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NON-AUTONOMOUS DIFFERENTIAL SUBORDINATIONS RELATED TO A SECTOR

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ABSTRACT. Let $\lambda(z)$ be a complex valued function defined in the unit disc E and let $p(z)$ be a function analytic in E with $p(0) = 1$ and $p(z) \neq 0$ in E. In this article, we determine the largest constants $\gamma_k, k = 1, 2, 3, \ldots$ and conditions on $\lambda(z)$ such that for given α, β and δ, the non-autonomous differential subordination

$$(p(z))^\beta \left[1 + \lambda(z)\frac{zp'(z)}{p^k(z)}\right]^\alpha \prec \left(\frac{1+z}{1-z}\right)^{\gamma_k}, \; z \in E,$$

implies

$$p(z) \prec \left(\frac{1+z}{1-z}\right)^\delta$$

in E. Here the symbol ‘ $\prec$ ’ stands for subordination. Almost all the previously known results on differential subordination concerning a sector follow as particular cases of our results.

Key words and phrases: Univalent function, Starlike function, Subordination, Differential subordination.

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1. INTRODUCTION

Let $\mathcal{A}$ denote the class of functions f which are analytic in the unit disc $E = \{z : |z| < 1\}$ and satisfy $f(0) = 0, f'(0) = 1$. Denote by $\mathcal{A}'$, the class of functions p analytic in E for which $p(0) = 1$ and $p(z) \neq 0$ in E.

If f and g are analytic in E, we say that f is subordinate to g in E, written as $f(z) \prec g(z)$ in E, if g is univalent in E, $f(0) = g(0)$ and $f(E) \subset g(E)$.

Let h be a univalent function in E and let $\psi : C^2 \to C$, where C is the complex plane. If an analytic function p satisfies the differential subordination

$$\psi(p(z), zp'(z)) \prec h(z), h(0) = \psi(p(0), 0), \; z \in E, \tag{1.1}$$

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then a univalent function q is said to be the dominant of the differential subordination (1.1), if $p(0) = q(0)$ and $p \prec q$ for all p satisfying (1.1). Differential subordination (1.1) is said to be non-autonomous if a function of z is allowed to be present on the left hand side, in addition to the terms $p(z)$ and $zp'(z)$.

Since 1981, when a formal study of differential subordination started with a remarkable paper of Miller and Mocanu [5], several results concerning differential subordination in a sector have been proved (e.g. see [6], [9] and [3]).

In the present paper, we establish the following two theorems.

Theorem 1.1. *Let $\alpha \in [0,1]$ be fixed and let $\delta \in (0, \delta_0]$, where δ_0 is the solution of the equation*

$$\beta\delta\pi = 2\pi - \alpha\left(\frac{\pi}{2} + \arctan\eta\right)$$

for $\beta \geq 0$ and for a suitable fixed $\eta > 0$ such that $\lambda(z) : E \to C$ satisfies

$$\frac{\delta \operatorname{Re}\lambda(z)}{1 + \delta|\operatorname{Im}\lambda(z)|} \geq \eta,\ z \in E. \tag{1.2}$$

If $p \in \mathcal{A}'$ satisfies the non-autonomous differential subordination

$$(p(z))^\beta \left[1 + \lambda(z)\frac{zp'(z)}{p(z)}\right]^\alpha \prec \left(\frac{1+z}{1-z}\right)^{\gamma_1},\ z \in E, \tag{1.3}$$

then,

$$p(z) \prec \left(\frac{1+z}{1-z}\right)^\delta \text{ in } E,$$

where,

$$\gamma_1 = \beta\delta + \frac{2\alpha}{\pi}\arctan\eta,\ 0 < \delta \leq \delta_0. \tag{1.4}$$

Theorem 1.2. *Let $k = 2, 3, 4, \ldots, \alpha \in [0,1], \delta \in \left(0, \frac{1}{k-1}\right)$ be fixed. Also let $\beta \geq 0$ be such that $0 \leq \beta\delta \leq 2$. For a suitable fixed $\eta > 0$, let $\lambda(z) : E \to C$ be a function satisfying*

$$\frac{\delta|\lambda(z)|\cos\psi}{x + \delta|\lambda(z)||\sin\psi|} \geq \eta,\ z \in E, \tag{1.5}$$

where,

$$|\psi - Arg\lambda(z)| = (k-1)\frac{\delta\pi}{2} \tag{1.6}$$

and,

$$x = (1-(k-1)\delta)^{\frac{1-(k-1)\delta}{2}}(1+(k-1)\delta)^{\frac{1+(k-1)\delta}{2}}. \tag{1.7}$$

If $p \in \mathcal{A}'$ satisfies the non-autonomous differential subordination

$$(p(z))^\beta \left[1 + \lambda(z)\frac{zp'(z)}{p^k(z)}\right]^\alpha \prec \left(\frac{1+z}{1-z}\right)^{\gamma_k},\ z \in E, \tag{1.8}$$

then,

$$p(z) \prec \left(\frac{1+z}{1-z}\right)^\delta \quad \text{in } E,$$

where,

$$\gamma_k = \beta\delta + \frac{2\alpha}{\pi}\arctan\eta. \tag{1.9}$$

We claim that our results unify most of the previously known results related to differential subordinations in a sector. Some special cases of Theorem 1.1 and Theorem 1.2 have been discussed in Section 3. In Section 4, we give some applications of our results to the univalent functions.

We shall need the following lemma to prove our results.

Lemma 1.3. *Let F be analytic in E and let G be analytic and univalent in $\overline{E}$ except for points ζ such that $\lim_{z\to\zeta} F(z) = \infty$, with $F(0) = G(0)$. If $F \not\prec G$ in E, then there exist points $z_0 \in E, \zeta_0 \in \partial E$ (boundary of E) and an $m \geq 1$ for which*

(a) $F(|z| < |z_0|) \subset G(E)$,
(b) $F(z_0) = G(\zeta_0)$ *and*
(c) $z_0 F'(z_0) = m\zeta_0 G'(\zeta_0)$.

Lemma 1.3 is due to Miller and Mocanu [5].

2. PROOFS OF MAIN THEOREMS

Proof of Theorem 1.1. Let $q(z) = \left(\frac{1+z}{1-z}\right)^\delta$. Then we need to prove that (1.3) implies $p(z) \prec q(z)$ in E. Suppose, on the contrary, that $p \not\prec q$ in E. Then, by Lemma 1.3, there exist points $z_0 \in E, \zeta_0 \in \partial E$ and an $m \geq 1$ such that $p(z_0) = q(\zeta_0)$ and $z_0 p'(z_0) = m\zeta_0 q'(\zeta_0)$. Since $p(z_0) = q(\zeta_0) \neq 0$, it follows that $\zeta_0 \neq \pm 1$. Thus $\frac{1+\zeta_0}{1-\zeta_0} = ri$ for $r \neq 0$. Writing $\lambda(z_0) = Re^{i\phi}, |\phi| < \pi/2$, a simple calculation gives

$$\begin{aligned}(p(z_0))^\beta \left[1 + \lambda(z_0)\frac{z_0 p'(z_0)}{p(z_0)}\right]^\alpha &= (q(\zeta_0))^\beta \left[1 + \lambda(z_0)\frac{m\zeta_0 q'(\zeta_0)}{q(\zeta_0)}\right]^\alpha \\ &= (ri)^{\beta\delta}\left[1 + \frac{im\delta Re^{i\phi}}{2}\left(\frac{r^2+1}{r}\right)\right]^\alpha \\ &= \Psi_0 \quad \text{say.}\end{aligned}$$

Then,

$$Arg\Psi_0 = \pm\frac{\beta\delta\pi}{2} + \alpha \arctan\left[\frac{m\delta R\cos\phi}{\frac{2r}{r^2+1} - m\delta R\sin\phi}\right]. \tag{2.1}$$

Here a positive sign corresponds to $r > 0$ and a negative sign to $r < 0$.

Let $A = A(m,r) = \frac{m\delta R\cos\phi}{x(r) - m\delta R\sin\phi}$, where $x(r) = \frac{2r}{r^2+1}$. Then, Max$|x(r)| = 1$ for all values of r whether positive or negative.

If we define,

$$B = B(m,r) = \frac{m\delta R\cos\phi}{|x(r)| + m\delta R|\sin\phi|},$$

then B is an increasing function of m for each fixed r, and therefore, for $m \geq 1$, we have

$$B \geq \frac{\delta R\cos\phi}{|x(r)| + \delta R|\sin\phi|} \geq \frac{\delta R\cos\phi}{1 + \delta R|\sin\phi|} \geq \eta, \quad \text{(using (1.2)).}$$

Now, It is easy to check the following six cases:

(i) When $r > 0$ and $\sin\phi \leq 0$, we have $A = B$.
(ii) When $r > 0$ and $x(r) > m\delta R\sin\phi > 0$, we get $A > B$.
(iii) When $r > 0$ and $x(r) < m\delta R\sin\phi$, we get $A < -B$.
(iv) When $r < 0$ and $\sin\phi \geq 0$, we have $A = -B$.
(v) When $r < 0$ and $m\delta R\sin\phi < -|x(r)|$, we have $A > B$.
(vi) When $r < 0$ and $0 > m\delta R\sin\phi > -|x(r)|$, we get $A < -B$.

Since arctan is an increasing function of its argument, therefore, in view of cases (i) and (ii), we get from (2.1),

$$\begin{aligned} Arg\Psi_0 &\geq \frac{\beta\delta\pi}{2} + \alpha \arctan B \\ &\geq \frac{\beta\delta\pi}{2} + \alpha \arctan \eta \\ &= \frac{\gamma_1\pi}{2}. \end{aligned}$$

For the case (iii), we get

$$\begin{aligned} Arg\Psi_0 &< \frac{\beta\delta\pi}{2} + \alpha \arctan(-B) \\ &\leq \frac{\beta\delta\pi}{2} - \alpha \arctan \eta \\ &= \beta\delta\pi - \frac{\gamma_1\pi}{2} \\ &\leq 2\pi - \frac{\gamma_1\pi}{2}, \text{ since } 0 \leq \beta\delta \leq \beta\delta_0 < 2. \end{aligned}$$

For cases (iv) and (vi), we get

$$\begin{aligned} Arg\Psi_0 &\leq -\frac{\beta\delta\pi}{2} + \alpha \arctan(-B) \\ &\leq -\frac{\beta\delta\pi}{2} - \alpha \arctan \eta \\ &= -\frac{\gamma_1\pi}{2}. \end{aligned}$$

In case (v), we have

$$\begin{aligned} Arg\Psi_0 &> -\frac{\beta\delta\pi}{2} + \alpha \arctan B \\ &\geq -\frac{\beta\delta\pi}{2} + \alpha \arctan \eta \\ &= -\left(\beta\delta\pi - \frac{\gamma_1\pi}{2}\right) \\ &\geq -\left(2\pi - \frac{\gamma_1\pi}{2}\right), \text{ since } 0 \leq \beta\delta \leq \beta\delta_0 < 2. \end{aligned}$$

Combining all the above cases, we obtain

$$\frac{\gamma_1\pi}{2} \leq |\operatorname{Arg}\Psi_0| \leq 2\pi - \frac{\gamma_1\pi}{2},$$

which is a contradiction to (1.3). Hence $p(z) \prec \left(\frac{1+z}{1-z}\right)^{\delta}$. The proof of the theorem is complete. □

Proof of Theorem 1.2. Proceeding as in the proof of Theorem 1.1, we get

$$\begin{aligned} (p(z_0))^{\beta}\left[1+\lambda(z_0)\frac{z_0p'(z_0)}{p^k(z_0)}\right]^{\alpha} &= (ri)^{\beta\delta}\left[1+\frac{im\delta R}{2}(r^2+1)r^{-1-(k-1)\delta}e^{i(\phi-(k-1)\delta\pi/2)}\right]^{\alpha} \\ &= \Theta_0 \text{ say.} \end{aligned}$$

Now, if $\psi - \phi = -(k-1)\frac{\delta\pi}{2}$, then we get

$$\operatorname{Arg}\Theta_0 = \frac{\beta\delta\pi}{2} + \alpha \arctan\left[\frac{m\delta R\cos\psi}{\frac{2r^{1+(k-1)\delta}}{r^2+1} - m\delta R\sin\psi}\right]. \tag{2.2}$$

Let $A = A(m,r) = \left[\frac{m\delta R\cos\psi}{x(r)-m\delta R\sin\psi}\right]$, where $x(r) = \frac{2r^{1+(k-1)\delta}}{r^2+1}$. Since $(k-1)\delta < 1$, it is easy to verify that for $r > 0$ (and, of course, also for $r < 0$),

$$\max|x(r)| = (1-(k-1)\delta)^{\frac{1-(k-1)\delta}{2}}(1+(k-1)\delta)^{\frac{1+(k-1)\delta}{2}} = x, \text{ using (1.7).}$$

Now, if we define,

$$B = B(m,r) = \frac{m\delta R\cos\psi}{|x(r)| + m\delta R|\sin\psi|},$$

then, B is an increasing function of m and, therefore, for $m \geq 1$, we have

$$B \geq \frac{\delta R\cos\psi}{|x(r)| + \delta R|\sin\psi|} \geq \frac{\delta R\cos\psi}{x + \delta R|\sin\psi|} \geq \eta, \text{ using (1.5).}$$

Now, one can easily verify the following three cases:

Case (i). $A = B$, when $\sin\psi \leq 0$.
Case(ii). $A > B$, when $\sin\psi > 0$ and $x(r) > m\delta R\sin\psi$.
Case (iii). $A < -B$, when $\sin\psi > 0$ and $x(r) < m\delta R\sin\psi$.

Since arctan is an increasing function of its argument, therefore, in view of cases (i) and (ii), we get from (2.2)

$$\begin{aligned} Arg\Theta_0 &\geq \frac{\beta\delta\pi}{2} + \alpha\arctan\eta \\ &= \frac{\gamma_k\pi}{2}, \text{ using (1.9).} \end{aligned}$$

For the case (iii), we obtain

$$\begin{aligned} Arg\Theta_0 &< \frac{\beta\delta\pi}{2} + \alpha\arctan(-B) \\ &\leq \frac{\beta\delta\pi}{2} - \alpha\arctan\eta \\ &\leq \beta\delta\pi - \frac{\gamma_k\pi}{2} \\ &\leq 2\pi - \frac{\gamma_k\pi}{2}, \text{ since } \beta\delta \leq 2. \end{aligned}$$

Now, consider the case when $r < 0$. Writing $r = -a, a > 0$ we have

$$\begin{aligned} (p(z_0))^\beta\left[1+\lambda(z_0)\frac{z_0p'(z_0)}{p^k(z_0)}\right]^\alpha &= a^{\beta\delta}e^{\frac{-i\beta\delta\pi}{2}}\left[1-\frac{im\delta R}{2}\frac{(a^2+1)e^{i(\phi+(k-1)\delta\pi/2)}}{a^{1+(k-1)\delta}}\right]^\alpha \\ &= \Phi_0 \text{ say.} \end{aligned}$$

If $\psi - \phi = (k-1)\frac{\delta\pi}{2}$, then

$$Arg\Phi_0 = \frac{-\beta\delta\pi}{2} + \alpha\arctan C,$$

where $C = C(m,r) = \frac{m\delta R\cos\psi}{-|x(r)|-m\delta R\sin\psi}$.

It is now elementary to check the following three cases:

Case (a). When $\sin\psi \geq 0$, we have $C = -B$.
Case (b). When $\sin\psi < 0$ and $m\delta R\sin\psi < -|x(r)|$, we obtain $C > B$.
Case (c). For $\sin\psi < 0$ and $m\delta R\sin\psi > -|x(r)|$, we have $C < -B$.

For cases (a) and (c), we get

$$\begin{aligned} Arg\Phi_0 &\leq \frac{-\beta\delta\pi}{2} + \alpha \arctan(-B) \\ &\leq \frac{-\beta\delta\pi}{2} - \alpha \arctan \eta \\ &= -\frac{\gamma_k \pi}{2}. \end{aligned}$$

In case (b), we have

$$\begin{aligned} Arg\Phi_0 &> \frac{-\beta\delta\pi}{2} + \alpha \arctan B \\ &\geq \frac{-\beta\delta\pi}{2} + \alpha \arctan \eta \\ &= -(\beta\delta\pi - \frac{\gamma_k \pi}{2}) \\ &\geq -(2\pi - \frac{\gamma_k \pi}{2}), \text{ since } \beta\delta \leq 2. \end{aligned}$$

Combining the above six cases, three for $r > 0$ and three for $r < 0$, we have

$$\frac{\gamma_k \pi}{2} \leq \left| Arg \left((p(z_0))^\beta \left[1 + \lambda(z_0) \frac{z_0 p'(z_0)}{p^k(z_0)} \right]^\alpha \right) \right| \leq 2\pi - \frac{\gamma_k \pi}{2},$$

which is a contradiction to (1.8). Hence $p(z) \prec \left(\frac{1+z}{1-z}\right)^\delta$. This completes the proof. □

3. SPECIAL CASES

(i) Taking $\alpha = \beta = 1$ in Theorem 1.1, we get the result of S. Ponnusamy [9, p. 399, Lemma 1].

(ii) Setting $\alpha = \beta = 1$ and $\lambda(z) = \lambda$, a positive real number, in Theorem 1.1, we get Theorem 5 of Miller and Mocanu [6, p. 532].

(iii) Putting $\beta = 1$ and $\lambda(z) = 1$ in Theorem 1.1, we get the result of A. Lecko et.al. [4, p. 198, Theorem 2.1].

(iv) In Theorem 1.1, taking $\alpha = 1$, we get the following result:

Let $\beta \geq 0$ and let δ_0 be the solution of the equation

$$\beta\delta\pi = \frac{3\pi}{2} - \arctan\ \eta,$$

for a suitable fixed $\eta > 0$ such that $\lambda(z) : E \to C$ is a function satisfying

$$\frac{\delta \operatorname{Re} \lambda(z)}{1 + \delta |\operatorname{Im} \lambda(z)|} \geq \eta,\ z \in E.$$

If $p \in \mathcal{A}'$ satisfies the non-autonomous differential subordination

$$p^\beta(z) + \lambda(z) p^{\beta-1}(z) z p'(z) \prec \left(\frac{1+z}{1-z} \right)^\gamma,\ z \in E,$$

then,

$$p(z) \prec \left(\frac{1+z}{1-z} \right)^\delta \text{ in } E,$$

where,

$$\gamma = \beta\delta + \frac{2}{\pi} \arctan \eta,\ 0 < \delta \leq \delta_0.$$

(v) In Theorem 1.2, taking $\lambda(z) = 1$ and $\beta = 1$, we get the result of A. Lecko [3, p. 344, Theorem 2.1].

(vi) Putting $\alpha = \beta = 1$ and $k = 2$ in Theorem 1.2, we obtain the following result:

For $0 < \delta < 1$ and $\eta > 0$, let $\lambda(z) : E \to C$ be a function satisfying

$$\frac{\delta|\lambda(z)|\cos\psi}{x + \delta|\lambda(z)||\sin\psi|} \geq \eta,\ z \in E,$$

where $|\psi - Arg\lambda(z)| = \frac{\delta\pi}{2}$ and $x = (1+\delta)^{\frac{1+\delta}{2}}(1-\delta)^{\frac{1-\delta}{2}}$.

If $p \in \mathcal{A}'$ satisfies,

$$p(z) + \lambda(z)\frac{zp'(z)}{p(z)} \prec \left(\frac{1+z}{1-z}\right)^{\gamma_2}$$

in E, then,

$$p(z) \prec \left(\frac{1+z}{1-z}\right)^{\delta}$$

in E, where $\gamma_2 = \delta + \frac{2}{\pi}\arctan\eta$.

Taking $\lambda(z) = \lambda$, a positive real number in the above result, we get the well-known differential subordination (see [7, p. 268, 5.1-40]).

(vii) Taking $\beta + \alpha$ in place of β, $\lambda(z) = 1, k = 2$ and $p(z) = \frac{zf'(z)}{f(z)}$, where $f \in \mathcal{A}$, in Theorem 1.2, we get the following (Also see Darus and Thomas [2, p. 1050, Theorem 1]):

Let $\alpha \in [0,1], \delta \in (0,1)$ and $\beta \geq 0$ be such that $0 \leq (\beta+\alpha)\delta \leq 2$. If $f \in \mathcal{A}$ satisfies

$$\left(\frac{zf'(z)}{f(z)}\right)^{\beta}\left[1 + \frac{zf''(z)}{f'(z)}\right]^{\alpha} \prec \left(\frac{1+z}{1-z}\right)^{\gamma_2},$$

then,

$$\frac{zf'(z)}{f(z)} \prec \left(\frac{1+z}{1-z}\right)^{\delta} \text{ in } E,$$

where,

$$\gamma_2 = \beta\delta + \frac{2\alpha}{\pi}\arctan\left[\tan\frac{\delta\pi}{2} + \frac{\delta}{(1+\delta)^{\frac{1+\delta}{2}}(1-\delta)^{\frac{1-\delta}{2}}\cos\frac{\delta\pi}{2}}\right].$$

(viii) Writing $\beta = 1$, $\alpha = 1$, $\lambda(z) = 1$, $k = 2$ and $p(z) = zf'(z)/f(z)$, where $f \in \mathcal{A}$, in Theorem 1.2, we get well-known result of Nunokawa and Thomas [8, p. 364, Theorem1].

4. APPLICATIONS TO UNIVALENT FUNCTIONS

In this section, we give some applications of our results to univalent functions and obtain some new conditions for univalence, starlikeness and strongly starlikeness.

A function $f \in \mathcal{A}$ is said to be strongly starlike of order $\alpha, 0 < \alpha \leq 1$, if

$$\left|\arg\frac{zf'(z)}{f(z)}\right| < \frac{\alpha\pi}{2} \text{ in } E.$$

We denote the set of all such functions by $S^*(\alpha)$. The class $S^*(\alpha)$ was introduced and studied independently by Brannan and Kirwan [1] and Stankiewicz [10]. Note that $S^*(1)$ is the usual class of starlike functions f in $\mathcal{A}$ satisfying

$$\text{Re}\left[\frac{zf'(z)}{f(z)}\right] > 0,\ z \in E.$$

We denote this class by St.

First of all, we note that if $f \in \mathcal{A}$, then the functionals $\frac{f(z)}{z}$, $f'(z)$ and $\frac{zf'(z)}{f(z)}$ are all members of the class $\mathcal{A}'$.

Theorem 4.1. *Let* $\alpha \in [0,1]$ *and let* $\delta_0 = 0.6165\ldots$ *be the unique root of the equation*

$$2\arctan(1-\delta) + \pi(1-2\delta) = 0. \tag{4.1}$$

Further, let $\beta \geq 0$ *be such that* $\beta\delta \leq \beta\delta_0 \leq 2$ *for* $0 < \delta \leq \delta_0$*. If a function* $f \in \mathcal{A}$, $f'(z) \neq 0,\ z \in E$, *satisfies*

$$(f'(z))^\beta \left[1 + \frac{zf''(z)}{f'(z)}\right]^\alpha \prec \left(\frac{1+z}{1-z}\right)^\gamma,\ z \in E, \tag{4.2}$$

then $f \in St$ *where* $\gamma = \beta\delta + \frac{2\alpha}{\pi}\arctan\delta$.

Proof. In Theorem 1.1, taking $\lambda(z) = 1$ and $p(z) = f'(z)$, the subordination (4.2) implies

$$f'(z) \prec \left(\frac{1+z}{1-z}\right)^\delta,\ z \in E, \tag{4.3}$$

where $\gamma = \beta\delta + \frac{2\alpha}{\pi}\arctan\delta$. Again using Theorem 1.1 with $\alpha = \beta = \lambda(z) = 1$ and $p(z) = \frac{f(z)}{z}$, we get from (4.3),

$$\frac{f(z)}{z} \prec \left(\frac{1+z}{1-z}\right)^\mu,\ z \in E,$$

where $\delta = \mu + \frac{2}{\pi}\arctan\mu$ which is equivalent to (4.1) with $\mu = 1 - \delta$. Now

$$\begin{aligned}\left|\arg\frac{zf'(z)}{f(z)}\right| &\leq |\arg f'(z)| + \left|\arg\frac{f(z)}{z}\right| \\ &\leq (\delta+\mu)\frac{\pi}{2} = \frac{\pi}{2},\end{aligned}$$

and the desired result follows. □

Writing $p(z) = \frac{f(z)}{z}$ and $\lambda(z) = 1$ in Theorem 1.1, we get:

Lemma 4.2. *Let* $\alpha \in [0,1]$*be fixed and let* $\delta \in (0, \delta_0]$*, where* δ_0 *is the solution of the equation*

$$\beta\delta\pi = 2\pi - \alpha\left(\frac{\pi}{2} + \arctan\delta\right)$$

for fixed $\beta \geq 0$*. If* $f \in \mathcal{A}$, $\frac{f(z)}{z} \neq 0$, $z \in E$, *satisfies*

$$\left(\frac{f(z)}{z}\right)^{\beta-\alpha} (f'(z))^\alpha \prec \left(\frac{1+z}{1-z}\right)^\gamma \text{ in } E,$$

then,

$$\frac{f(z)}{z} \prec \left(\frac{1+z}{1-z}\right)^\delta,\ z \in E,$$

where $\gamma = \beta\delta + \frac{2\alpha}{\pi}\arctan\delta$.

Theorem 4.3. *Let* $\alpha \in [0,1]$, $\beta \geq 0$ *be fixed. Suppose that* $\gamma_0, \alpha/2 < \gamma_0 \leq \alpha$*, is the unique root of the equation*

$$\beta = (\alpha - \gamma)\cot\left[\left(\frac{2\gamma - \alpha}{\alpha}\right)\frac{\pi}{2}\right]. \tag{4.4}$$

Let $f \in \mathcal{A}$, $\frac{f(z)}{z} \neq 0,\ z \in E$, *satisfy*

$$\left(\frac{f(z)}{z}\right)^{\beta-\alpha} (f'(z))^{\alpha} \prec \left(\frac{1+z}{1-z}\right)^{\gamma} \quad \text{in } E. \tag{4.5}$$

Then $f \in St$.

Proof. In view of (4.5) and Lemma 4.2, we have

$$\frac{f(z)}{z} \prec \left(\frac{1+z}{1-z}\right)^{\delta},\ z \in E,$$

where δ is given by the equation

$$\gamma = \beta\delta + \frac{2\alpha}{\pi} \arctan \delta. \tag{4.6}$$

We observe that (4.6) is equivalent to (4.4) with $\delta\beta = \alpha - \gamma$. Now

$$\begin{aligned}\alpha \left|\arg \frac{zf'(z)}{f(z)}\right| &\leq \left|\arg (f'(z))^{\alpha} \left(\frac{f(z)}{z}\right)^{\beta-\alpha}\right| + \left|\arg \left(\frac{f(z)}{z}\right)^{\beta}\right| \\ &\leq \frac{\gamma\pi}{2} + \frac{\beta\delta\pi}{2} = \frac{\alpha\pi}{2},\end{aligned}$$

and the conclusion follows. □

Remark 4.4. Taking $\alpha = 1$ in Theorem 4.3, we get the Theorem 1 of Ponnusamy [9, p. 403].

Taking $\alpha = \beta = \frac{1}{2}$ in Theorem 4.3, we get

Example 4.1. For $f \in \mathcal{A}$, the differential subordination

$$\sqrt{f'(z)} \prec \left(\frac{1+z}{1-z}\right)^{\gamma} \quad \text{in } E,$$

implies that f is starlike in E where $\gamma = 0.3082\ldots$ is given by $2 \arctan (1-2\gamma)+\pi(1-4\gamma) = 0$.

Setting $p(z) = f'(z)$ and $\alpha = \lambda(z) = 1$ in Theorem 1.1, we obtain

Corollary 4.5. *For* $\beta \geq 0$, *if an analytic function* f *in* $\mathcal{A}$ *satisfies*

$$(f'(z))^{\beta} + (f'(z))^{\beta-1} z f''(z) \prec \left(\frac{1+z}{1-z}\right)^{\frac{2\beta+1}{2}} \quad \text{in } E,$$

then

$$f'(z) \prec \frac{1+z}{1-z} \ \text{ in } \ E,$$

and, hence, f *is univalent in* E.

Writing $p(z) = \frac{zf'(z)}{f(z)}$, $k = 2$ and $(\beta + \alpha)$ in place of β in Theorem 1.2, we get the following result:

Theorem 4.6. *Let* $\alpha \in [0, 1]$ *and* $\beta \geq 0$ *be fixed. Let* $\delta \in (0, 1)$ *be such that* $0 \leq (\beta + \alpha)\delta \leq 2$. *For a fixed* $\eta > 0$, *let* $\lambda(z) : E \to C$ *be a function satisfying*

$$\frac{\delta|\lambda(z)|\cos\psi}{x + \delta|\lambda(z)||\sin\psi|} \geq \eta,\ z \in E,$$

where $|\psi - \operatorname{Arg} \lambda(z)| = \frac{\delta\pi}{2}$ *and* $x = (1+\delta)^{\frac{1+\delta}{2}}(1-\delta)^{\frac{1-\delta}{2}}$.

If $f \in \mathcal{A}$ satisfies the differential subordination

$$\left(\frac{zf'(z)}{f(z)}\right)^{\beta}\left[(1-\lambda(z))\frac{zf'(z)}{f(z)}+\lambda(z)\left(1+\frac{zf''(z)}{f'(z)}\right)\right]^{\alpha} \prec \left(\frac{1+z}{1-z}\right)^{\gamma_2}$$

in E, then

$$\frac{zf'(z)}{f(z)} \prec \left(\frac{1+z}{1-z}\right)^{\delta}$$

in E i.e. $f \in S^*(\delta)$ in E, where $\gamma_2 = (\beta+\alpha)\delta + \frac{2\alpha}{\pi}\arctan\eta$.

Taking $\beta = 0,\ \alpha = 1$, $\lambda(z) = \lambda,\ \lambda$ real, in Theorem 4.6, we obtain the well-known result [7, p. 266, Cor. 5. 1i. 1].

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