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APPROXIMATION OF FIXED POINTS OF ASYMPTOTICALLY DEMICONTRACTIVE MAPPINGS IN ARBITRARY BANACH SPACES

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ABSTRACT. Let E be a real Banach Space and K a nonempty closed convex (not necessarily bounded) subset of E. Iterative methods for the approximation of fixed points of asymptotically demicontractive mappings $T : K \to K$ are constructed using the more general modified Mann and Ishikawa iteration methods with errors.

Our results show that a recent result of Osilike [3] (which is itself a generalization of a theorem of Qihou [4]) can be extended from real q-uniformly smooth Banach spaces, $1 < q < \infty$, to arbitrary real Banach spaces, and to the more general Modified Mann and Ishikawa iteration methods with errors. Furthermore, the boundedness assumption imposed on the subset K in ([3, 4]) are removed in our present more general result. Moreover, our iteration parameters are independent of any geometric properties of the underlying Banach space.

Key words and phrases: Asymptotically Demicontractive Maps, Fixed Points, Modified Mann and Ishikawa Iteration Methods with Errors.

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1. INTRODUCTION

Let E be an arbitrary real Banach space and let J denote the normalized duality mapping from E into 2^{E^*} given by $J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\}$, where E^* denotes the dual space of E and $\langle\ ,\ \rangle$ denotes the generalized duality pairing. If E^* is strictly convex, then J is single-valued. In the sequel, we shall denote the single-valued duality mapping by j.

Let K be a nonempty subset of E. A mapping $T : K \to K$ is called k-*strictly asymptotically pseudocontractive* mapping, with sequence $\{k_n\} \subseteq [1, \infty)$, $\lim_n k_n = 1$ (see for example [3, 4]), if for all $x, y \in K$ there exists $j(x - y) \in J(x - y)$ and a constant $k \in [0, 1)$ such that

$$\langle (I - T^n)x - (I - T^n)y, j(x - y)\rangle \geq \frac{1}{2}(1 - k)\,\|(I - T^n)x - (I - T^n)y\|^2 - \frac{1}{2}(k_n^2 - 1)\|x - y\|^2, \tag{1.1}$$

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for all $n \in \mathbb{N}$. T is called an *asymptotically demicontractive* mapping with sequence $k_n \subseteq [0,\infty)$, $\lim_n k_n = 1$ (see for example [3, 4]) if $F(T) = \{x \in K : Tx = x\} \neq \emptyset$ and for all $x \in K$ and $x^* \in F(T)$, there exists $k \in [0,1)$ and $j(x-x^*) \in J(x-x^*)$ such that

$$\langle x - T^n x, j(x-x^*)\rangle \geq \frac{1}{2}(1-k)\|x-T^n x\|^2 - \frac{1}{2}(k_n^2-1)\|x-x^*\|^2 \tag{1.2}$$

for all $n \in \mathbb{N}$. Furthermore, T is *uniformly L-Lipschitzian*, if there exists a constant $L > 0$, such that

$$\|T^n x - T^n y\| \leq L\|x-y\|, \tag{1.3}$$

for all x , $y \in K$ and $n \in \mathbb{N}$.

It is clear that a k-strictly asymptotically pseudocontrative mapping with a nonempty fixed point set $F(T)$ is asymptotically demicontrative. The classes of k-strictly asymptotically pseudocontractive and asymptotically demicontractive maps were first introduced in Hilbert spaces by Qihou [4]. In a Hilbert space, j is the identity and it is shown in [3] that (1.1) and (1.2) are respectively equivalent to the inequalities:

$$\|T^n x - T^n y\| \leq k_n^2\|x-y\|^2 + k\|(I-T^n)x - (I-T^n)y\|^2 \tag{1.4}$$

and

$$\|T^n x - T^n y\|^2 \leq k_n^2\|x-y\|^2 + \|x-T^n x\|^2 \tag{1.5}$$

which are the inequalities considered by Qihou [4].

In [4], Qihou using the *modified Mann* iteration method introduced by Schu [5], proved convergence theorem for the iterative approximation of fixed points of k-strictly asymptotically pseudocontractive mappings and asymptotically demicontractive mappings in Hilbert spaces. Recently, Osilike [3], extended the theorems of Qihou [4] concerning the iterative approximation of fixed points of k- strictly asymptotically demicontractive mappings from Hilbert spaces to much more general real q-uniformly smooth Banach spaces, $1 < q < \infty$, and to the much more general *modified Ishikawa iteration method*. More precisely, he proved the following:

Theorem 1.1. *(Osilike* [3, p. 1296]*): Let $q > 1$ and let E be a real q-uniformly smooth Banach space. Let K be a closed convex and bounded subset of E and $T : K \to K$ a completely continuous uniformly L-Lipschitzian asymptotically demicontractive mapping with a sequence $k_n \subseteq [1,\infty)$ satisfying $\sum_{n=1}^{\infty}(k_n^2-1) < \infty$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences satisfying the conditions.*

(i) $0 \leq \alpha_n, \beta_n \leq 1$, $n \geq 1$;
(ii) $0 < \in \leq c_q \alpha_n^{q-1}(1+L\beta_n)^q \leq \frac{1}{2}\{q(1-k)(1+L)^{-(q-2)}\} - \in$, *for all* $n \geq 1$ *and for some* $\in > 0$; *and*
(iii) $\sum_{n=0}^{\infty}\beta_n < \infty$.

Then the sequence $\{x_n\}$ generated from an arbitrary $x_1 \in K$ by

$$y_n = (1-\beta_n)x_n + \beta_n T^n x_n,\ n \geq 1,$$
$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n y_n,\ n \geq 1$$

converges strongly to a fixed point of T.

In Theorem 1.1, c_q is a constant appearing in an inequality which characterizes q-uniformly smooth Banach spaces. In Hilbert spaces, $q = 2$, $c_q = 1$ and with $\beta_n = 0\ \forall n$, Theorems 1 and 2 of Qihou [4] follow from Theorem 1.1 (see Remark 2 of [3]).

It is our purpose in this paper to extend Theorem 1.1 from real q-uniformly smooth Banach spaces to arbitrary real Banach spaces using the more general modified Ishikawa iteration method with errors in the sense of Xu [7] given by

$$y_n = a_n x_n + b_n T^n x_n + c_n u_n,\ n \geq 1, \\ x_{n+1} = a'_n x_n + b'_n T^n y_n + c'_n v_n,\ n \geq 1, \tag{1.6}$$

where $\{a_n\},\ \{b_n\},\ \{c_n\},\ \{a'_n\},\ \{b'_n\},\ \{c'_n\}$ are real sequences in $[0,1]$. $a_n + b_n + c_n = 1 = a'_n + b'_n + c'_n$, $\{u_n\}$ and $\{v_n\}$ are bounded sequences in K. If we set $b_n = c_n = 0$ in (1.6) we obtain the *modified Mann iteration method with errors* in the sense of Xu [7] given by

$$x_{n+1} = a'_n x_n + b'_n T^n x_n + c'_n v_n,\ n \geq 1. \tag{1.7}$$

2. MAIN RESULTS

In the sequel, we shall need the following:

Lemma 2.1. *Let E be a normed space, and K a nonempty convex subset of E. Let $T : K \to K$ be uniformly L-Lipschitzian mapping and let $\{a_n\},\ \{b_n\},\ \{c_n\},\ \{a'_n\},\ \{b'_n\}$ and $\{c'_n\}$ be sequences in $[0,1]$ with $a_n + b_n + c_n = a'_n + b'_n + c'_n = 1$. Let $\{u_n\}$, $\{v_n\}$ be bounded sequences in K. For arbitrary $x_1 \in K$, generate the sequence $\{x_n\}$ by*

$$y_n = a_n x_n + b_n T^n x_n + c_n u_n,\ n \geq 1 \\ x_{n+1} = a'_n x_n + b'_n T^n y_n + c'_n v_n,\ n \geq 0.$$

Then

$$\begin{aligned} \|x_n - Tx_n\| \leq \|x_n - T^n x_n\| &+ L(1+L)^2 \|x_{n+1} - T^{n-1} x_{n-1}\| \\ &+ L(1+L)c'_{n-1}\|v_{n-1} - x_{n-1}\| + L^2(1+L)c_{n-1}\|u_{n-1} - x_n\| \\ &+ Lc'_{n-1}\|x_{n-1} - T^{n-1}x_{n-1}\|. \end{aligned} \tag{2.1}$$

Proof. Set $\lambda_n = \|x_n - T^n x_n\|$. Then

$$\begin{aligned} \|x_n - Tx_n\| &\leq \|x_n - T^n x_n\| + L\|T^{n-1}x_n - x_n\| \\ &\leq \lambda_n + L^2\|x_n - x_{n-1}\| + L\|T^{n-1}x_{n-1} - x_n\| \\ &= \lambda_n + L^2\|a'_{n-1}x_n + b'_{n-1}T^{n-1}y_{n-1} + c'_{n-1}v_{n-1} - x_{n-1}\| \\ &\quad + L\|a'_{n-1}x_{n-1} + b'_{n-1}T^{n-1}y_{n-1} + c'_{n-1}v_{n-1} - T^{n-1}x_{n-1}\| \\ &= \lambda_n + L^2\|b'_{n-1}(T^{n-1}y_{n-1} - x_{n-1}) + c'_{n-1}(v_{n-1} - x_{n-1})\| \\ &\quad + L\|a'_{n-1}(x_{n-1} - T^{n-1}x_{n-1}) + b'_{n-1}(T^{n-1}y_{n-1} - T^{n-1}x_{n-1}) \\ &\quad + c'_{n-1}(v_{n-1} - T^{n-1}x_{n-1})\| \\ &\leq \lambda_n + L^2\|T^{n-1}y_{n-1} - x_{n-1}\| + L^2 c'_{n-1}\|v_{n-1} - x_{n-1}\| \\ &\quad + L\|x_{n-1} - T^{n-1}x_{n-1}\| + L^2\|y_{n-1} - x_{n-1}\| + Lc'_{n-1}\|v_{n-1} - x_{n-1}\| \\ &\quad + Lc'_{n-1}\|x_{n-1} - T^{n-1}x_{n-1}\| \\ &= \lambda_n + L\lambda_{n-1} + L(1+L)c'_{n-1}\|v_{n-1} - x_{n-1}\| + L^2\|T^{n-1}y_{n-1} - x_{n-1}\| \\ &\quad + L^2\|y_{n-1} - x_{n-1}\| + Lc'_{n-1}\|x_{n-1} - T^{n-1}x_{n-1})\| \end{aligned}$$

$$\begin{aligned}
&\le \lambda_n + 2L\lambda_{n-1} + L(1+L)c'_{n-1}\|v_{n-1} - x_{n-1}\| + L^2(1+L)\|y_{n-1} - x_{n-1}\| \\
&\quad + L^2\|T^{n-1}x_{n-1} - x_{n-1}\| + Lc'_{n-1}\|x_{n-1} - T^{n-1}x_{n-1})\| \\
&= \lambda_n + L(1+L)\lambda_{n-1} + L(1+L)c'_{n-1}\|v_{n-1} - x_{n-1}\| \\
&\quad + L^2(1+L)\|b_{n-1}(T^{n-1}x_{n-1} - x_{n-1}) + c_{n-1}(u_{n-1} - x_{n-1})\| \\
&\quad + Lc'_{n-1}\|x_{n-1} - T^{n-1}x_{n-1}\| \\
&\le \lambda_n + L(L^2 + 2L + 1)\lambda_{n-1} + L(1+L)c'_{n-1}\|v_{n-1} - x_{n-1}\| \\
&\quad + L^2(1+L)c_{n-1}\|u_{n-1} - x_{n-1}\| + Lc'_{n-1}\|x_{n-1} - T^{n-1}x_{n-1}\|,
\end{aligned}$$

completing the proof of Lemma 1. □

Lemma 2.2. *Let* $\{a_n\},\ \{b_n\}$ *and* $\{\delta_n\}$ *be sequences of nonnegative real numbers satisfying*

(2.2) $$a_{n+1} \le (1+\delta_n)a_n + b_n,\ n \ge 1.$$

If $\sum_{n=1}^{\infty}\delta_n < \infty$ *and* $\sum_{n=1}^{\infty} b_n < \infty$ *then* $\lim_{n\to\infty} a_n$ *exists. If in addition* $\{a_n\}$ *has a subsequence which converges strongly to zero then* $\lim_{n\to\infty} a_n = 0$.

Proof. Observe that

$$\begin{aligned}
a_{n+1} &\le (1+\delta_n)a_n + b_n \\
&\le (1+\delta_n)[(1+\delta_{n-1})a_{n-1} + b_{n-1}] + b_n \\
&\le \dots \le \prod_{j=1}^{n}(1+\delta_j)a_1 + \prod_{j=1}^{n}(1+\delta_j)\sum_{j=1}^{n} b_j \\
&\le \dots \le \prod_{j=1}^{\infty}(1+\delta_j)a_1 + \prod_{j=1}^{\infty}(1+\delta_j)\sum_{j=1}^{\infty} b_j < \infty.
\end{aligned}$$

Hence $\{a_n\}$ is bounded. Let $M > 0$ be such that $a_n \le M,\ n \ge 1$. Then

$$a_{n+1} \le (1+\delta_n)a_n + b_n \le a_n + M\delta_n + b_n = a_n + \sigma_n$$

where $\sigma_n = M\delta_n + b_n$. It now follows from Lemma 2.1 of ([6, p. 303]) that $\lim_n a_n$ exists. Consequently, if $\{a_n\}$ has a subsequence which converges strongly to zero then $\lim_n a_n = 0$ completing the proof of Lemma 2.2. □

Lemma 2.3. *Let* E *be a real Banach space and* K *a nonempty convex subset of* E. *Let* $T : K \to K$ *be uniformly* L*-Lipschitzian asymptotically demicontractive mapping with a sequence* $\{k_n\} \subseteq [1,\infty)$, *such that* $\lim_n k_n = 1$, *and* $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$. *Let* $\{a_n\},\ \{b_n\},\ \{c_n\}$, $\{a'_n\}, \{b'_n\}, \{c'_n\}$ *be real sequences in* $[0,1]$ *satisfying:*

(i) $a_n + b_n + c_n = 1 = a'_n + b'_n + c'_n$,
(ii) $\sum_{n=1}^{\infty} b'_n = \infty$,
(iii) $\sum_{n=1}^{\infty}(b'_n)^2 < \infty$, $\sum_{n=1}^{\infty} c'_n < \infty$, $\sum_{n=1}^{\infty} b_n < \infty$, *and* $\sum_{n=1}^{\infty} c_n < \infty$.

Let $\{u_n\}$ *and* $\{v_n\}$ *be bounded sequences in* K *and let* $\{x_n\}$ *be the sequence generated from an arbitrary* $x_1 \in K$ *by*

$$\begin{aligned}
y_n &= a_n x_n + b_n T^n x_n + c_n u_n,\ n \ge 1, \\
x_{n+1} &= a'_n + b'_n T^n y_n + c'_n v_n,\ n \ge 1,
\end{aligned}$$

then $\liminf_n \|x_n - Tx_n\| = 0$.

Proof. It is now well-known (see e.g. [1]) that for all $x, y \in E$, there exists $j(x+y) \in J(x+y)$ such that

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle. \tag{2.3}$$

Let $x^* \in F(T)$ and let $M > 0$ be such that $\|u_n - x^*\| \leq M,\ \|v_n - x^*\| \leq M,\ n \geq 1$. Using (1.6) and (2.3) we obtain

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|(1 - b_n' - c_n')x_n + b_n' T^n y_n + c_n' v_n - x^*\|^2 \\
&= \|(x_n - x^*) + b_n'(T^n y_n - x_n) + c_n'(v_n - x_n)\|^2 \\
&\leq \|(x_n - x^*)\|^2 + 2\langle b_n'(T^n y_n - x_n) + c_n'(v_n - x_n), j(x_{n+1} - x^*)\rangle \\
&= \|(x_n - x^*)\|^2 - 2b_n'\langle x_{n+1} - T^n x_{n+1}, j(x_{n+1} - x^*)\rangle + \\
&\quad + 2b_n'\langle x_{n+1} - T^n x_{n+1}, j(x_{n+1} - x^*)\rangle + 2b_n'\langle T^n y_n - x_n, j(x_{n+1} - x^*)\rangle \\
&\quad + 2c_n'\langle v_n - x_n, j(x_{n+1} - x^*)\rangle \\
&= \|(x_n - x^*)\|^2 - 2b_n'\langle x_{n+1} - T^n x_{n+1}, j(x_{n+1} - x^*)\rangle \\
&\quad + 2b_n'\langle x_{n+1} - x_n, j(x_{n+1} - x^*)\rangle + 2b_n'\langle T^n y_n - T^n x_{n+1}, j(x_{n+1} - x^*)\rangle \\
&\quad + 2c_n'\langle v_n - x_n, j(x_{n+1} - x^*)\rangle.
\end{aligned} \tag{2.4}$$

Observe that

$$x_{n+1} - x_n = b_n'(T^n y_n - x_n) + c_n'(v_n - x_n).$$

Using this and (1.2) in (2.4) we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leq \|(x_n - x^*)\|^2 - b_n'(1-k)\|x_{n+1} - T^n x_{n+1}\|^2 \\
&\quad + b_n'(k_n^2 - 1)\|x_{n+1} - x^*\|^2 + 2(b_n')^2\langle T^n y_n - x_n, j(x_{n+1} - x^*)\rangle \\
&\quad + 2b_n'\langle T^n y_n - T^n x_{n+1}, j(x_{n+1} - x^*)\rangle + 3c_n'\langle v_n - x_n, j(x_{n+1} - x^*)\rangle \\
&\leq \|(x_n - x^*)\|^2 - b_n'(1-k)\|x_{n+1} - T^n x_{n+1}\|^2 + (k_n^2 - 1)\|x_{n+1} - x^*\|^2 \\
&\quad + 2(b_n')^2\|T^n y_n - x_n\|\|x_{n+1} - x^*\| + 2b_n' L\|x_{n+1} - y_n\|\|x_{n+1} - x^*\| \\
&\quad + 3c_n'\|v_n - x_n\|\|x_{n+1} - x^*\| \\
&= \|(x_n - x^*)\|^2 - b_n'(1-k)\|x_{n+1} - T^n x_{n+1}\|^2 + (k_n^2 - 1)\|x_{n+1} - x^*\|^2 \\
&\quad + [2(b_n')^2\|T^n y_n - x_n\| + 2b_n' L\|x_{n+1} - y_n\| + 3c_n'\|v_n - x_n\|]\|x_{n+1} - x^*\|.
\end{aligned} \tag{2.5}$$

Observe that

$$\begin{aligned}
\|y_n - x^*\| &= \|a_n(x_n - x^*) + b_n(T^n x_n - x^*) + c_n(u_n - x^*)\| \\
&\leq \|x_n - x^*\| + L\|x_n - x^*\| + M \\
&= (1+L)\|x_n - x^*\| + M,
\end{aligned} \tag{2.6}$$

so that

$$\begin{aligned}
\|T^n y_n - x_n\| &\leq L\|y_n - x^*\| + \|x_n - x^*)\| \\
&\leq L[(1+L)\|x_n - x^*\| + M] + \|x_n - x^*\| \\
&\leq [1 + L(1+L)]\|x_n - x^*\| + ML,
\end{aligned} \tag{2.7}$$

$$\begin{aligned}
\|x_{n+1} - x^*\| &= \|a_n'(x_n - x^*) + b_n'(T^n y_n - x^*) + c_n'(v_n - x^*)\| \\
&\leq \|x_n - x^*\| + L\|y_n - x^*\| + M \\
&\leq \|x_n - x^*\| + L[(1+L)\|x_n - x^*\| + M] + M \\
&= [1 + L(1+L)]\|x_n - x^*\| + (1+L)M,
\end{aligned} \tag{2.8}$$

and

$$\begin{aligned}\|x_{n+1} - y_n\| &= \|a'_n(x_n - y_n) + b'_n(T^n y_n - y_n) + c'_n(v_n - y_n)\| \\ &\leq \|x_n - y_n\| + b'_n[\|T^n y_n - x^*\| + \|y_n - x^*\|] \\ &\quad + c'_n[\|v_n - x^*\| + \|y_n - x^*\|] \\ &= \|b_n(T^n x_n - x_n) + c_n(u_n - x_n)\| + b'_n[L\|y_n - x_n\| + \|y_n - x^*\|] \\ &\quad + c'_n M + c'_n\|y_n - x^*\| \\ &\leq b_n(1+L)\|x_n - x^*\| + c_n M + c_n\|x_n - x^*\| \\ &\quad + [b'_n(1+L) + c'_n]\|y_n - x^*\| + c'_n M \\ &\leq [b_n(1+L) + c_n]\|x_n - x^*\| + c_n M \\ &\quad + [b'_n(1+L) + c'_n][(1+L)\|x_n - x^*\| + M] + c'_n M \\ &\leq \{[b_n(1+L) + c_n] + [b'_n(1+L) + c'_n](1+L)\}\|x_n - x^*\| \\ &\quad + M[b'_n(1+L) + 2c'_n + c_n]. \end{aligned} \tag{2.9}$$

Substituting (2.7)-(2.9) in (2.5) we obtain,

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\leq \|(x_n - x^*)\|^2 - b'_n(1-k)\|x_{n+1} - T^n x_{n+1}\|^2 \\ &\quad + (k_n^2 - 1)\{[1 + L(1+L)]\|x_{n+1} - x^*\| + M(1+L)\}^2 \\ &\quad + \{(b'_n[[1 + L(1+L)]\|x_n - x^*\| + ML] + 3c'_n[M + \|x_n - x^*\|] \\ &\quad + 2b'_n L[[b_n(1+L) + c_n] + [b'_n(1 + L + c'_n](1+L)]\|x_n - x^*\| \\ &\quad + M[b'_n(1+L) + 2c'_n + c_n]\}\{[1 + L(1+L)]\|x_n - x^*\| + M(1+L)\} \\ &\leq \|(x_n - x^*)\|^2 - b'_n(1-k)\|x_{n+1} - T^n x_{n+1}\|^2 \\ &\quad + (k_n^2 - 1)\{[1 + L(1+L)]^2\|x_{n+1} - x^*\|^2 \\ &\quad + 2M(1+L)[1 + L(1+L)]\|x_n - x^*\| + M^2(1+L)^2\} \\ &\quad + 2(b'_n)^2[[1 + L(1+L)]\|x_n - x^*\| + ML][[1 + L(1+L)]\|x_n - x^*\| \\ &\quad + M(1+L)] + 3c'_n[M + \|x_n - x^*\|][[1 + L(1+L)]\|x_n - x^*\| \\ &\quad + M(1+L)] + 2b'_n L\{[[b_n(1+L) + c_n] \\ &\quad + [b'_n(1+L) + c'_n](1+L)]\|x_n - x^*\| \\ &\quad + M[b'_n(1+L) + 2c'_n + c_n]\}\{[1 + L(1+L)]\|x_n - x^*\| + M(1+L)\}. \end{aligned}$$

Since $\|x_n - x^*\| \leq 1 + \|x_n - x^*\|^2$, we have

$$\|x_{n+1} - x^*\|^2 \leq [1 + \delta_n]\|x_n - x^*\|^2 + \sigma_n - b'_n(1-k)\|x_{n+1} - T^n x_{n+1}\|^2, \tag{2.10}$$

where

$$\begin{aligned}\delta_n &= (k_n^2 - 1)\{[1 + L(1+L)]^2 + 2M(1+L)[1 + L(1+L)]\} \\ &\quad + 2(b'_n)^2\{[1 + L(1+L)]^2 + M(1+L)[1 + L(1+L)] + ML[1 + L(1+L)]\} \\ &\quad + 3c'_n\{[1 + L(1+L)] + M[1 + L(1+L)] + M(1+L)\} \\ &\quad + 2b'_n L\{\{[b_n(1+L) + c_n] + [b'_n(1+L) + c'_n](1+L)\}\{[1 + L(1+L)] + M(1+L)\} \\ &\quad + M[b'_n(1+L) + 2c'_n + c_n][1 + L(1+L)]\}\end{aligned}$$

and

$$\begin{aligned}\sigma_n &= (k_n^2-1)\{2M(1+L)[1+L(1+L)]+M^2(1+L)^2\}\\ &\quad+2(b_n')^2\{[1+L(1+L)]M(1+L)+ML[1+L(1+L)]+M^2L(1+L)\}\\ &\quad+3c_n'\{M[1+L(1+L)]+M^2(1+L)+M(1+L)\\ &\quad+2b_n'L\{[[b_n(1+L)+c_n]+[b_n'(1+L)+c_n'](1+L)][M(1+L)]\\ &\quad+M[b_n'(1+L)+2c_n'+c_n][[1+L(1+L)]+M(1+L)]\}.\end{aligned}$$

Since $\sum_{n=1}^{\infty}(k_n^2-1)<\infty$, condition (iii) implies that $\sum_{n=1}^{\infty}\delta_n<\infty$ and $\sum_{n=1}^{\infty}\sigma_n<\infty$. From (2.10) we obtain

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\le [1+\delta_n]\|x_n-x^*\|+\sigma_n\\ &\le \dots\le\prod_{j=1}^{n}[1+\delta_j]\|x_1-x^*\|^2+\prod_{j=1}^{n}[1+\delta_j]\sum_{j=1}^{n}\sigma_j\\ &\le \prod_{j=1}^{\infty}[1+\delta_j]\|x_1-x^*\|^2+\prod_{j=1}^{\infty}[1+\delta_j]\sum_{j=1}^{\infty}\sigma_j<\infty,\end{aligned}$$

since $\sum_{n=1}^{\infty}\delta_n<\infty$ and $\sum_{n=1}^{\infty}\sigma_n<\infty$. Hence $\{\|x_n-x^*\|\}_{n=1}^{\infty}$ is bounded. Let $\|x_n-x^*\|\le M,\ n\ge 1$. Then it follows from (2.10) that

$$\|x_{n+1}-x^*\|^2\le\|x_n-x^*\|^2+M^2\delta_n+\sigma_n-b_n'(1-k)\|x_{n+1}-T^nx_{n+1}\|^2,\ n\ge 1 \tag{2.11}$$

Hence,

$$b_n'(1-k)\|x_{n+1}-T^nx_{n+1}\|^2\le\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2+\mu_n,$$

where $\mu_n=M^2\delta_n+\sigma_n$ so that,

$$(1-k)\sum_{j=1}^{n}b_j'\|x_{j+1}-T^jx_{j+1}\|^2\le\|x_1-x^*\|^2+\sum_{j=1}^{n}\mu_j<\infty,$$

Hence,

$$\sum_{n=1}^{\infty}b_n'\|x_{n+1}-T^nx_{n+1}\|^2<\infty,$$

and condition (ii) implies that $\liminf_{n\to\infty}\|x_{n+1}-T^nx_{n+1}\|=0$. Observe that

$$\begin{aligned}\|x_{n+1}-T^nx_{n+1}\|^2 &= \|(1-b_n'-c_n')x_n+b_n'T^ny_n+c_n'v_n-T^nx_{n+1}\|^2\\ &= \|x_n-T^nx_n+b_n'(T^ny_n-x_n)+T^nx_n-T^nx_{n+1}\\ &\qquad +c_n'(v_n-x_n)\|^2.\end{aligned}\tag{2.12}$$

For arbitrary $u,\ v\in E$, set $x=u+v$ and $y=-v$ in (2.3) to obtain

$$\|v+u\|^2\ge\|u\|^2+2\langle v,j(u)\rangle.\tag{2.13}$$

From (2.12) and (2.13), we have

$$\begin{aligned}\|x_{n+1}-T^nx_{n+1}\|^2 &= \|x_n-T^nx_n+b_n'(T^ny_n-x_n)+T^nx_n-T^nx_{n+1}+c_n'(v_n-x_n)\|^2\\ &\ge \|x_n-T^nx_n\|^2+2\langle b_n'(T^ny_n-x_n)+T^nx_n-T^nx_{n+1}\\ &\qquad +c_n'(v_n-x_n),j(x_n-T^nx_n)\rangle.\end{aligned}$$

Hence

$$\begin{aligned}
\|x_n - T^n x_n\|^2 &\leq \|x_{n+1} - T^n x_{n+1}\|^2 + 2\|b'_n(T^n y_n - x_n) \\
&\qquad + T^n x_n - T^n x_{n+1} + c'_n(v_n - x_n)\|\|x_n - T^n x_n\| \\
&\leq \|x_{n+1} - T^n x_{n+1}\|^2 + 2\{b'_n\|T^n y_n - x_n\| + L\|x_{n+1} - x_n\| \\
&\quad + c'_n\|v_n - x_n\|\}\|x_n - T^n x_n\| \\
&\leq \|x_{n+1} - T^n x_{n+1}\|^2 + 2\{b'_n\|T^n y_n - x_n\| + Lb'_n\|T^n y_n - x_n\| \\
&\qquad + Lc'_n\|v_n - x_n\| + c'_n\|v_n - x_n\|\}\|x_n - T^n x_n\| \\
&\leq \|x_{n+1} - T^n x_{n+1}\|^2 + 2(1+L)\|x_n - x^*\| \\
&\qquad \times\{(1+L)b'_n\|T^n y_n - x_n\| + (1+L)c'_n\|v_n - x_n\|\} \\
&\leq \|x_{n+1} - T^n x_{n+1}\|^2 \\
&\qquad + 2(1+L)\|x_n - x^*\|\{(1+L)b'_n[[1 + L(1+L)]\|x_n - x^*\| + ML] \\
&\qquad + (1+L)c'_n[M + \|x_n - x^*\|], \quad \text{(using (2.6))} \\
&\leq \|x_{n+1} - T^n x_{n+1}\|^2 + 2(1+L)M\{(1+L)b'_n[[1 + L(1+L)]M + ML] \\
&\qquad + (1+L)c'_n[M + M]\}, \quad \text{(since } \|x_n - x^*\| \leq M) \\
&= \|x_{n+1} - T^n x_{n+1}\|^2 + 2b'_n(1+L)^4 M^2 + 4c'_n(1+L)^2 M. \qquad (2.14)
\end{aligned}$$

Since $\lim_{n\to\infty} b'_n = 0$, $\lim_{n\to\infty} c'_n = 0$ and $\liminf_{n\to\infty} \|x_{n+1} - T^n x_{n+1}\| = 0$, it follows from (2.14) that $\liminf_{n\to\infty} \|x_n - T^n x_n\| = 0$. It then follows from Lemma 1 that $\liminf_{n\to\infty} \|x_n - Tx_n\| = 0$, completing the proof of Lemma 2.3. □

Corollary 2.4. *Let E be a real Banach space and K a nonempty convex subset of E. Let $T : K \to K$ be a k-strictly asymptotically pseudocontractive map with $F(T) \neq \emptyset$ and sequence $\{k_n\} \subset [1, \infty)$ such that $\lim_n k_n = 1$, $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{a'_n\}$, $\{b'_n\}$, $\{c'_n\}$, $\{u_n\}$, and $\{v_n\}$ be as in Lemma 2.3 and let $\{x_n\}$ be the sequence generated from an arbitrary $x_1 \in K$ by*

$$\begin{aligned}
y_n &= a_n x_n + b_n T^n x_n + c_n u_n,\ n \geq 1, \\
x_{n+1} &= a'_n + b'_n T^n y_n + c'_n v_n,\ n \geq 1,
\end{aligned}$$

Then $\liminf_{n\to\infty} \|x_n - Tx_n\| = 0$.

Proof. From (1.1) we obtain

$$\begin{aligned}
&\|(I - T^n)x - (I - T^n)y\|\|x - y\| \\
&\qquad \geq \frac{1}{2}\{(1-k)\|(I - T^n)x - (I - T^n)y\|^2 - (k_n^2 - 1)\|x - y\|^2\} \\
&\qquad = \frac{1}{2}[\sqrt{1-k}\|(I - T^n)x - (I - T^n)y\| \\
&\qquad\qquad + \sqrt{k_n^2 - 1}\|x - y\|][\sqrt{1-k}\|(I - T^n)x - (I - T^n)y\| \\
&\qquad\qquad - \sqrt{k^2 - 1}\|x - y\|] \\
&\qquad \geq \frac{1}{2}[\sqrt{1-k}\|(I - T^n)x - (I - T^n)y\|] \\
&\qquad\qquad [\sqrt{1-k}\|(I - T^n)x - (I - T^n)y\| - \sqrt{k^2 - 1}\|x - y\|]
\end{aligned}$$

so that

$$\frac{1}{2}\sqrt{1-k}[\sqrt{1-k}\|(I-T^n)x-(I-T^n)y\|]-\sqrt{k^2-1}\|x-y\| \leq \|x-y\|.$$

Hence

$$\|(I-T^n)x-(I-T^n)y\| \leq [\frac{2+\sqrt{\{(1-k)(k_n^2-1)\}}}{1-k}]\|x-y\|.$$

Furthermore,

$$\begin{aligned}\|T^nx-T^ny\|-\|x-y\| &\leq \|(I-T^n)x-(I-T^n)y\| \\ &\leq [\frac{2+\sqrt{\{(1-k)(k_n^2-1)\}}}{1-k}]\|x-y\|,\end{aligned}$$

from which it follows that

$$\|T^nx-T^ny\| \leq [1+\frac{2+\sqrt{\{(1-k)(k_n^2-1)\}}}{1-k}]\|x-y\|.$$

Since $\{k_n\}$ is bounded, let $k_n \leq D,\ \forall\, n \geq 1$. Then

$$\begin{aligned}\|T^nx-T^ny\| &\leq [1+\frac{2+\sqrt{\{(1-k)(D^2-1)\}}}{1-k}]\|x-y\| \\ &\leq L\|x-y\|,\end{aligned}$$

where

$$L = 1+\frac{2+\sqrt{\{(1-k)(D^2-1)\}}}{1-k}.$$

Hence T is uniformly L-Lipschitzian. Since $F(T) \neq \emptyset$, then T is uniformly L-Lipschitzian and asymptotically demicontractive and hence the result follows from Lemma 2.3. □

Remark 2.5. It is shown in [3] that if E is a Hilbert space and $T: K \to K$ is k-asymptotically pseudocontractive with sequence $\{k_n\}$ then

$$\|T^nx-T^ny\| \leq \frac{D+\sqrt{k}}{1-\sqrt{k}}\|x-y\|\ \forall\, x,\ y \in K, \text{ where } k_n \leq D,\ \forall\, n \geq 1.$$

Theorem 2.6. *Let E be a real Banach space and K a nonempty closed convex subset of E. Let $T: K \to K$ be a completely continuous uniformly L-Lipschitzian asymptotically demicontractive mapping with sequence $\{k_n\} \subset [1,\infty)$ such that $\lim_n k_n = 1$ and $\sum_{n=1}^{\infty}(k_n^2-1) < \infty$. Let $\{a_n\},\ \{b_n\},\ \{c_n\}, \{a'_n\},\ \{b'_n\},\ \{c'_n\},\ \{u_n\}$, and $\{v_n\}$ be as in Lemma 2.3. Then the sequence $\{x_n\}$ generated from an arbitrary $x_1 \in K$ by*

$$\begin{aligned}y_n &= a_nx_n+b_nT^nx_n+c_nu_n,\ n \geq 1, \\ x_{n+1} &= a'_nx_n+b'_nT^ny_n+c'_nv_n,\ n \geq 1,\end{aligned}$$

converges strongly to a fixed point of T.

Proof. From Lemma 2.3, $\liminf_n \|x_n-T^nx_n\| = 0$, hence there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\lim_n \|x_{n_j}-Tx_{n_j}\| = 0$.

Since $\{x_{n_j}\}$ is bounded and T is completely continuous, then $\{Tx_{n_j}\}$ has a subsequence $\{Tx_{j_k}\}$ which converges strongly. Hence $\{x_{n_{j_k}}\}$ converges strongly. Suppose $\lim_{k\to\infty} x_{n_{j_k}} = p$. Then $\lim_{k\to\infty} Tx_{n_{j_k}} = Tp$. $\lim_{k\to\infty}\|x_{n_{j_k}}-Tx_{n_{j_k}}\| = \|p-Tp\| = 0$ so that $p \in F(T)$. It follows from (2.11) that

$$\|x_{n+1}-p\|^2 \leq \|x_n-p\|^2+\mu_n$$

Lemma 2.2 now implies $\lim_{n\to\infty} \|x_n - p\| = 0$ completing the proof of Theorem 2.6. □

Corollary 2.7. *Let E be an arbitrary real Banach space and K a nonempty closed convex subset of E. let $T : K \to K$ be a k-strictly asymptotically pseudocontractive mapping with $F(T) \neq \emptyset$ and sequence $\{k_n\} \subset [1,\infty)$ such that $\lim_n k_n = 1$, and $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$. Let $\{a_n\},\ \{b_n\},\ \{c_n\}, \{a'_n\},\ \{b'_n\},\ \{c'_n\},\ \{u_n\}$, and $\{v_n\}$ be as in Lemma 2.3. Then the sequence $\{x_n\}$ generated from an arbitrary $x_1 \in K$ by*

$$\begin{aligned} y_n &= a_n x_n + b_n T^n x_n + c_n u_n,\ n \geq 1, \\ x_{n+1} &= a'_n x_n + b'_n T^n y_n + c'_n v_n,\ n \geq 1, \end{aligned}$$

converges strongly to a fixed point of T.

Proof. As shown in Corollary 2.4, T is uniformly L-Lipschitzian and since $F(T) \neq \emptyset$ then T is asymptotically demicontractive and the result follows from Theorem 2.6. □

Remark 2.8. If we set $b_n = c_n = 0,\ \forall\, n \geq 1$ in Lemma 2.3, Theorem 2.6 and Corollaries 2.4 and 2.7, we obtain the corresponding results for the modified Mann iteration method with errors in the sense of Xu [7].

Remark 2.9. Theorem 2.6 extends the results of Osilike [3] (which is itself a generalization of a theorem of Qihou [4]) from real q-uniformly smooth Banach space to arbitrary real Banach space.

Furthermore, our Theorem 2.6 is proved without the boundedness condition imposed on the subset K in ([3, 4]) and using the more general modified Ishikawa Iteration method with errors in the sense of Xu [7]. Also our iteration parameters $\{a_n\},\ \{b_n\},\ \{c_n\}, \{a'_n\},\ \{b'_n\},\ \{c'_n\},\ \{u_n\}$, and $\{v_n\}$ are completely independent of any geometric properties of underlying Banach space.

Remark 2.10. Prototypes for our iteration parameters are:

$$\begin{aligned} b'_n &= \frac{1}{3(n+1)},\ c'_n = \frac{1}{3(n+1)^2}, a'_n = 1 - (b'_n + c'_n), \\ b_n &= c_n = \frac{1}{3(n+1)^2}, a_n = 1 - \frac{1}{3(n+1)^2},\ n \geq 1. \end{aligned}$$

The proofs of the following theorems and corollaries for the Ishikawa iteration method with errors in the sense of Liu [2] are omitted because the proofs follow by a straightforward modifications of the proofs of the corresponding results for the Ishikawa iteration method with errors in the sense of Xu [7].

Theorem 2.11. *Let E be a real Banach space and let $T : E \to E$ be a uniformly L-Lipschitzian asymptotically demicontractive mapping with sequence $\{k_n\} \subset [1,\infty)$ such that $\lim_n k_n = 1$, and $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$. Let $\{u_n\}$ and $\{v_n\}$ be sequences in E such that $\sum_{n=1}^{\infty}\|u_n\| < \infty$ and $\sum_{n=1}^{\infty}\|v_n\| < \infty$, and let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0,1]$ satisfying the conditions:*

(i) $0 \leq \alpha_n, \beta_n \leq 1,\ n \geq 1$;
(ii) $\sum_{n=1}^{\infty}\alpha_n = \infty$
(iii) $\sum_{n=1}^{\infty}\alpha_n^2 < \infty$ *and* $\sum_{n=1}^{\infty}\beta_n < \infty$.

Let $\{x_n\}$ be the sequence generated from an arbitrary $x_1 \in E$ by

$$\begin{aligned} y_n &= (1-\beta_n)x_n + \beta_n T^n x_n + u_n,\ n \geq 1, \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T^n y_n + v_n,\ n \geq 1, \end{aligned}$$

Then $\liminf_{n\to\infty}\|x_n - Tx_n\| = 0$.

Corollary 2.12. *Let E be a real Banach space and let $T : E \to E$ be a k-strictly asymptotically pseudocontractive map with $F(T) \neq \emptyset$ and sequence $\{k_n\} \subset [1, \infty)$ such that $\lim_n k_n = 1$, and $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$. Let $\{u_n\}, \{v_n\}, \{\alpha_n\}$ and $\{\beta_n\}$ be as in Theorem 2.11 and let $\{x_n\}$ be the sequence generated from an arbitrary $x_1 \in E$ by*

$$\begin{aligned} y_n &= (1-\beta_n)x_n + \beta_n T^n x_n + u_n, \ n \geq 1, \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T^n y_n + v_n, \ n \geq 1, \end{aligned}$$

Then $\liminf_{n\to\infty} \|x_n - Tx_n\| = 0$.

Theorem 2.13. *Let E, T, $\{u_n\}, \{v_n\}, \{\alpha_n\}$ and $\{\beta_n\}$ be as in Theorem 2.11. If in addition $T : E \to E$ is completely continuous then the sequence $\{x_n\}$ generated from an arbitrary $x_1 \in E$ by*

$$\begin{aligned} y_n &= (1-\beta_n)x_n + \beta_n T^n x_n + u_n, \ n \geq 1, \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T^n y_n + v_n, \ n \geq 1, \end{aligned}$$

converges strongly to a fixed point of T.

Corollary 2.14. *Let E, T, $\{u_n\}, \{v_n\}, \{\alpha_n\}$ and $\{\beta_n\}$ be as in Corollary 2.12. If in addition T is completely continuous, then the sequence $\{x_n\}$ generated from an arbitrary $x, y \in E$ by*

$$\begin{aligned} y_n &= (1-\beta_n)x_n + \beta_n T^n x_n + u_n, \ n \geq 1, \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T^n y_n + v_n, \ n \geq 1, \end{aligned}$$

converges strongly to a fixed point of T.

Remark 2.15.
(a) If K is a nonempty closed convex subset of E and $T : K \to K$, then Theorems 2.11 and 2.13 and Corollaries 2.12 and 2.14 also hold provided that in each case the sequence $\{x_n\}$ lives in K.
(b) If we set $\beta_n = 0$, $\forall\, n \geq 1$ in Theorems 2.11 and 2.13 and Corollaries 2.12 and 2.14, we obtain the corresponding results for modified Mann iteration method with errors in the sense of Liu [2].

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