

MUGUR ACU AND SHIGEYOSHI OWA

University "Lucian Blaga" of Sibiu
Department of Mathematics
Str. Dr. I. Rațiu, No. 5-7
550012 - Sibiu, Romania.

Department of Mathematics
School of Science and Engineering
Kinki University
Higashi-Osaka, Osaka 577-8502, Japan
EMail: owa@math.kindai.ac.jp



volume 6, issue 3, article 70,
2005.

*Received 17 February, 2005;
accepted 01 June, 2005.*

Communicated by: N.E. Cho

[Abstract](#)

[Contents](#)



[Home Page](#)

[Go Back](#)

[Close](#)

[Quit](#)

Abstract

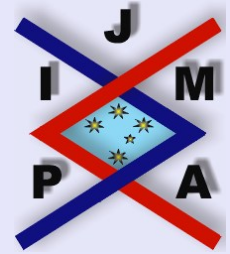
In 1999, S. Kanas and F. Ronning introduced the classes of functions starlike and convex, which are normalized with $f(w) = f'(w) - 1 = 0$ and w is a fixed point in \mathbb{U} . The aim of this paper is to continue the investigation of the univalent functions normalized with $f(w) = f'(w) - 1 = 0$, where w is a fixed point in \mathbb{U} .

2000 Mathematics Subject Classification: 30C45.

Key words: Close-to-convex functions, α -convex functions, Briot-Bouquet differential subordination.

Contents

1	Introduction	3
2	Preliminary Results	5
3	Main Results	7
	References	



On Some Subclasses of Univalent Functions

Mugur Acu and Shigeyoshi Owa

Title Page

Contents



Go Back

Close

Quit

Page 2 of 14

1. Introduction

Let $\mathcal{H}(\mathbb{U})$ be the set of functions which are regular in the unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$,

$$A = \{f \in \mathcal{H}(\mathbb{U}) : f(0) = f'(0) - 1 = 0\} \quad \text{and}$$

$$S = \{f \in A : f \text{ is univalent in } \mathbb{U}\}.$$

We recall here the definitions of the well-known classes of starlike, convex, close-to-convex and α -convex functions:

$$S^* = \left\{ f \in A : \operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) > 0, z \in \mathbb{U} \right\},$$

$$S^c = \left\{ f \in A : \operatorname{Re} \left(1 + \frac{zf''(z)}{f'(z)} \right) > 0, z \in \mathbb{U} \right\},$$

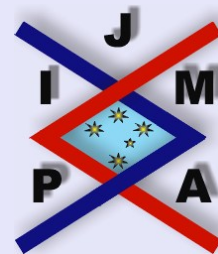
$$CC = \left\{ f \in A : \exists g \in S^*, \operatorname{Re} \left(\frac{zf'(z)}{g(z)} \right) > 0, z \in \mathbb{U} \right\},$$

$$M_\alpha = \left\{ f \in A : \frac{f(z)f'(z)}{z} \neq 0, \operatorname{Re} J(\alpha, f; z) > 0, z \in \mathbb{U} \right\},$$

where

$$J(\alpha, f; z) = (1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \left(1 + \frac{zf''(z)}{f'(z)} \right).$$

Let w be a fixed point in \mathbb{U} and $A(w) = \{f \in \mathcal{H}(\mathbb{U}) : f(w) = f'(w) - 1 = 0\}$.



On Some Subclasses of Univalent Functions

Mugur Acu and Shigeyoshi Owa

Title Page

Contents



Go Back

Close

Quit

Page 3 of 14

In [3], S. Kanas and F. Ronning introduced the following classes:

$$S(w) = \{f \in A(w) : f \text{ is univalent in } \mathbb{U}\}$$

$$ST(w) = S^*(w) = \left\{ f \in S(w) : \operatorname{Re} \left(\frac{(z-w)f'(z)}{f(z)} \right) > 0, z \in \mathbb{U} \right\}$$

$$CV(w) = S^c(w) = \left\{ f \in S(w) : 1 + \operatorname{Re} \left(\frac{(z-w)f''(z)}{f'(z)} \right) > 0, z \in \mathbb{U} \right\}.$$

The class $S^*(w)$ is defined by the geometric property that the image of any circular arc centered at w is starlike with respect to $f(w)$ and the corresponding class $S^c(w)$ is defined by the property that the image of any circular arc centered at w is convex. We observe that the definitions are somewhat similar to the ones for uniformly starlike and convex functions introduced by A. W. Goodman in [1] and [2], except that in this case the point w is fixed.

It is obvious that there exists a natural "Alexander relation" between the classes $S^*(w)$ and $S^c(w)$:

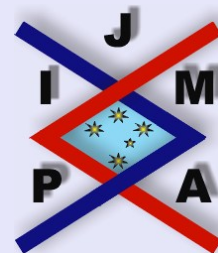
$$g \in S^c(w) \text{ if and only if } f(z) = (z-w)g'(z) \in S^*(w).$$

Let $\mathcal{P}(w)$ denote the class of all functions

$$p(z) = 1 + \sum_{n=1}^{\infty} B_n(z-w)^n$$

that are regular in U and satisfy $p(w) = 1$ and $\operatorname{Re} p(z) > 0$ for $z \in \mathbb{U}$.

The purpose of this note is to define the classes of close to convex and α -convex functions normalized with $f(w) = f'(w) - 1 = 0$, where w is a fixed point in \mathbb{U} , and to obtain some results concerning these classes.



On Some Subclasses of Univalent Functions

Mugur Acu and Shigeyoshi Owa

Title Page

Contents



Go Back

Close

Quit

Page 4 of 14

2. Preliminary Results

It is easy to see that a function $f \in A(w)$ has the series expansion:

$$f(z) = (z - w) + a_2(z - w)^2 + \dots$$

In [7], J.K. Wald gives the sharp bounds for the coefficients B_n of the function $p \in \mathcal{P}(w)$ as follows.

Theorem 2.1. *If $p \in \mathcal{P}(w)$,*

$$p(z) = 1 + \sum_{n=1}^{\infty} B_n(z - w)^n,$$

then

$$(2.1) \quad |B_n| \leq \frac{2}{(1+d)(1-d)^n},$$

where $d = |w|$ and $n \geq 1$.

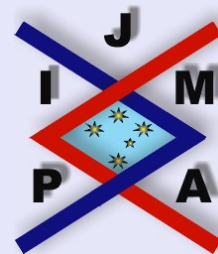
Using the above result, S. Kanas and F. Ronning [3] obtain the following:

Theorem 2.2. *Let $f \in S^*(w)$ and $f(z) = (z - w) + a_2(z - w)^2 + \dots$. Then*

$$(2.2) \quad |a_2| \leq \frac{2}{1-d^2}, \quad |a_3| \leq \frac{3+d}{(1-d^2)^2},$$

$$(2.3) \quad |a_4| \leq \frac{2(2+d)(3+d)}{3(1-d^2)^3}, \quad |a_5| \leq \frac{1(2+d)(3+d)(3d+5)}{6(1-d^2)^4}$$

where $d = |w|$.



On Some Subclasses of
Univalent Functions

Mugur Acu and Shigeyoshi Owa

Title Page

Contents



Go Back

Close

Quit

Page 5 of 14

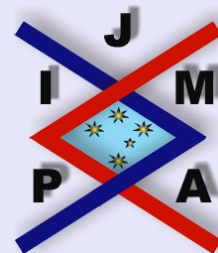
Remark 1. *It is clear that the above theorem also provides bounds for the coefficients of functions in $S^c(w)$, due to the relation between $S^c(w)$ and $S^*(w)$.*

The next theorem is the result of the so called "admissible functions method" introduced by P.T. Mocanu and S.S. Miller (see [4], [5], [6]).

Theorem 2.3. *Let h be convex in \mathbb{U} and $\operatorname{Re}[\beta h(z) + \gamma] > 0$, $z \in \mathbb{U}$. If $p \in \mathcal{H}(\mathbb{U})$ with $p(0) = h(0)$ and p satisfies the Briot-Bouquet differential subordination*

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z), \quad z \in \mathbb{U},$$

then $p(z) \prec h(z)$, $z \in \mathbb{U}$.



On Some Subclasses of
Univalent Functions

Mugur Acu and Shigeyoshi Owa

Title Page

Contents



Go Back

Close

Quit

Page 6 of 14

3. Main Results

Let us consider the integral operator $L_a : A(w) \rightarrow A(w)$ defined by

$$(3.1) \quad f(z) = L_a F(z) = \frac{1+a}{(z-w)^a} \int_w^z F(t)(t-w)^{a-1} dt, \quad a \in \mathbb{R}, a \geq 0.$$

We denote by

$$D(w) = \left\{ z \in \mathbb{U} : \operatorname{Re} \left(\frac{w}{z} \right) < 1 \quad \text{and} \quad \operatorname{Re} \left(\frac{z(1+z)}{(z-w)(1-z)} \right) > 0 \right\},$$

with $D(0) = \mathbb{U}$, and

$$s(w) = \{ f : D(w) \rightarrow \mathbb{C} \} \cap S(w),$$

where w is a fixed point in \mathbb{U} . Denoting $s^*(w) = S^*(w) \cap s(w)$, where w is a fixed point in \mathbb{U} , we obtain

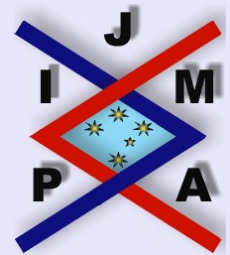
Theorem 3.1. *Let w be a fixed point in \mathbb{U} and $F(z) \in s^*(w)$. Then $f(z) = L_a F(z) \in S^*(w)$, where the integral operator L_a is defined by (3.1).*

Proof. By differentiating (3.1), we obtain

$$(3.2) \quad (1+a)F(z) = af(z) + (z-w)f'(z).$$

From (3.2), we also have

$$(3.3) \quad (1+a)F'(z) = (1+a)f'(z) + (z-w)f''(z).$$



On Some Subclasses of
Univalent Functions

Mugur Acu and Shigeyoshi Owa

Title Page

Contents



Go Back

Close

Quit

Page 7 of 14

Using (3.2) and (3.3), we obtain

$$(3.4) \quad \frac{(z-w)F'(z)}{F(z)} = \frac{(1+a)(z-w)\frac{f'(z)}{f(z)} + (z-w)^2\frac{f''(z)}{f(z)}}{a + (z-w)\frac{f'(z)}{f(z)}}.$$

Letting

$$p(z) = \frac{(z-w)f'(z)}{f(z)},$$

where $p \in \mathcal{H}(\mathbb{U})$ and $p(0) = 1$, we have

$$(z-w)p'(z) = p(z) + (z-w)^2 \cdot \frac{f''(z)}{f(z)} - [p(z)]^2$$

and thus

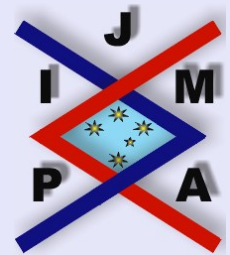
$$(3.5) \quad (z-w)^2 \frac{f''(z)}{f(z)} = (z-w)p'(z) - p(z)[1 - p(z)].$$

Using (3.4) and (3.5), we obtain

$$(3.6) \quad \frac{(z-w)F'(z)}{F(z)} = p(z) + \frac{(z-w)p'(z)}{a + p(z)}.$$

Since $F \in s^*(w)$, from (3.6), we have

$$p(z) + \frac{z-w}{a + p(z)}p'(z) \prec \frac{1+z}{1-z} \equiv h(z)$$



On Some Subclasses of Univalent Functions

Mugur Acu and Shigeyoshi Owa

Title Page

Contents



Go Back

Close

Quit

Page 8 of 14

or

$$p(z) + \frac{1 - \frac{w}{z}}{a + p(z)} zp'(z) \prec \frac{1 + z}{1 - z}.$$

From the hypothesis, we have

$$\operatorname{Re} \left(\frac{1}{1 - \frac{w}{z}} h(z) + \frac{a}{1 - \frac{w}{z}} \right) > 0$$

and thus from Theorem 2.3, we obtain

$$p(z) \prec \frac{1 + z}{1 - z}, \quad z \in \mathbb{U}$$

or

$$\operatorname{Re} \left(\frac{(z - w)f'(z)}{f(z)} \right) > 0, \quad z \in \mathbb{U}.$$

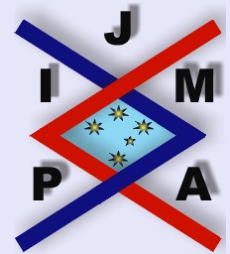
This means that $f \in S^*(w)$. □

Definition 3.1. Let $f \in S(w)$ where w is a fixed point in \mathbb{U} . We say that f is w -close-to-convex if there exists a function $g \in S^*(w)$ such that

$$\operatorname{Re} \left(\frac{(z - w)f'(z)}{g(z)} \right) > 0, \quad z \in \mathbb{U}.$$

We denote this class by $CC(w)$.

Remark 2. If we consider $f = g$, $g \in S^*(w)$, then we have $S^*(w) \subset CC(w)$. If we take $w = 0$, then we obtain the well-known close-to-convex functions.



On Some Subclasses of
Univalent Functions

Mugur Acu and Shigeyoshi Owa

Title Page

Contents



Go Back

Close

Quit

Page 9 of 14

Theorem 3.2. Let w be a fixed point in \mathbb{U} and $f \in CC(w)$, where

$$f(z) = (z - w) + \sum_{n=2}^{\infty} b_n(z - w)^n,$$

with respect to the function $g \in S^*(w)$, where

$$g(z) = (z - w) + \sum_{n=2}^{\infty} a_n(z - w)^n.$$

Then

$$|b_n| \leq \frac{1}{n} \left[|a_n| + \sum_{k=1}^{n-1} |a_k| \cdot \frac{2}{(1+d)(1-d)^{n-k}} \right],$$

where $d = |w|$, $n \geq 2$ and $a_1 = 1$.

Proof. Let $f \in CC(w)$ with respect to the function $g \in S^*(w)$. Then there exists a function $p \in \mathcal{P}(w)$ such that

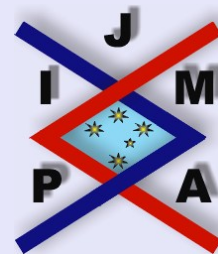
$$\frac{(z - w)f'(z)}{g(z)} = p(z),$$

where

$$p(z) = 1 + \sum_{n=1}^{\infty} B_n(z - w)^n.$$

Using the hypothesis through identification of $(z - w)^n$ coefficients, we obtain

$$(3.7) \quad nb_n = a_n + \sum_{k=1}^{n-1} a_k B_{n-k},$$



On Some Subclasses of
Univalent Functions

Mugur Acu and Shigeyoshi Owa

Title Page

Contents



Go Back

Close

Quit

Page 10 of 14

where $a_1 = 1$ and $n \geq 2$. From (3.7), we have

$$|b_n| \leq \frac{1}{n} \left[|a_n| + \sum_{k=1}^{n-1} |a_k| \cdot |B_{n-k}| \right], \quad a_1 = 1, \quad n \geq 2.$$

Applying the above and the estimates (2.1), we obtain the result. \square

Remark 3. If we use the estimates (2.2), we obtain the same estimates for the coefficients b_n , $n = 2, 3, 4, 5$.

Definition 3.2. Let $\alpha \in \mathbb{R}$ and w be a fixed point in \mathbb{U} . For $f \in S(w)$, we define

$$J(\alpha, f, w; z) = (1 - \alpha) \frac{(z - w)f'(z)}{f(z)} + \alpha \left[1 + \frac{(z - w)f''(z)}{f'(z)} \right].$$

We say that f is $w - \alpha$ -convex function if

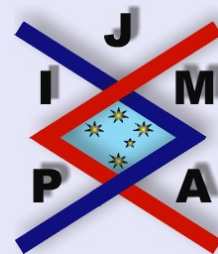
$$\frac{f(z)f'(z)}{z - w} \neq 0, \quad z \in \mathbb{U}$$

and $\operatorname{Re} J(\alpha, f, w; z) > 0$, $z \in \mathbb{U}$. We denote this class by $M_\alpha(w)$.

Remark 4. It is easy to observe that $M_\alpha(0)$ is the well-known class of α -convex functions.

Theorem 3.3. Let w be a fixed point in \mathbb{U} , $\alpha \in \mathbb{R}$, $\alpha \geq 0$ and $m_\alpha(w) = M_\alpha(w) \cap s(w)$. Then we have

1. If $f \in m_\alpha(w)$ then $f \in S^*(w)$. This means $m_\alpha(w) \subset S^*(w)$.



On Some Subclasses of
Univalent Functions

Mugur Acu and Shigeyoshi Owa

Title Page

Contents



Go Back

Close

Quit

Page 11 of 14

2. If $\alpha, \beta \in \mathbb{R}$, with $0 \leq \beta/\alpha < 1$, then $m_\alpha(w) \subset m_\beta(w)$.

Proof. From $f \in m_\alpha(w)$, we have $\operatorname{Re} J(\alpha, f, w; z) > 0$, $z \in \mathbb{U}$. Putting

$$p(z) = \frac{(z-w)f'(z)}{f(z)},$$

with $p \in \mathcal{H}(\mathbb{U})$ and $p(0) = 1$, we obtain

$$\operatorname{Re} J(\alpha, f, w; z) = \operatorname{Re} \left[p(z) + \alpha \frac{(z-w)p'(z)}{p(z)} \right] > 0, \quad z \in \mathbb{U}$$

or

$$p(z) + \frac{\alpha \left(1 - \frac{w}{z}\right)}{p(z)} zp'(z) \prec \frac{1+z}{1-z} \equiv h(z).$$

In particular, for $\alpha = 0$, we have

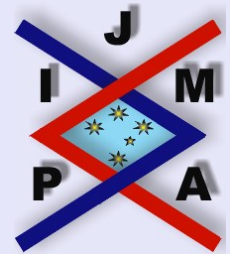
$$p(z) \prec \frac{1+z}{1-z}, \quad z \in \mathbb{U}.$$

Using the hypothesis, we have for $\alpha > 0$,

$$\operatorname{Re} \left(\frac{1}{\alpha \left(1 - \frac{w}{z}\right)} h(z) \right) > 0, \quad z \in \mathbb{U}$$

and from Theorem 2.3, we obtain

$$p(z) \prec \frac{1+z}{1-z}, \quad z \in \mathbb{U}.$$



On Some Subclasses of
Univalent Functions

Mugur Acu and Shigeyoshi Owa

Title Page

Contents



Go Back

Close

Quit

Page 12 of 14

This means that

$$\operatorname{Re} \left(\frac{(z-w)f'(z)}{f(z)} \right) > 0, \quad z \in \mathbb{U}$$

for $\alpha \geq 0$ or $f \in S^*(w)$.

If we denote by $A = \operatorname{Re} p(z)$ and $B = \operatorname{Re} ((z-w)p'(z)/p(z))$, then we have $A > 0$ and $A + B\alpha > 0$, where $\alpha \geq 0$. Using the geometric interpretation of the equation $y(x) = A + Bx$, $x \in [0, \alpha]$, we obtain

$$y(\beta) = A + B\beta > 0 \quad \text{for every } \beta \in [0, \alpha].$$

This means that

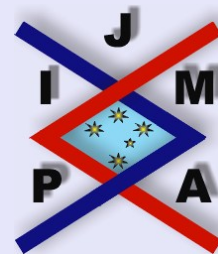
$$\operatorname{Re} \left(p(z) + \beta \frac{(z-w)p'(z)}{p(z)} \right) > 0, \quad z \in \mathbb{U}$$

or $f \in m_\beta(w)$. □

Remark 5. From Theorem 3.3, we have

$$m_1(w) \subseteq s^c(w) \subseteq m_\alpha(w) \subseteq s^*(w),$$

where $0 \leq \alpha \leq 1$ and $s^c(w) = S^c(w) \cap s(w)$.



On Some Subclasses of Univalent Functions

Mugur Acu and Shigeyoshi Owa

Title Page

Contents



Go Back

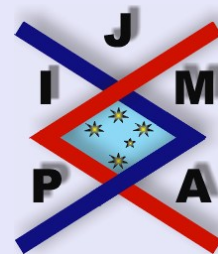
Close

Quit

Page 13 of 14

References

- [1] A.W. GOODMAN, On Uniformly Starlike Functions, *J. Math. Anal. Appl.*, **155** (1991), 364–370.
- [2] A.W. GOODMAN, On uniformly convex functions, *Ann. Polon. Math.*, **56** (1991), 87–92.
- [3] S. KANAS AND F. RONNING, Uniformly starlike and convex functions and other related classes of univalent functions, *Ann. Univ. Mariae Curie - Sklodowska Section A*, **53** (1999), 95–105.
- [4] S.S. MILLER AND P.T. MOCANU, Differential subordinations and univalent functions, *Michigan Math. J.*, **28** (1981), 157–171.
- [5] S.S. MILLER AND P.T. MOCANU, Univalent solutions of Briot-Bouquet differential equations, *J. Diff. Eqns.*, **56** (1985), 297–309.
- [6] S.S. MILLER AND P.T. MOCANU, On some classes of first-order differential subordinations, *Michigan Math. J.*, **32** (1985), 185–195.
- [7] J.K. WALD, *On Starlike Functions*, Ph. D. thesis, University of Delaware, Newark, Delaware (1978).



On Some Subclasses of
Univalent Functions

Mugur Acu and Shigeyoshi Owa

Title Page

Contents



Go Back

Close

Quit

Page 14 of 14