

INTEGRAL MEANS FOR STARLIKE AND CONVEX FUNCTIONS WITH NEGATIVE COEFFICIENTS

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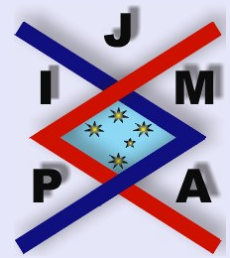
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Abstract

Let \mathcal{T} be the class of functions $f(z)$ with negative coefficients which are analytic and univalent in the open unit disk \mathbb{U} with $f(0) = 0$ and $f'(0) = 1$. The classes \mathcal{T}^* and \mathcal{C} are defined as the subclasses of \mathcal{T} which are starlike and convex in \mathbb{U} , respectively. In view of the interesting results for integral means given by H. Silverman (*Houston J. Math.* **23**(1977)), some generalization theorems are discussed in this paper.

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Memorial Paper for Professor Nicolae N. Pascu

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1. Introduction

Let \mathcal{A} denote the class of functions $f(z)$ of the form

$$(1.1) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

that are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Let \mathcal{S} be the subclass of \mathcal{A} consisting of all univalent functions $f(z)$ in \mathbb{U} . A function $f(z) \in \mathcal{A}$ is said to be starlike with respect to the origin in \mathbb{U} if it satisfies

$$(1.2) \quad \operatorname{Re} \left\{ \frac{z f'(z)}{f(z)} \right\} > 0 \quad (z \in \mathbb{U}).$$

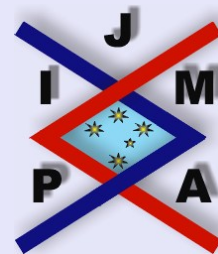
We denote by \mathcal{S}^* the subclass of \mathcal{S} consisting of all starlike functions $f(z)$ with respect to the origin in \mathbb{U} . Further, a function $f(z) \in \mathcal{A}$ is said to be convex in \mathbb{U} if it satisfies

$$(1.3) \quad \operatorname{Re} \left\{ 1 + \frac{z f''(z)}{f'(z)} \right\} > 0 \quad (z \in \mathbb{U}).$$

We also denote by \mathcal{K} the subclass of \mathcal{S} consisting of $f(z)$ which are convex in \mathbb{U} . By the above definitions, we know that $f(z) \in \mathcal{K}$ if and only if $z f'(z) \in \mathcal{S}^*$, and that $\mathcal{K} \subset \mathcal{S}^* \subset \mathcal{S} \subset \mathcal{A}$.

The class \mathcal{T} is defined as the subclass of \mathcal{S} consisting of all functions $f(z)$ which are given by

$$(1.4) \quad f(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad (a_n \geq 0).$$



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Further, we denote by $\mathcal{T}^* = \mathcal{S}^* \cap \mathcal{T}$ and $\mathcal{C} = \mathcal{K} \cap \mathcal{T}$. It is well-known by Silverman [6] that

Remark 1. A function $f(z) \in \mathcal{T}^*$ if and only if

$$(1.5) \quad \sum_{n=2}^{\infty} n a_n \leq 1.$$

A function $f(z) \in \mathcal{C}$ if and only if

$$(1.6) \quad \sum_{n=2}^{\infty} n^2 a_n \leq 1.$$

For $f(z) \in \mathcal{A}$ and $g(z) \in \mathcal{A}$, $f(z)$ is said to be subordinate to $g(z)$ in \mathbb{U} if there exists an analytic function $\omega(z)$ in \mathbb{U} such that $\omega(0) = 0$, $|\omega(z)| < 1$ ($z \in \mathbb{U}$), and $f(z) = g(\omega(z))$. We denote this subordination by

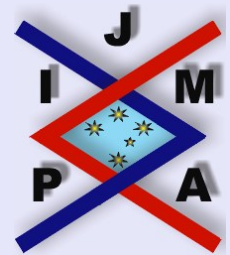
$$(1.7) \quad f(z) \prec g(z). \quad (\text{cf. Duren [1]}).$$

For subordinations, Littlewood [2] has given the following integral mean.

Theorem A. If $f(z)$ and $g(z)$ are analytic in \mathbb{U} with $f(z) \prec g(z)$, then, for $\lambda > 0$ and $|z| = r$ ($0 < r < 1$),

$$(1.8) \quad \int_0^{2\pi} |f(re^{i\theta})|^\lambda d\theta \leq \int_0^{2\pi} |g(re^{i\theta})|^\lambda d\theta.$$

Furthermore, Silverman [6] has shown that



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Remark 2. $f_1(z) = z$ and $f_n(z) = z - \frac{z^n}{n}$ ($n \geq 2$) are extreme points of the class \mathcal{T}^* (or \mathcal{T}). $f_1(z) = z$ and $f_n(z) = z - \frac{z^n}{n^2}$ ($n \geq 2$) are extreme points of the class \mathcal{C} .

Applying Theorem A with extreme points of \mathcal{T} , Silverman [7] has proved the following results.

Theorem B. Suppose that $f(z) \in \mathcal{T}^*$, $\lambda > 0$ and $f_2(z) = z - \frac{z^2}{2}$. Then, for $z = re^{i\theta}$ ($0 < r < 1$),

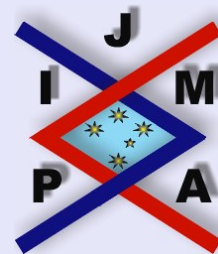
$$(1.9) \quad \int_0^{2\pi} |f(z)|^\lambda d\theta \leq \int_0^{2\pi} |f_2(z)|^\lambda d\theta.$$

Theorem C. If $f(z) \in \mathcal{T}^*$, $\lambda > 0$, and $f_2(z) = z - \frac{z^2}{2}$, then, for $z = re^{i\theta}$ ($0 < r < 1$),

$$(1.10) \quad \int_0^{2\pi} |f'(z)|^\lambda d\theta \leq \int_0^{2\pi} |f_2'(z)|^\lambda d\theta.$$

In the present paper, we consider the generalization properties for Theorem B and Theorem C with $f(z) \in \mathcal{T}^*$ and $f(z) \in \mathcal{C}$.

Remark 3. More recently, applying Theorem A by Littlewood [2], Sekine, Tsurumi and Srivastava [4]; and Sekine, Tsurumi, Owa and Srivastava [5] have discussed some interesting properties of integral means inequalities for fractional derivatives of some general subclasses of analytic functions $f(z)$ in the open unit disk \mathbb{U} . Further, Owa and Sekine [3] have considered the integral means with some coefficient inequalities for certain analytic functions $f(z)$ in \mathbb{U} .



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2. Generalization Properties

Our first result for the generalization properties is contained in

Theorem 2.1. Let $f(z) \in \mathcal{T}^*$, $\lambda > 0$, and $f_k(z) = z - \frac{z^k}{k}$ ($k \geq 2$). If $f(z)$ satisfies

$$(2.1) \quad \sum_{j=0}^{k-3} \frac{j+1}{k} (a_{2k+j-1} + a_{k+j+1} - a_{k-j-1}) \geq 0$$

for $k \geq 3$, and if there exists an analytic function $\omega(z)$ in \mathbb{U} given by

$$(\omega(z))^{k-1} = k \left(\sum_{n=2}^{\infty} a_n z^{n-1} \right),$$

then, for $z = re^{i\theta}$ ($0 < r < 1$),

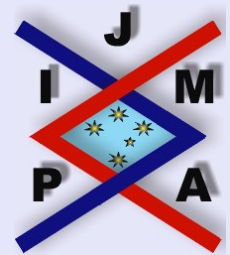
$$(2.2) \quad \int_0^{2\pi} |f(z)|^\lambda d\theta \leq \int_0^{2\pi} |f_k(z)|^\lambda d\theta.$$

Proof. For $f(z) \in \mathcal{T}^*$, we have to show that

$$\int_0^{2\pi} \left| 1 - \sum_{n=2}^{\infty} a_n z^{n-1} \right|^\lambda d\theta \leq \int_0^{2\pi} \left| 1 - \frac{z^{k-1}}{k} \right|^\lambda d\theta.$$

By Theorem A, it suffices to prove that

$$1 - \sum_{n=2}^{\infty} a_n z^{n-1} \prec 1 - \frac{z^{k-1}}{k}.$$



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Let us define the function $\omega(z)$ by

$$(2.3) \quad 1 - \sum_{n=2}^{\infty} a_n z^{n-1} = 1 - \frac{1}{k} (\omega(z))^{k-1}.$$

It follows from (2.3) that

$$|\omega(z)|^{k-1} = \left| k \sum_{n=2}^{\infty} a_n z^{n-1} \right| \leq |z| \left(\sum_{n=2}^{\infty} k a_n \right).$$

Thus, we only show that

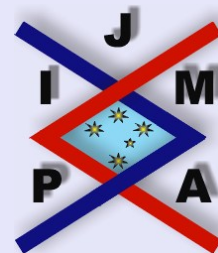
$$\sum_{n=2}^{\infty} k a_n \leq \sum_{n=2}^{\infty} n a_n,$$

or

$$\sum_{n=2}^{\infty} a_n \leq \frac{1}{k} \left(\sum_{n=2}^{\infty} n a_n \right).$$

Indeed, we see that

$$\begin{aligned} \frac{1}{k} \left(\sum_{n=2}^{\infty} n a_n \right) &= \left(1 - \frac{k-2}{k} \right) a_2 + \left(1 - \frac{k-3}{k} \right) a_3 + \cdots + \left(1 - \frac{2}{k} \right) a_{k-2} \\ &+ \left(1 - \frac{1}{k} \right) a_{k-1} + a_k + \left(1 + \frac{1}{k} \right) a_{k+1} + \left(1 + \frac{2}{k} \right) a_{k+2} \\ &+ \cdots + \left(1 + \frac{k+1}{k} \right) a_{2k+1} + \left(1 + \frac{k+2}{k} \right) a_{2k+2} + \cdots \end{aligned}$$



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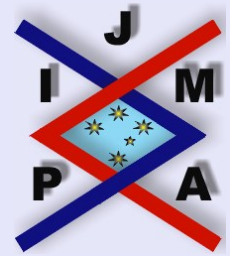
$$\begin{aligned}
&= \frac{k-2}{k}(a_{2k-2} - a_2) + \frac{k-3}{k}(a_{2k-3} - a_3) + \dots \\
&\quad + \frac{2}{k}(a_{k+2} - a_{k-2}) + \frac{1}{k}(a_{k+1} - a_{k-1}) + \left(1 + \frac{k-1}{k}\right) a_{2k-1} \\
&\quad + \left(1 + \frac{k}{k}\right) a_{2k} + \left(1 + \frac{k+1}{k}\right) a_{2k+1} + \dots + \sum_{n=2}^{2k-2} a_n.
\end{aligned}$$

Noting that

$$1 + \frac{k+j}{k} \geq 1 + \frac{2+j}{k}, \quad (j = -1, 0, 1, \dots),$$

we obtain

$$\begin{aligned}
(2.4) \quad &\frac{1}{k} \left(\sum_{n=2}^{\infty} n a_n \right) \\
&\geq \frac{k-2}{k}(a_{2k-2} - a_2) + \frac{k-3}{k}(a_{2k-3} - a_3) + \dots \\
&\quad + \frac{2}{k}(a_{k+2} - a_{k-2}) + \frac{1}{k}(a_{k+1} - a_{k-1}) \\
&\quad + \left(1 + \frac{1}{k}\right) a_{2k-1} + \left(1 + \frac{2}{k}\right) a_{2k} + \dots \\
&\quad + \left(1 + \frac{k-3}{k}\right) a_{3k-5} + \left(1 + \frac{k-2}{k}\right) a_{3k-4} + \dots + \sum_{n=2}^{2k-2} a_n
\end{aligned}$$



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$$\begin{aligned}
&\geq \frac{1}{k}(a_{2k-1} + a_{k+1} - a_{k-1}) + \frac{2}{k}(a_{2k} + a_{k+2} - a_{k-2}) + \cdots \\
&\quad + \frac{k-2}{k}(a_{3k-4} + a_{2k-2} - a_2) + \sum_{n=2}^{\infty} a_n \\
&= \sum_{j=0}^{k-3} \frac{j+1}{k}(a_{2k+j-1} + a_{k+j+1} - a_{k-j-1}) + \sum_{n=2}^{\infty} a_n \\
&\geq \sum_{n=2}^{\infty} a_n
\end{aligned}$$

with the following condition

$$\sum_{j=0}^{k-3} \frac{j+1}{k}(a_{2k+j-1} + a_{k+j+1} - a_{k-j-1}) \geq 0.$$

Thus, we observe that the function $\omega(z)$ defined by (2.3) is analytic in \mathbb{U} with $\omega(0) = 0$, $|\omega(z)| < 1$ ($z \in \mathbb{U}$). This completes the proof of the theorem. \square

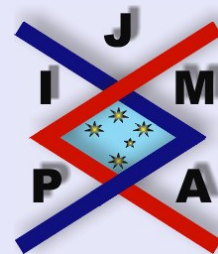
Remark 4. Taking $k = 2$ in Theorem 2.1, we have Theorem B by Silverman [7].

Example 2.1. Let us define

$$(2.5) \quad f(z) = z - \frac{37}{1200}z^2 - \frac{1}{18}z^3 - \frac{1}{48}z^4 - \frac{1}{100}z^5$$

and

$$(2.6) \quad f_3(z) = z - \frac{1}{3}z^3$$



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with $k = 3$ in Theorem 2.1. Since $f(z)$ satisfies

$$\sum_{n=2}^{\infty} na_n = \frac{217}{600} < 1,$$

we have $f(z) \in \mathcal{T}^*$. Furthermore, $f(z)$ satisfies,

$$\frac{1}{3}(a_5 + a_4 - a_2) = \frac{1}{3} \left(\frac{1}{100} + \frac{1}{48} - \frac{37}{1200} \right) = 0.$$

Thus, $f(z)$ satisfies the conditions in Theorem 2.1 with $k = 3$.

If we take $\lambda = 2$, then we have

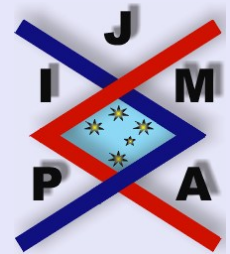
$$\int_0^{2\pi} |f(z)|^2 d\theta \leq 2\pi r^2 \left(1 + \frac{1}{9}r^4 \right) < \frac{20}{9}\pi = 6.9813\dots$$

Corollary 2.2. Let $f(z) \in \mathcal{T}^*$, $0 < \lambda \leq 2$, and $f_k(z) = z - \frac{z^k}{k}$ ($k \geq 2$). If $f(z)$ satisfies the conditions in Theorem 2.1, then, for $z = re^{i\theta}$ ($0 < r < 1$),

$$(2.7) \quad \int_0^{2\pi} |f(z)|^\lambda d\theta \leq 2\pi r^\lambda \left(1 + \frac{1}{k^2}r^{2(k-1)} \right)^{\frac{\lambda}{2}} < 2\pi \left(1 + \frac{1}{k^2} \right)^{\frac{\lambda}{2}}.$$

Proof. It follows that

$$\int_0^{2\pi} |f_k(z)|^\lambda d\theta = \int_0^{2\pi} |z|^\lambda \left| 1 - \frac{z^{k-1}}{k} \right|^\lambda d\theta.$$



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Applying Hölder's inequality for $0 < \lambda < 2$, we obtain that

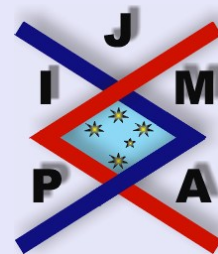
$$\begin{aligned}
 \int_0^{2\pi} |z|^\lambda \left| 1 - \frac{z^{k-1}}{k} \right|^\lambda d\theta &\leq \left(\int_0^{2\pi} (|z|^\lambda)^{\frac{2}{2-\lambda}} d\theta \right)^{\frac{2-\lambda}{2}} \left(\int_0^{2\pi} \left(\left| 1 - \frac{z^{k-1}}{k} \right|^\lambda \right)^{\frac{2}{\lambda}} d\theta \right)^{\frac{\lambda}{2}} \\
 &= \left(\int_0^{2\pi} |z|^{\frac{2\lambda}{2-\lambda}} d\theta \right)^{\frac{2-\lambda}{2}} \left(\int_0^{2\pi} \left| 1 - \frac{z^{k-1}}{k} \right|^2 d\theta \right)^{\frac{\lambda}{2}} \\
 &= \left(2\pi r^{\frac{2\lambda}{2-\lambda}} \right)^{\frac{2-\lambda}{2}} \left(2\pi \left(1 + \frac{1}{k^2} r^{2(k-1)} \right) \right)^{\frac{\lambda}{2}} \\
 &= 2\pi r^\lambda \left(1 + \frac{1}{k^2} r^{2(k-1)} \right)^{\frac{\lambda}{2}} \\
 &< 2\pi \left(1 + \frac{1}{k^2} \right)^{\frac{\lambda}{2}}.
 \end{aligned}$$

Further, it is clear for $\lambda = 2$. □

For the generalization of Theorem C by Silverman [7], we have

Theorem 2.3. *Let $f(z) \in \mathcal{T}^*$, $\lambda > 0$, and $f_k(z) = z - \frac{z^k}{k}$ ($k \geq 2$). If there exists an analytic function $\omega(z)$ in \mathbb{U} given by*

$$(\omega(z))^{k-1} = \sum_{n=2}^{\infty} n a_n z^{n-1},$$



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then, for $z = re^{i\theta}$ ($0 < r < 1$),

$$(2.8) \quad \int_0^{2\pi} |f'(z)|^\lambda d\theta \leq \int_0^{2\pi} |f'_k(z)|^\lambda d\theta.$$

Proof. For $f(z) \in \mathcal{T}^*$, it is sufficient to show that

$$(2.9) \quad 1 - \sum_{n=2}^{\infty} na_n z^{n-1} \prec 1 - z^{k-1}.$$

Let us define the function $\omega(z)$ by

$$(2.10) \quad 1 - \sum_{n=2}^{\infty} na_n z^{n-1} = 1 - \omega(z)^{k-1},$$

or, by

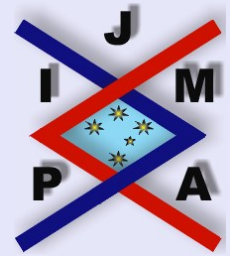
$$\omega(z)^{k-1} = \sum_{n=2}^{\infty} na_n z^{n-1}.$$

Since $f(z)$ satisfies

$$\sum_{n=2}^{\infty} na_n \leq 1,$$

the function $\omega(z)$ is analytic in \mathbb{U} , $\omega(0) = 0$, and $|\omega(z)| < 1$ ($z \in \mathbb{U}$). \square

Remark 5. If we take $k = 2$ in Theorem 2.3, then we have Theorem C by Silverman [7].



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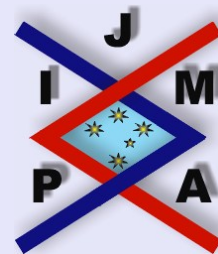
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Using the Hölder inequality for Theorem 2.3, we have

Corollary 2.4. Let $f(z) \in \mathcal{T}^*$, $0 < \lambda \leq 2$, and $f_k(z) = z - \frac{z^k}{k}$ ($k \geq 2$). If $f(z)$ satisfies the conditions in Theorem 2.3, then, for $z = re^{i\theta}$ ($0 < r < 1$),

$$\int_0^{2\pi} |f'(z)|^\lambda d\theta \leq 2\pi (1 + r^{2(k-1)})^{\frac{\lambda}{2}} < 2^{\frac{2+\lambda}{2}} \pi.$$



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3. Integral Means for Functions in the Class \mathcal{C}

In this section, we discuss the integral means for functions $f(z)$ in the class \mathcal{C} .

Theorem 3.1. Let $f(z) \in \mathcal{C}$, $\lambda > 0$, and $f_k(z) = z - \frac{z^k}{k^2}$ ($k \geq 2$). If $f(z)$ satisfies

$$(3.1) \quad \sum_{j=2}^{k-1} \frac{(k+j)(k-j)}{k^2} (a_{2k-j} - a_j) \geq 0$$

for $k \geq 3$, and if there exists an analytic function $\omega(z)$ in \mathbb{U} given by

$$(\omega(z))^{k-1} = k^2 \sum_{n=2}^{\infty} a_n z^{n-1},$$

then, for $z = re^{i\theta}$ ($0 < r < 1$),

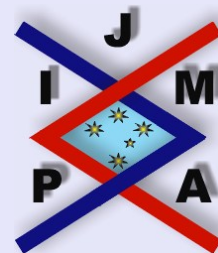
$$(3.2) \quad \int_0^{2\pi} |f(z)|^\lambda d\theta \leq \int_0^{2\pi} |f_k(z)|^\lambda d\theta.$$

Proof. For the proof, we need to show that

$$(3.3) \quad 1 - \sum_{n=2}^{\infty} a_n z^{n-1} \prec 1 - \frac{z^{k-1}}{k^2}$$

by Theorem A. Define the function $\omega(z)$ by

$$(3.4) \quad 1 - \sum_{n=2}^{\infty} a_n z^{n-1} = 1 - \frac{1}{k^2} \omega(z)^{k-1},$$



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or by

$$(3.5) \quad (\omega(z))^{k-1} = k^2 \left(\sum_{n=2}^{\infty} a_n z^{n-1} \right).$$

Therefore, we have to show that

$$\sum_{n=2}^{\infty} a_n \leq \frac{1}{k^2} \left(\sum_{n=2}^{\infty} n^2 a_n \right).$$

Using the same technique as in the proof of Theorem 2.1, we see that

$$\begin{aligned} \frac{1}{k^2} \left(\sum_{n=2}^{\infty} n^2 a_n \right) &\geq \sum_{j=2}^{k-1} \frac{(k+j)(k-j)}{k^2} (a_{2k-j} - a_j) + \sum_{n=2}^{\infty} a_n \\ &\geq \sum_{n=2}^{\infty} a_n. \end{aligned}$$

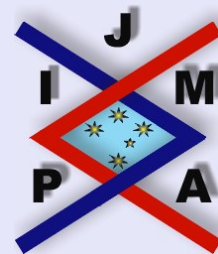
□

Example 3.1. Consider the functions

$$(3.6) \quad f(z) = z - \frac{1}{40}z^2 - \frac{1}{18}z^3 - \frac{1}{40}z^4$$

and

$$(3.7) \quad f_3(z) = z - \frac{1}{9}z^3$$



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with $k = 3$ in Theorem 3.1. Then we have that

$$\sum_{n=2}^{\infty} n^2 a_n = \frac{4}{40} + \frac{9}{18} + \frac{16}{40} = 1,$$

which implies $f(z) \in \mathcal{C}$, and that

$$\frac{5}{9}(a_4 - a_2) = 0.$$

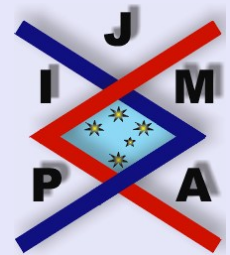
Thus $f(z)$ satisfies the conditions of Theorem 3.1. If we make $\lambda = 2$, then we see that

$$\int_0^{2\pi} |f(z)|^2 d\theta \leq 2\pi r^2 \left(1 + \frac{1}{81} r^4\right) < \frac{164}{81} \pi = 6.3607 \dots$$

Corollary 3.2. Let $f(z) \in \mathcal{C}$, $0 < \lambda \leq 2$, and $f_k(z) = z - \frac{z^k}{k^2}$ ($k \geq 2$). If $f(z)$ satisfies the condition in Theorem 3.1, then, for $k \geq 3$, then, for $z = re^{i\theta}$ ($0 < r < 1$),

$$(3.8) \quad \int_0^{2\pi} |f(z)|^\lambda d\theta \leq 2\pi r^\lambda \left(1 + \frac{1}{k^4} r^{2(k-1)}\right)^{\frac{\lambda}{2}} < 2\pi \left(1 + \frac{1}{k^4}\right)^{\frac{\lambda}{2}}.$$

Further, we may have



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Theorem 3.3. Let $f(z) \in \mathcal{C}$, $\lambda > 0$, and $f_k(z) = z - \frac{z^k}{k^2}$ ($k \geq 2$). If $f(z)$ satisfies

$$(3.9) \quad \sum_{j=2}^{2k-2} j(k-j)a_j \leq 0,$$

and if there exists an analytic function $\omega(z)$ in \mathbb{U} given by

$$(\omega(z))^{k-1} = k \sum_{n=2}^{\infty} n a_n z^{n-1},$$

then, for $z = re^{i\theta}$ ($0 < r < 1$),

$$(3.10) \quad \int_0^{2\pi} |f'(z)|^\lambda d\theta \leq \int_0^{2\pi} |f'_k(z)|^\lambda d\theta.$$

Example 3.2. Take the functions

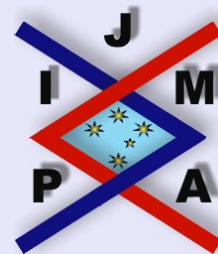
$$(3.11) \quad f(z) = z - \frac{1}{24}z^2 - \frac{1}{18}z^3 - \frac{1}{48}z^4$$

and

$$(3.12) \quad f_3(z) = z - \frac{1}{9}z^3$$

with $k = 3$ in Theorem 3.3. Since

$$\sum_{n=2}^{\infty} n^2 a_n = \frac{4}{24} + \frac{9}{18} + \frac{16}{48} = \frac{5}{6} < 1$$



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and

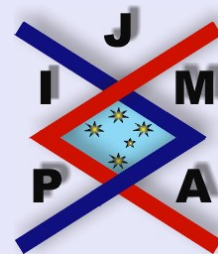
$$2(3-2)a_2 + 3(3-3)a_3 + 4(3-4)a_4 = \frac{1}{12} - \frac{1}{12} = 0,$$

$f(z)$ satisfies the conditions in Theorem 3.3. If we take $\lambda = 2$, then we have

$$\int_0^{2\pi} |f'(z)|^2 d\theta \leq 2\pi \left(1 + \frac{1}{9}r^4\right) < \frac{20}{9}\pi.$$

Corollary 3.4. Let $f(z) \in \mathcal{C}$, $0 < \lambda \leq 2$, and $f_k(z) = z - \frac{z^k}{k^2}$ ($k \geq 2$). If $f(z)$ satisfies the condition in Theorem 3.3, then, for $k \geq 2$, then, for $z = re^{i\theta}$ ($0 < r < 1$),

$$\int_0^{2\pi} |f'(z)|^\lambda d\theta \leq 2\pi \left(1 + \frac{1}{k}r^{2(k-1)}\right)^{\frac{\lambda}{2}} < 2\pi \left(1 + \frac{1}{k}\right)^{\frac{\lambda}{2}}.$$



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4. Applications for the Integrated Functions

For $f(z) \in \mathcal{T}$, we define

$$I_0 f(z) = f(z) = z - \sum_{n=2}^{\infty} a_n z^n$$

$$I f(z) = I_1 f(z) = \int_0^z f(t) dt = \frac{1}{2} z^2 - \sum_{n=2}^{\infty} \frac{a_n}{n+1} z^{n+1}$$

$$I_k f(z) = I(I_{k-1} f(z)) = \frac{1}{(k+1)!} z^{k+1} - \sum_{n=2}^{\infty} \frac{n!}{(n+k)!} a_n z^{n+k} \quad (k = 1, 2, 3, \dots).$$

Theorem 4.1. Let $f(z) \in \mathcal{T}^*$, $\lambda > 0$, and $f_j(z) = z - \frac{z^j}{j}$ ($j = 2, 3, 4, \dots$).

If $f(z)$ satisfies

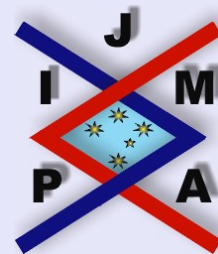
$$(4.1) \quad \sum_{k=2}^{j^2+j-1} \frac{j^2+j-k}{j(j+1)} (a_{2j^2+2j-k} - a_k) \geq 0$$

for $j = 2, 3, 4, \dots$, and if there exists an analytic function $\omega(z)$ in \mathbb{U} given by

$$(\omega(z))^{j-1} = j(j+1) \left(\sum_{n=2}^{\infty} \frac{1}{n+1} a_n z^{n-1} \right),$$

then

$$(4.2) \quad \int_0^{2\pi} |I f(z)|^\lambda d\theta \leq \int_0^{2\pi} |I f_j(z)|^\lambda d\theta.$$



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Proof. We have to prove

$$\int_0^{2\pi} \left| 1 - \sum_{n=2}^{\infty} \frac{2}{n+1} a_n z^{n-1} \right|^\lambda d\theta \leq \int_0^{2\pi} \left| 1 - \frac{2}{j(j+1)} z^{j-1} \right|^\lambda d\theta.$$

If

$$1 - \sum_{n=2}^{\infty} \frac{2}{n+1} a_n z^{n-1} \prec 1 - \frac{2}{j(j+1)} z^{j-1},$$

then the proof is completed by Theorem A.

Let us define the function $\omega(z)$ by

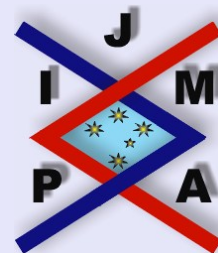
$$1 - \sum_{n=2}^{\infty} \frac{2}{n+1} a_n z^{n-1} = 1 - \frac{2}{j(j+1)} (\omega(z))^{j-1}.$$

Then

$$\begin{aligned} |\omega(z)|^{j-1} &= \left| j(j+1) \sum_{n=2}^{\infty} \frac{1}{n+1} a_n z^{n-1} \right| \\ &\leq |z| \left(j(j+1) \sum_{n=2}^{\infty} \frac{1}{n+1} a_n \right). \end{aligned}$$

Thus, we only show that

$$j(j+1) \sum_{n=2}^{\infty} \frac{1}{n+1} a_n \leq \sum_{n=2}^{\infty} n a_n$$



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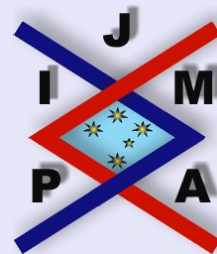
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or

$$\sum_{n=2}^{\infty} a_n \leq \sum_{n=2}^{\infty} n \left(\frac{1}{j(j+1)} + \frac{1}{n+1} \right) a_n.$$

Indeed,

$$\begin{aligned} & \sum_{n=2}^{\infty} n \left(\frac{1}{j(j+1)} + \frac{1}{n+1} \right) a_n \\ &= 2 \left(\frac{1}{j(j+1)} + \frac{1}{3} \right) a_2 + 3 \left(\frac{1}{j(j+1)} + \frac{1}{4} \right) a_3 + \cdots \\ &+ (j-1) \left(\frac{1}{j(j+1)} + \frac{1}{j} \right) a_{j-1} + j \left(\frac{1}{j(j+1)} + \frac{1}{j+1} \right) a_j \\ &+ (j+1) \left(\frac{1}{j(j+1)} + \frac{1}{j+2} \right) a_{j+1} \\ &+ \cdots + (2j^2 + 2j - 3) \left(\frac{1}{j(j+1)} + \frac{1}{2j^2 + 2j - 2} \right) a_{2j^2 + 2j - 3} \\ &+ (2j^2 + 2j - 2) \left(\frac{1}{j(j+1)} + \frac{1}{2j^2 + 2j - 1} \right) a_{2j^2 + 2j - 1} + \cdots \\ &\geq \left(1 - \frac{j(j+1) - 2}{j(j+1)} \right) a_2 + \left(1 - \frac{j(j+1) - 3}{j(j+1)} \right) a_3 + \cdots \\ &+ \left(1 - \frac{j(j+1) - (j-1)}{j(j+1)} \right) a_{j-1} + \left(1 - \frac{j(j+1) - j}{j(j+1)} \right) a_j \\ &+ \left(1 - \frac{j(j+1) - (j+1)}{j(j+1)} \right) a_{j+1} \end{aligned}$$



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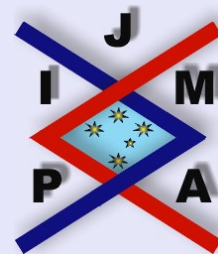


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$$\begin{aligned}
 & + \cdots + \left(1 - \frac{j(j+1) - (2j^2 + 2j - 3)}{j(j+1)}\right) a_{2j^2+2j-3} \\
 & + \left(1 - \frac{j(j+1) - (2j^2 + 2j - 2)}{j(j+1)}\right) a_{2j^2+2j-2} + \cdots \\
 = & \frac{j^2 + j - 2}{j(j+1)}(a_{2j^2+2j-2} - a_2) + \frac{j^2 + j - 3}{j(j+1)}(a_{2j^2+2j-3} - a_3) \\
 & + \cdots + \frac{j^2 + 1}{j(j+1)}(a_{2j^2+j+1} - a_{j-1}) + \frac{j^2}{j(j+1)}(a_{2j^2+j} - a_j) \\
 & + \frac{j^2 - 1}{j(j+1)}(a_{2j^2+j-1} - a_{j+1}) + \cdots + a_2 + a_3 + \cdots + a_{2j^2+2j-2} + \cdots \\
 = & \sum_{k=2}^{j^2+j-1} \frac{j^2 + j - k}{j(j+1)}(a_{2j^2+2j-k} - a_k) + \sum_{n=2}^{\infty} a_n \\
 \geq & \sum_{n=2}^{\infty} a_n
 \end{aligned}$$

for

$$\sum_{k=2}^{j^2+j-1} \frac{j^2 + j - k}{j(j+1)}(a_{2j^2+2j-k} - a_k) \geq 0.$$

This completes the proof of Theorem 4.1. □

Finally, we derive

Theorem 4.2. Let $f(z) \in T^*$, $\lambda > 0$, and $f_j(z) = z - \frac{z^j}{j}$ ($j = 2, 3, 4, \dots$). If

$f(z)$ satisfies

$$(4.3) \quad \sum_{n=2}^{\infty} a_n \geq \frac{6}{5} \sum_{n=2}^{\frac{(j+k)!}{2(j-1)!}-1} \left(1 - \frac{2n(j-1)!}{(j+k)!}\right) \left(a_n - a_{\frac{(j+k)!}{(j-1)!}-n}\right)$$

for $k = 2, 3, 4, \dots$, and if there exists an analytic function $\omega(z)$ in \mathbb{U} given by

$$(\omega(z))^{j-1} = \frac{(j+k)!}{(j-1)!} \sum_{n=2}^{\infty} \frac{n!}{(n+k)!} a_n z^{n-1},$$

then

$$(4.4) \quad \int_0^{2\pi} |I_k f(z)|^\lambda d\theta \leq \int_0^{2\pi} |I_k f_j(z)|^\lambda d\theta.$$

Proof. We have to show that

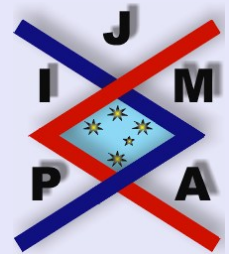
$$1 - \sum_{n=2}^{\infty} \frac{n!(k+1)!}{(n+k)!} a_n z^{n-1} \prec 1 - \frac{(j-1)!(k+1)!}{(j+k)!} z^{j-1}.$$

Define $\omega(z)$ by

$$1 - \sum_{n=2}^{\infty} \frac{n!(k+1)!}{(n+k)!} a_n z^{n-1} = 1 - \frac{(j-1)!(k+1)!}{(j+k)!} (\omega(z))^{j-1}$$

or by

$$(\omega(z))^{j-1} = \frac{(j+k)!}{(j-1)!} \sum_{n=2}^{\infty} \frac{n!}{(n+k)!} a_n z^{n-1}.$$



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Then we have to show that

$$\frac{(j+k)!}{(j-1)!} \sum_{n=2}^{\infty} \frac{n!}{(n+k)!} a_n \leq \sum_{n=2}^{\infty} n a_n,$$

that is, that

$$\sum_{n=2}^{\infty} \frac{n!}{(n+k)!} a_n \leq \frac{(j-1)!}{(j+k)!} \sum_{n=2}^{\infty} n a_n.$$

Since

$$\begin{aligned} \sum_{n=2}^{\infty} \frac{n!}{(n+k)!} a_n &= \sum_{n=2}^{\infty} \frac{1}{(n+1)(n+2)\cdots(n+k)} a_n \\ &= \sum_{n=2}^{\infty} \left\{ \left(\frac{1}{n+1} - \frac{1}{n+2} \right) \left(\frac{1}{n+3} - \frac{1}{n+4} \right) \cdots \right\} a_n \\ &\leq \sum_{n=2}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+2} \right)^{\lfloor \frac{k}{2} \rfloor} a_n \\ &\leq \sum_{n=2}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+2} \right) a_n, \end{aligned}$$

we obtain

$$\sum_{n=2}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+2} \right) a_n \leq \frac{(j-1)!}{(j+k)!} \sum_{n=2}^{\infty} n a_n.$$

Furthermore, we have

$$\sum_{n=2}^{\infty} a_n \leq \sum_{n=2}^{\infty} \left(\frac{2n(j-1)!}{(j+k)!} + \frac{2n}{n+1} - \frac{n}{n+2} \right) a_n.$$



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Let the function $h(n)$ be given by

$$h(n) = \frac{2n}{n+1} - \frac{n}{n+2} = 1 - \frac{2}{n^2 + 3n + 2}.$$

Since $h(n)$ is increasing for $n \geq 2$,

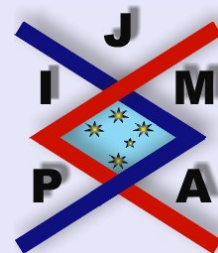
$$h(n) \geq \frac{5}{6}.$$

Thus, we only show that

$$\sum_{n=2}^{\infty} a_n \leq \sum_{n=2}^{\infty} \left(\frac{11}{6} - \frac{(j+k)! - 2n(j-1)!}{(j+k)!} \right) a_n.$$

In fact,

$$\begin{aligned} & \sum_{n=2}^{\infty} \left(\frac{11}{6} - \frac{(j+k)! - 2n(j-1)!}{(j+k)!} \right) a_n \\ &= \left(\frac{11}{6} - \frac{(j+k)! - 4(j-1)!}{(j+k)!} \right) a_2 \\ &+ \left(\frac{11}{6} - \frac{(j+k)! - 6(j-1)!}{(j+k)!} \right) a_3 + \dots \\ &+ \left(\frac{11}{6} - \frac{4(j-1)!}{(j+k)!} \right) a_{\frac{(j+k)!}{2(j-1)!} - 2} + \left(\frac{11}{6} - \frac{2(j-1)!}{(j+k)!} \right) a_{\frac{(j+k)!}{2(j-1)!} - 1} \end{aligned}$$



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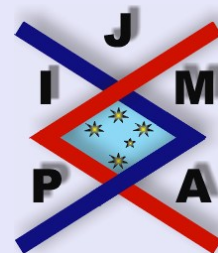


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$$\begin{aligned}
 & + \left(\frac{11}{6} - 0 \right) a_{\frac{(j+k)!}{2(j-1)!}} + \left(\frac{11}{6} - \frac{2(j-1)!}{(j+k)!} \right) a_{\frac{(j+k)!}{2(j-1)!}+1} \\
 & + \left(\frac{11}{6} + \frac{4(j-1)!}{(j+k)!} \right) a_{\frac{(j+k)!}{2(j-1)!}+2} + \cdots \\
 & + \left(\frac{11}{6} + \frac{(j+k)! - 6(j-1)!}{(j+k)!} \right) a_{\frac{(j+k)!}{(j-1)!}-3} \\
 & + \left(\frac{11}{6} + \frac{(j+k)! - 4(j-1)!}{(j+k)!} \right) a_{\frac{(j+k)!}{(j-1)!}-2} + \cdots \\
 \geq & \frac{11}{6} \sum_{n=2}^{\infty} a_n + \frac{(j+k)! - 4(j-1)!}{(j+k)!} \left(a_{\frac{(j+k)!}{(j-1)!}-2} - a_2 \right) \\
 & + \frac{(j+k)! - 6(j-1)!}{(j+k)!} \left(a_{\frac{(j+k)!}{(j-1)!}-3} - a_3 \right) \\
 & + \frac{4(j-1)!}{(j+k)!} \left(a_{\frac{(j+k)!}{2(j-1)!}+2} - a_{\frac{(j+k)!}{2(j-1)!}-2} \right) \\
 & + \frac{2(j-1)!}{(j+k)!} \left(a_{\frac{(j+k)!}{2(j-1)!}+1} - a_{\frac{(j+k)!}{2(j-1)!}-1} \right) \\
 = & \sum_{n=2}^{\infty} a_n + \frac{5}{6} \sum_{n=2}^{\infty} a_n + \frac{(j+k)! - 4(j-1)!}{(j+k)!} \left(a_{\frac{(j+k)!}{2(j-1)!}-2} - a_2 \right) \\
 & + \frac{(j+k)! - 6(j-1)!}{(j+k)!} \left(a_{\frac{(j+k)!}{2(j-1)!}-3} - a_3 \right) + \cdots \\
 & + \frac{(j+k)! - \{(j+k)! - 4(j-1)!\}}{(j+k)!} \left(a_{\frac{(j+k)!}{2(j-1)!}+2} - a_{\frac{(j+k)!}{2(j-1)!}-2} \right)
 \end{aligned}$$

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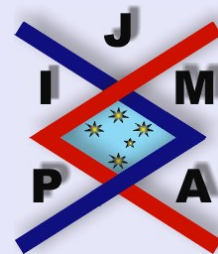


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$$\begin{aligned}
 & + \frac{(j+k)! - \{(j+k)! - 2(j-1)!\}}{(j+k)!} \left(a_{\frac{(j+k)!}{2(j-1)!}+1} - a_{\frac{(j+k)!}{2(j-1)!}-1} \right) \\
 & = \sum_{n=2}^{\infty} a_n + \frac{5}{6} \sum_{n=2}^{\infty} a_n + \sum_{n=2}^{\frac{(j+k)!}{2(j-1)!}-1} \frac{(j+k)! - 2n(j-1)!}{(j+k)!} \left(a_{\frac{(j+k)!}{(j-1)!}-n} - a_n \right) \\
 & \geq \sum_{n=2}^{\infty} a_n
 \end{aligned}$$

for

$$\sum_{n=2}^{\infty} a_n \geq \frac{6}{5} \sum_{n=2}^{\frac{(j+k)!}{2(j-1)!}-1} \left(1 - \frac{2n(j-1)!}{(j+k)!} \right) \left(a_n - a_{\frac{(j+k)!}{(j-1)!}-n} \right).$$

This completes the proof of Theorem 4.2. □

Remark 6. Letting $k = 2$, if $f(z)$ satisfies,

$$(4.5) \quad \sum_{n=2}^{\infty} a_n \geq \frac{6}{5} \sum_{n=2}^{\frac{j(j+1)(j+2)}{2}-1} \left(1 - \frac{2n}{j(j+1)(j+2)} \right) (a_n - a_{j(j+1)(j+2)-n})$$

for $j = 2, 3, 4, \dots$, then

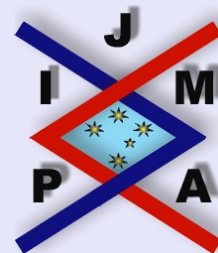
$$(4.6) \quad \int_0^{2\pi} |I_2 f(z)|^\lambda d\theta \leq \int_0^{2\pi} |I_2 f_j(z)|^\lambda d\theta.$$

Remark 7. Letting $k = 3$, if $f(z)$ satisfies,

$$(4.7) \quad \sum_{n=2}^{\infty} a_n \geq \frac{6}{5} \sum_{n=2}^{\frac{j(j+1)(j+2)(j+3)}{2}-1} \left(1 - \frac{2n}{j(j+1)(j+2)(j+3)} \right) \times (a_n - a_{j(j+1)(j+2)(j+3)-n})$$

for $j = 2, 3, 4, \dots$, then

$$(4.8) \quad \int_0^{2\pi} |I_3 f(z)|^\lambda d\theta \leq \int_0^{2\pi} |I_3 f_j(z)|^\lambda d\theta.$$



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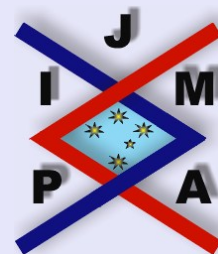
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Integral Means for Starlike and Convex Functions with Negative Coefficients

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