k(z)|^\lambda d\theta. \tag{2.2}$$

Proof. For $f(z) \in \mathcal{T}^*$, we have to show that

$$\int_0^{2\pi} \left|1 - \sum_{n=2}^{\infty} a_n z^{n-1}\right|^\lambda d\theta \leq \int_0^{2\pi} \left|1 - \frac{z^{k-1}}{k}\right|^\lambda d\theta.$$

By Theorem A, it suffices to prove that

$$1-\sum_{n=2}^{\infty}a_n z^{n-1}\prec 1-\frac{z^{k-1}}{k}.$$

Let us define the function $\omega(z)$ by

$$1-\sum_{n=2}^{\infty}a_n z^{n-1}=1-\frac{1}{k}(\omega(z))^{k-1}. \tag{2.3}$$

It follows from (2.3) that

$$|\omega(z)|^{k-1}=\left|k\sum_{n=2}^{\infty}a_n z^{n-1}\right|\leq |z|\left(\sum_{n=2}^{\infty}ka_n\right).$$

Thus, we only show that

$$\sum_{n=2}^{\infty}ka_n\leq\sum_{n=2}^{\infty}na_n,$$

or

$$\sum_{n=2}^{\infty}a_n\leq\frac{1}{k}\left(\sum_{n=2}^{\infty}na_n\right).$$

Indeed, we see that

$$\begin{aligned}\frac{1}{k}\left(\sum_{n=2}^{\infty}na_n\right)&=\left(1-\frac{k-2}{k}\right)a_2+\left(1-\frac{k-3}{k}\right)a_3+\cdots+\left(1-\frac{2}{k}\right)a_{k-2}\\&\quad+\left(1-\frac{1}{k}\right)a_{k-1}+a_k+\left(1+\frac{1}{k}\right)a_{k+1}+\left(1+\frac{2}{k}\right)a_{k+2}\\&\quad+\cdots+\left(1+\frac{k+1}{k}\right)a_{2k+1}+\left(1+\frac{k+2}{k}\right)a_{2k+2}+\cdots\\&=\frac{k-2}{k}(a_{2k-2}-a_2)+\frac{k-3}{k}(a_{2k-3}-a_3)+\cdots\\&\quad+\frac{2}{k}(a_{k+2}-a_{k-2})+\frac{1}{k}(a_{k+1}-a_{k-1})+\left(1+\frac{k-1}{k}\right)a_{2k-1}\\&\quad+\left(1+\frac{k}{k}\right)a_{2k}+\left(1+\frac{k+1}{k}\right)a_{2k+1}+\cdots+\sum_{n=2}^{2k-2}a_n.\end{aligned}$$

Noting that

$$1+\frac{k+j}{k}\geq 1+\frac{2+j}{k},\quad (j=-1,0,1,\dots),$$

we obtain

$$\begin{aligned}
(2.4)\qquad \frac{1}{k}\left(\sum_{n=2}^{\infty} n a_n\right) &\geq \frac{k-2}{k}(a_{2k-2}-a_2)+\frac{k-3}{k}(a_{2k-3}-a_3)+\cdots \\
&\quad + \frac{2}{k}(a_{k+2}-a_{k-2})+\frac{1}{k}(a_{k+1}-a_{k-1}) \\
&\quad + \left(1+\frac{1}{k}\right)a_{2k-1}+\left(1+\frac{2}{k}\right)a_{2k}+\cdots \\
&\quad + \left(1+\frac{k-3}{k}\right)a_{3k-5}+\left(1+\frac{k-2}{k}\right)a_{3k-4}+\cdots+\sum_{n=2}^{2k-2} a_n \\
&\geq \frac{1}{k}(a_{2k-1}+a_{k+1}-a_{k-1})+\frac{2}{k}(a_{2k}+a_{k+2}-a_{k-2})+\cdots \\
&\quad + \frac{k-2}{k}(a_{3k-4}+a_{2k-2}-a_2)+\sum_{n=2}^{\infty} a_n \\
&= \sum_{j=0}^{k-3}\frac{j+1}{k}(a_{2k+j-1}+a_{k+j+1}-a_{k-j-1})+\sum_{n=2}^{\infty} a_n \\
&\geq \sum_{n=2}^{\infty} a_n
\end{aligned}$$

with the following condition

$$\sum_{j=0}^{k-3}\frac{j+1}{k}(a_{2k+j-1}+a_{k+j+1}-a_{k-j-1}) \geq 0.$$

Thus, we observe that the function $\omega(z)$ defined by (2.3) is analytic in $\mathbb{U}$ with $\omega(0) = 0$, $|\omega(z)| < 1$ $(z \in \mathbb{U})$. This completes the proof of the theorem. □

Remark 2.2. Taking $k = 2$ in Theorem 2.1, we have Theorem B by Silverman [7].

Example 2.1. Let us define

$$(2.5)\qquad f(z) = z - \frac{37}{1200}z^2 - \frac{1}{18}z^3 - \frac{1}{48}z^4 - \frac{1}{100}z^5$$

and

$$(2.6)\qquad f_3(z) = z - \frac{1}{3}z^3$$

with $k = 3$ in Theorem 2.1. Since $f(z)$ satisfies

$$\sum_{n=2}^{\infty} n a_n = \frac{217}{600} < 1,$$

we have $f(z) \in \mathcal{T}^*$. Furthermore, $f(z)$ satisfies,

$$\frac{1}{3}(a_5 + a_4 - a_2) = \frac{1}{3}\left(\frac{1}{100} + \frac{1}{48} - \frac{37}{1200}\right) = 0.$$

Thus, $f(z)$ satisfies the conditions in Theorem 2.1 with $k = 3$.

If we take $\lambda = 2$, then we have

$$\int_0^{2\pi} |f(z)|^2 d\theta \leq 2\pi r^2 \left(1 + \frac{1}{9} r^4\right) < \frac{20}{9}\pi = 6.9813\ldots.$$

Corollary 2.3. *Let $f(z) \in \mathcal{T}^*$, $0 < \lambda \leq 2$, and $f_k(z) = z - \frac{z^k}{k}$ $(k \geq 2)$. If $f(z)$ satisfies the conditions in Theorem 2.1, then, for $z = re^{i\theta}$ $(0 < r < 1)$,*

$$\int_0^{2\pi} |f(z)|^\lambda d\theta \leq 2\pi r^\lambda \left(1 + \frac{1}{k^2} r^{2(k-1)}\right)^{\frac{\lambda}{2}} < 2\pi \left(1 + \frac{1}{k^2}\right)^{\frac{\lambda}{2}}. \tag{2.7}$$

Proof. It follows that

$$\int_0^{2\pi} |f_k(z)|^\lambda d\theta = \int_0^{2\pi} |z|^\lambda \left|1 - \frac{z^{k-1}}{k}\right|^\lambda d\theta.$$

Applying Hölder's inequality for $0 < \lambda < 2$, we obtain that

$$\begin{aligned}
\int_0^{2\pi} |z|^\lambda \left|1 - \frac{z^{k-1}}{k}\right|^\lambda d\theta &\leq \left(\int_0^{2\pi} (|z|^\lambda)^{\frac{2}{2-\lambda}} d\theta\right)^{\frac{2-\lambda}{2}} \left(\int_0^{2\pi} \left(\left|1 - \frac{z^{k-1}}{k}\right|^\lambda\right)^{\frac{2}{\lambda}} d\theta\right)^{\frac{\lambda}{2}} \\
&= \left(\int_0^{2\pi} |z|^{\frac{2\lambda}{2-\lambda}} d\theta\right)^{\frac{2-\lambda}{2}} \left(\int_0^{2\pi} \left|1 - \frac{z^{k-1}}{k}\right|^2 d\theta\right)^{\frac{\lambda}{2}} \\
&= \left(2\pi r^{\frac{2\lambda}{2-\lambda}}\right)^{\frac{2-\lambda}{2}} \left(2\pi \left(1 + \frac{1}{k^2} r^{2(k-1)}\right)\right)^{\frac{\lambda}{2}} \\
&= 2\pi r^\lambda \left(1 + \frac{1}{k^2} r^{2(k-1)}\right)^{\frac{\lambda}{2}} \\
&< 2\pi \left(1 + \frac{1}{k^2}\right)^{\frac{\lambda}{2}}.
\end{aligned}$$

Further, it is clear for $\lambda = 2$. □

For the generalization of Theorem C by Silverman [7], we have

Theorem 2.4. *Let $f(z) \in \mathcal{T}^*$, $\lambda > 0$, and $f_k(z) = z - \frac{z^k}{k}$ $(k \geq 2)$. If there exists an analytic function $\omega(z)$ in $\mathbb{U}$ given by*

$$\left(\omega(z)\right)^{k-1} = \sum_{n=2}^{\infty} n a_n z^{n-1},$$

then, for $z = re^{i\theta}$ $(0 < r < 1)$,

$$\int_0^{2\pi} |f'(z)|^\lambda d\theta \leq \int_0^{2\pi} |f_k'(z)|^\lambda d\theta. \tag{2.8}$$

Proof. For $f(z) \in \mathcal{T}^*$, it is sufficient to show that

$$1 - \sum_{n=2}^{\infty} n a_n z^{n-1} \prec 1 - z^{k-1}. \tag{2.9}$$

Let us define the function $\omega(z)$ by

$$1-\sum_{n=2}^{\infty} na_n z^{n-1} = 1-\omega(z)^{k-1}, \tag{2.10}$$

or, by

$$\omega(z)^{k-1} = \sum_{n=2}^{\infty} na_n z^{n-1}.$$

Since $f(z)$ satisfies

$$\sum_{n=2}^{\infty} na_n \leq 1,$$

the function $\omega(z)$ is analytic in $\mathbb{U}$, $\omega(0)=0$, and $|\omega(z)|<1$ $(z\in\mathbb{U})$. □

Remark 2.5. If we take $k=2$ in Theorem 2.4, then we have Theorem C by Silverman [7].

Using the Hölder inequality for Theorem 2.4, we have

Corollary 2.6. *Let $f(z)\in\mathcal{T}^*$, $0<\lambda\leq 2$, and $f_k(z)=z-\frac{z^k}{k}$ $(k\geq 2)$. If $f(z)$ satisfies the conditions in Theorem 2.4, then, for $z=re^{i\theta}$ $(0<r<1)$,*

$$\int_0^{2\pi} |f'(z)|^{\lambda} d\theta \leq 2\pi\left(1+r^{2(k-1)}\right)^{\frac{\lambda}{2}} < 2^{\frac{2+\lambda}{2}}\pi.$$

3. Integral Means for Functions in the Class $\mathcal{C}$

In this section, we discuss the integral means for functions $f(z)$ in the class $\mathcal{C}$.

Theorem 3.1. *Let $f(z)\in\mathcal{C}$, $\lambda>0$, and $f_k(z)=z-\frac{z^k}{k^2}$ $(k\geq 2)$. If $f(z)$ satisfies*

$$\sum_{j=2}^{k-1} \frac{(k+j)(k-j)}{k^2}(a_{2k-j}-a_j) \geq 0 \tag{3.1}$$

for $k\geq 3$, and if there exists an analytic function $\omega(z)$ in $\mathbb{U}$ given by

$$(\omega(z))^{k-1} = k^2\sum_{n=2}^{\infty} a_n z^{n-1},$$

then, for $z=re^{i\theta}$ $(0<r<1)$,

$$\int_0^{2\pi} |f(z)|^{\lambda} d\theta \leq \int_0^{2\pi} |f_k(z)|^{\lambda} d\theta. \tag{3.2}$$

Proof. For the proof, we need to show that

$$1-\sum_{n=2}^{\infty} a_n z^{n-1} \prec 1-\frac{z^{k-1}}{k^2} \tag{3.3}$$

by Theorem A. Define the function $\omega(z)$ by

$$1-\sum_{n=2}^{\infty} a_n z^{n-1} = 1-\frac{1}{k^2}\omega(z)^{k-1}, \tag{3.4}$$

or by

$$(\omega(z))^{k-1} = k^2 \left(\sum_{n=2}^{\infty} a_n z^{n-1} \right). \tag{3.5}$$

Therefore, we have to show that

$$\sum_{n=2}^{\infty} a_n \leq \frac{1}{k^2} \left(\sum_{n=2}^{\infty} n^2 a_n \right).$$

Using the same technique as in the proof of Theorem 2.1, we see that

$$\begin{aligned} \frac{1}{k^2} \left(\sum_{n=2}^{\infty} n^2 a_n \right) &\geq \sum_{j=2}^{k-1} \frac{(k+j)(k-j)}{k^2} (a_{2k-j} - a_j) + \sum_{n=2}^{\infty} a_n \\ &\geq \sum_{n=2}^{\infty} a_n. \end{aligned}$$

□

Example 3.1. Consider the functions

$$f(z) = z - \frac{1}{40} z^2 - \frac{1}{18} z^3 - \frac{1}{40} z^4 \tag{3.6}$$

and

$$f_3(z) = z - \frac{1}{9} z^3 \tag{3.7}$$

with $k = 3$ in Theorem 3.1. Then we have that

$$\sum_{n=2}^{\infty} n^2 a_n = \frac{4}{40} + \frac{9}{18} + \frac{16}{40} = 1,$$

which implies $f(z) \in \mathcal{C}$, and that

$$\frac{5}{9}(a_4 - a_2) = 0.$$

Thus $f(z)$ satisfies the conditions of Theorem 3.1. If we make $\lambda = 2$, then we see that

$$\int_0^{2\pi} |f(z)|^2 d\theta \leq 2\pi r^2 \left(1 + \frac{1}{81} r^4 \right) < \frac{164}{81} \pi = 6.3607 \cdots .$$

Corollary 3.2. *Let $f(z) \in \mathcal{C}$, $0 < \lambda \leq 2$, and $f_k(z) = z - \frac{z^k}{k^2}$ $(k \geq 2)$. If $f(z)$ satisfies the condition in Theorem 3.1, then, for $k \geq 3$, then, for $z = re^{i\theta}$ $(0 < r < 1)$,*

$$\begin{aligned} \int_0^{2\pi} |f(z)|^\lambda d\theta &\leq 2\pi r^\lambda \left(1 + \frac{1}{k^4} r^{2(k-1)} \right)^{\frac{\lambda}{2}} \\ &< 2\pi \left(1 + \frac{1}{k^4} \right)^{\frac{\lambda}{2}}. \end{aligned} \tag{3.8}$$

Further, we may have

Theorem 3.3. *Let* $f(z) \in \mathcal{C}$, $\lambda > 0$, *and* $f_k(z) = z - \frac{z^k}{k^2}$ $(k \geq 2)$. *If* $f(z)$ *satisfies*

$$\sum_{j=2}^{2k-2} j(k-j)a_j \leq 0, \tag{3.9}$$

and if there exists an analytic function $\omega(z)$ *in* $\mathbb{U}$ *given by*

$$(\omega(z))^{k-1} = k\sum_{n=2}^{\infty} na_n z^{n-1},$$

then, for $z = re^{i\theta}$ $(0 < r < 1)$,

$$\int_0^{2\pi} |f'(z)|^\lambda d\theta \leq \int_0^{2\pi} |f_k'(z)|^\lambda d\theta. \tag{3.10}$$

Example 3.2. Take the functions

$$f(z) = z - \frac{1}{24}z^2 - \frac{1}{18}z^3 - \frac{1}{48}z^4 \tag{3.11}$$

and

$$f_3(z) = z - \frac{1}{9}z^3 \tag{3.12}$$

with $k = 3$ in Theorem 3.3. Since

$$\sum_{n=2}^{\infty} n^2 a_n = \frac{4}{24} + \frac{9}{18} + \frac{16}{48} = \frac{5}{6} < 1$$

and

$$2(3-2)a_2 + 3(3-3)a_3 + 4(3-4)a_4 = \frac{1}{12} - \frac{1}{12} = 0,$$

$f(z)$ satisfies the conditions in Theorem 3.3. If we take $\lambda = 2$, then we have

$$\int_0^{2\pi} |f'(z)|^2 d\theta \leq 2\pi\left(1 + \frac{1}{9}r^4\right) < \frac{20}{9}\pi.$$

Corollary 3.4. *Let* $f(z) \in \mathcal{C}$, $0 < \lambda \leq 2$, *and* $f_k(z) = z - \frac{z^k}{k^2}$ $(k \geq 2)$. *If* $f(z)$ *satisfies the condition in Theorem 3.3, then, for* $k \geq 2$, *then, for* $z = re^{i\theta}$ $(0 < r < 1)$,

$$\int_0^{2\pi} |f'(z)|^\lambda d\theta \leq 2\pi\left(1 + \frac{1}{k}r^{2(k-1)}\right)^{\frac{\lambda}{2}} < 2\pi\left(1 + \frac{1}{k}\right)^{\frac{\lambda}{2}}.$$

4. APPLICATIONS FOR THE INTEGRATED FUNCTIONS

For $f(z) \in \mathcal{T}$, we define

$$I_0 f(z) = f(z) = z - \sum_{n=2}^{\infty} a_n z^n$$

$$If(z) = I_1 f(z) = \int_0^z f(t)dt = \frac{1}{2}z^2 - \sum_{n=2}^{\infty} \frac{a_n}{n+1} z^{n+1}$$

$$I_k f(z) = I(I_{k-1}f(z)) = \frac{1}{(k+1)!}z^{k+1} - \sum_{n=2}^{\infty} \frac{n!}{(n+k)!} a_n z^{n+k} \quad (k = 1, 2, 3, \dots).$$

Theorem 4.1. *Let $f(z) \in \mathcal{T}^*$, $\lambda > 0$, and $f_j(z) = z - \frac{z^j}{j}$ $(j = 2, 3, 4, \dots)$.*

If $f(z)$ satisfies

$$\sum_{k=2}^{j^2+j-1} \frac{j^2 + j - k}{j(j+1)} (a_{2j^2+2j-k} - a_k) \geq 0 \tag{4.1}$$

for $j = 2, 3, 4, \dots$, and if there exists an analytic function $\omega(z)$ in $\mathbb{U}$ given by

$$(\omega(z))^{j-1} = j(j+1) \left(\sum_{n=2}^{\infty} \frac{1}{n+1} a_n z^{n-1} \right),$$

then

$$\int_0^{2\pi} |If(z)|^\lambda d\theta \leq \int_0^{2\pi} |If_j(z)|^\lambda d\theta. \tag{4.2}$$

Proof. We have to prove

$$\int_0^{2\pi} \left| 1 - \sum_{n=2}^{\infty} \frac{2}{n+1} a_n z^{n-1} \right|^\lambda d\theta \leq \int_0^{2\pi} \left| 1 - \frac{2}{j(j+1)} z^{j-1} \right|^\lambda d\theta.$$

If

$$1 - \sum_{n=2}^{\infty} \frac{2}{n+1} a_n z^{n-1} \prec 1 - \frac{2}{j(j+1)} z^{j-1},$$

then the proof is completed by Theorem A.

Let us define the function $\omega(z)$ by

$$1 - \sum_{n=2}^{\infty} \frac{2}{n+1} a_n z^{n-1} = 1 - \frac{2}{j(j+1)} (\omega(z))^{j-1}.$$

Then

$$\begin{aligned} |\omega(z)|^{j-1} &= \left| j(j+1) \sum_{n=2}^{\infty} \frac{1}{n+1} a_n z^{n-1} \right| \\ &\leq |z| \left(j(j+1) \sum_{n=2}^{\infty} \frac{1}{n+1} a_n \right). \end{aligned}$$

Thus, we only show that

$$j(j+1) \sum_{n=2}^{\infty} \frac{1}{n+1} a_n \leq \sum_{n=2}^{\infty} n a_n$$

or

$$\sum_{n=2}^{\infty} a_n \leq \sum_{n=2}^{\infty} n \left(\frac{1}{j(j+1)} + \frac{1}{n+1} \right) a_n.$$

Indeed,

$$
\begin{aligned}
&\sum_{n=2}^{\infty} n\left(\frac{1}{j(j+1)}+\frac{1}{n+1}\right)a_n\\
&\quad= 2\left(\frac{1}{j(j+1)}+\frac{1}{3}\right)a_2+3\left(\frac{1}{j(j+1)}+\frac{1}{4}\right)a_3+\cdots\\
&\qquad+(j-1)\left(\frac{1}{j(j+1)}+\frac{1}{j}\right)a_{j-1}+j\left(\frac{1}{j(j+1)}+\frac{1}{j+1}\right)a_j\\
&\qquad+(j+1)\left(\frac{1}{j(j+1)}+\frac{1}{j+2}\right)a_{j+1}\\
&\qquad+\cdots+(2j^2+2j-3)\left(\frac{1}{j(j+1)}+\frac{1}{2j^2+2j-2}\right)a_{2j^2+2j-3}\\
&\qquad+(2j^2+2j-2)\left(\frac{1}{j(j+1)}+\frac{1}{2j^2+2j-1}\right)a_{2j^2+2j-1}+\cdots\\
&\quad\geq \left(1-\frac{j(j+1)-2}{j(j+1)}\right)a_2+\left(1-\frac{j(j+1)-3}{j(j+1)}\right)a_3+\cdots\\
&\qquad+\left(1-\frac{j(j+1)-(j-1)}{j(j+1)}\right)a_{j-1}+\left(1-\frac{j(j+1)-j}{j(j+1)}\right)a_j\\
&\qquad+\left(1-\frac{j(j+1)-(j+1)}{j(j+1)}\right)a_{j+1}\\
&\qquad+\cdots+\left(1-\frac{j(j+1)-(2j^2+2j-3)}{j(j+1)}\right)a_{2j^2+2j-3}\\
&\qquad+\left(1-\frac{j(j+1)-(2j^2+2j-2)}{j(j+1)}\right)a_{2j^2+2j-2}+\cdots\\
&\quad=\frac{j^2+j-2}{j(j+1)}(a_{2j^2+2j-2}-a_2)+\frac{j^2+j-3}{j(j+1)}(a_{2j^2+2j-3}-a_3)\\
&\qquad+\cdots+\frac{j^2+1}{j(j+1)}(a_{2j^2+j+1}-a_{j-1})+\frac{j^2}{j(j+1)}(a_{2j^2+j}-a_j)\\
&\qquad+\frac{j^2-1}{j(j+1)}(a_{2j^2+j-1}-a_{j+1})+\cdots+a_2+a_3+\cdots+a_{2j^2+2j-2}+\cdots\\
&\quad=\sum_{k=2}^{j^2+j-1}\frac{j^2+j-k}{j(j+1)}(a_{2j^2+2j-k}-a_k)+\sum_{n=2}^{\infty}a_n\\
&\quad\geq\sum_{n=2}^{\infty}a_n
\end{aligned}
$$

for

$$\sum_{k=2}^{j^2+j-1}\frac{j^2+j-k}{j(j+1)}(a_{2j^2+2j-k}-a_k)\geq 0.$$

This completes the proof of Theorem 4.1. □

Finally, we derive

Theorem 4.2. *Let* $f(z) \in \mathcal{T}^*$*,*$\lambda > 0$*,and* $f_j(z) = z - \frac{z^j}{j} \quad (j = 2, 3, 4, \dots)$*. If* $f(z)$ *satisfies*

$$\sum_{n=2}^{\infty} a_n \geq \frac{6}{5} \sum_{n=2}^{\frac{(j+k)!}{2(j-1)!}-1} \left(1 - \frac{2n(j-1)!}{(j+k)!}\right) \left(a_n - a_{\frac{(j+k)!}{(j-1)!}-n}\right) \tag{4.3}$$

for $k = 2, 3, 4, \dots$*, and if there exists an analytic function* $\omega(z)$ *in* $\mathbb{U}$ *given by*

$$(\omega(z))^{j-1} = \frac{(j+k)!}{(j-1)!} \sum_{n=2}^{\infty} \frac{n!}{(n+k)!} a_n z^{n-1},$$

then

$$\int_0^{2\pi} |I_k f(z)|^{\lambda} d\theta \leq \int_0^{2\pi} |I_k f_j(z)|^{\lambda} d\theta. \tag{4.4}$$

Proof. We have to show that

$$1 - \sum_{n=2}^{\infty} \frac{n!(k+1)!}{(n+k)!} a_n z^{n-1} \prec 1 - \frac{(j-1)!(k+1)!}{(j+k)!} z^{j-1}.$$

Define $\omega(z)$ by

$$1 - \sum_{n=2}^{\infty} \frac{n!(k+1)!}{(n+k)!} a_n z^{n-1} = 1 - \frac{(j-1)!(k+1)!}{(j+k)!} (\omega(z))^{j-1}$$

or by

$$(\omega(z))^{j-1} = \frac{(j+k)!}{(j-1)!} \sum_{n=2}^{\infty} \frac{n!}{(n+k)!} a_n z^{n-1}.$$

Then we have to show that

$$\frac{(j+k)!}{(j-1)!} \sum_{n=2}^{\infty} \frac{n!}{(n+k)!} a_n \leq \sum_{n=2}^{\infty} n a_n,$$

that is, that

$$\sum_{n=2}^{\infty} \frac{n!}{(n+k)!} a_n \leq \frac{(j-1)!}{(j+k)!} \sum_{n=2}^{\infty} n a_n.$$

Since

$$\begin{aligned} \sum_{n=2}^{\infty} \frac{n!}{(n+k)!} a_n &= \sum_{n=2}^{\infty} \frac{1}{(n+1)(n+2)\cdots(n+k)} a_n \\ &= \sum_{n=2}^{\infty} \left\{ \left(\frac{1}{n+1} - \frac{1}{n+2}\right) \left(\frac{1}{n+3} - \frac{1}{n+4}\right) \cdots \right\} a_n \\ &\leq \sum_{n=2}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+2}\right)^{\left[\frac{k}{2}\right]} a_n \\ &\leq \sum_{n=2}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+2}\right) a_n, \end{aligned}$$

we obtain

$$\sum_{n=2}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+2}\right) a_n \leq \frac{(j-1)!}{(j+k)!} \sum_{n=2}^{\infty} n a_n.$$

Furthermore, we have

$$\sum_{n=2}^{\infty} a_n \leq \sum_{n=2}^{\infty} \left(\frac{2n(j-1)!}{(j+k)!} + \frac{2n}{n+1} - \frac{n}{n+2} \right) a_n.$$

Let the function $h(n)$ be given by

$$h(n) = \frac{2n}{n+1} - \frac{n}{n+2} = 1 - \frac{2}{n^2+3n+2}.$$

Since $h(n)$ is increasing for $n \geq 2$,

$$h(n) \geq \frac{5}{6}.$$

Thus, we only show that

$$\sum_{n=2}^{\infty} a_n \leq \sum_{n=2}^{\infty} \left(\frac{11}{6} - \frac{(j+k)! - 2n(j-1)!}{(j+k)!} \right) a_n.$$

In fact,

$$\begin{aligned}
&\sum_{n=2}^{\infty} \left(\frac{11}{6} - \frac{(j+k)! - 2n(j-1)!}{(j+k)!} \right) a_n \\
&\quad = \left(\frac{11}{6} - \frac{(j+k)! - 4(j-1)!}{(j+k)!} \right) a_2 + \left(\frac{11}{6} - \frac{(j+k)! - 6(j-1)!}{(j+k)!} \right) a_3 + \cdots \\
&\qquad + \left(\frac{11}{6} - \frac{4(j-1)!}{(j+k)!} \right) a_{\frac{(j+k)!}{2(j-1)!}-2} + \left(\frac{11}{6} - \frac{2(j-1)!}{(j+k)!} \right) a_{\frac{(j+k)!}{2(j-1)!}-1} \\
&\qquad + \left(\frac{11}{6} - 0 \right) a_{\frac{(j+k)!}{2(j-1)!}} + \left(\frac{11}{6} - \frac{2(j-1)!}{(j+k)!} \right) a_{\frac{(j+k)!}{2(j-1)!}+1} \\
&\qquad + \left(\frac{11}{6} + \frac{4(j-1)!}{(j+k)!} \right) a_{\frac{(j+k)!}{2(j-1)!}+2} + \cdots + \left(\frac{11}{6} + \frac{(j+k)! - 6(j-1)!}{(j+k)!} \right) a_{\frac{(j+k)!}{(j-1)!}-3} \\
&\qquad + \left(\frac{11}{6} + \frac{(j+k)! - 4(j-1)!}{(j+k)!} \right) a_{\frac{(j+k)!}{(j-1)!}-2} + \cdots \\
&\quad \geq \frac{11}{6} \sum_{n=2}^{\infty} a_n + \frac{(j+k)! - 4(j-1)!}{(j+k)!} \left(a_{\frac{(j+k)!}{(j-1)!}-2} - a_2 \right) \\
&\qquad + \frac{(j+k)! - 6(j-1)!}{(j+k)!} \left(a_{\frac{(j+k)!}{(j-1)!}-3} - a_3 \right) + \frac{4(j-1)!}{(j+k)!} \left(a_{\frac{(j+k)!}{2(j-1)!}+2} - a_{\frac{(j+k)!}{2(j-1)!}-2} \right) \\
&\qquad + \frac{2(j-1)!}{(j+k)!} \left(a_{\frac{(j+k)!}{2(j-1)!}+1} - a_{\frac{(j+k)!}{2(j-1)!}-1} \right) \\
&\quad = \sum_{n=2}^{\infty} a_n + \frac{5}{6} \sum_{n=2}^{\infty} a_n + \frac{(j+k)! - 4(j-1)!}{(j+k)!} \left(a_{\frac{(j+k)!}{2(j-1)!}-2} - a_2 \right) \\
&\qquad + \frac{(j+k)! - 6(j-1)!}{(j+k)!} \left(a_{\frac{(j+k)!}{2(j-1)!}-3} - a_3 \right) + \cdots \\
&\qquad + \frac{(j+k)! - \{(j+k)! - 4(j-1)!\}}{(j+k)!} \left(a_{\frac{(j+k)!}{2(j-1)!}+2} - a_{\frac{(j+k)!}{2(j-1)!}-2} \right) \\
&\qquad + \frac{(j+k)! - \{(j+k)! - 2(j-1)!\}}{(j+k)!} \left(a_{\frac{(j+k)!}{2(j-1)!}+1} - a_{\frac{(j+k)!}{2(j-1)!}-1} \right)
\end{aligned}$$

$$= \sum_{n=2}^{\infty} a_n + \frac{5}{6}\sum_{n=2}^{\infty} a_n + \sum_{n=2}^{\frac{(j+k)!}{2(j-1)!}-1} \frac{(j+k)! - 2n(j-1)!}{(j+k)!}\left(a_{\frac{(j+k)!}{(j-1)!}-n} - a_n\right)$$
$$\geq \sum_{n=2}^{\infty} a_n$$

for

$$\sum_{n=2}^{\infty} a_n \geq \frac{6}{5}\sum_{n=2}^{\frac{(j+k)!}{2(j-1)!}-1}\left(1 - \frac{2n(j-1)!}{(j+k)!}\right)\left(a_n - a_{\frac{(j+k)!}{(j-1)!}-n}\right).$$

This completes the proof of Theorem 4.2. □

Remark 4.3. Letting $k = 2$, if $f(z)$ satisfies,

$$\sum_{n=2}^{\infty} a_n \geq \frac{6}{5}\sum_{n=2}^{\frac{j(j+1)(j+2)}{2}-1}\left(1 - \frac{2n}{j(j+1)(j+2)}\right)(a_n - a_{j(j+1)(j+2)-n}) \tag{4.5}$$

for $j = 2, 3, 4, \ldots$, then

$$\int_0^{2\pi} |I_2 f(z)|^{\lambda} d\theta \leq \int_0^{2\pi} |I_2 f_j(z)|^{\lambda} d\theta. \tag{4.6}$$

Remark 4.4. Letting $k = 3$, if $f(z)$ satisfies,

$$\sum_{n=2}^{\infty} a_n \geq \frac{6}{5}\sum_{n=2}^{\frac{j(j+1)(j+2)(j+3)}{2}-1}\left(1 - \frac{2n}{j(j+1)(j+2)(j+3)}\right)(a_n - a_{j(j+1)(j+2)(j+3)-n}) \tag{4.7}$$

for $j = 2, 3, 4, \ldots$, then

$$\int_0^{2\pi} |I_3 f(z)|^{\lambda} d\theta \leq \int_0^{2\pi} |I_3 f_j(z)|^{\lambda} d\theta. \tag{4.8}$$

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