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NONLINEAR DELAY INTEGRAL INEQUALITIES FOR PIECEWISE CONTINUOUS FUNCTIONS AND APPLICATIONS

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ABSTRACT. Nonlinear integral inequalities of Gronwall-Bihari type for piecewise continuous functions are solved. Inequalities for functions with delay as well as functions without delays are considered. Some of the obtained results are applied in the deriving of estimates for the solutions of impulsive integral, impulsive integro-differential and impulsive differential-difference equations.

Key words and phrases: Integral inequalities, Piecewise continuous functions, Impulsive differential equations.

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1. INTRODUCTION

Integral inequalities are a powerful mathematical apparatus, by the aid of which, various properties of the solutions of differential and integral equations can be studied, such as uniqueness of the solutions, boundedness, stability, etc. This leads to the necessity of solving various types of linear and nonlinear inequalities, which generalize the classical inequalities of Gronwall and Bihari. In recent years, many authors, such as S.S. Dragomir ([2] – [4]) and B.G. Pachpatte ([5] – [10]) have discovered and applied several new integral inequalities for continuous functions.

The development of the qualitative theory of impulsive differential equations, whose solutions are piecewise continuous functions, is connected with the preliminary deriving of results on integral inequalities for such types of functions (see [1] and references there).

In the present paper we prove some generalizations of the classical Bihari inequality for piecewise continuous functions. Two main types of nonlinear inequalities are considered – inequalities with constant delay of the argument as well as inequalities without delays. The obtained inequalities are used to investigate some properties of the solutions of impulsive integral equations, impulsive integro-differential and impulsive differential-difference equations.

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2. PRELIMINARY NOTES AND DEFINITIONS

Let the points $t_k \in (0,\infty)$, $k = 1, 2, \dots$ are fixed such that $t_{k+1} > t_k$ and $\lim_{k\downarrow\infty} t_k = \infty$.

We consider the set $PC(X, Y)$ of all functions $u : X \to Y$, $(X \subset \mathbb{R}, Y \subset \mathbb{R}^n)$ which are piecewise continuous in X with points of discontinuity of the first kind at the points $t_k \in X$, i.e. there exist the limits $\lim_{t\downarrow t_k} u(t) = u(t_k+) < \infty$ and $\lim_{t\uparrow t_k} u(t) = u(t_k-) = u(t_k) < \infty$.

Definition 2.1. We will say that the function $G(u)$ belongs to the class W_1 if

(1) $G \in C([0,\infty), [0,\infty))$.
(2) $G(u)$ is a nondecreasing function.

Definition 2.2. We will say that the function $G(u)$ belongs to the class $W_2(\varphi)$ if

(1) $G \in W_1$.
(2) There exists a function $\varphi \in C([0,\infty), [0,\infty))$ such that $G(uv) \le \varphi(u)G(v)$ for $u, v \ge 0$.

We note that if the function $G \in W_1$ and satisfies the inequality $G(uv) \le G(u)G(v)$ for $u, v \ge 0$ then $G \in W_2(G)$.

Further we will use the following notations $\sum_{i=1}^{k} \alpha_k = 0$ and $\prod_{i=1}^{k} \alpha_k = 1$ for $k \le 0$.

3. MAIN RESULTS

As a first result we will consider integral inequalities with delay for piecewise continuous functions.

Theorem 3.1. *Let the following conditions be fulfilled:*

(1) *The functions* $f_1, f_2, f_3, p, g \in C([0,\infty), [0,\infty))$.
(2) *The function* $\psi \in C([-h, 0], [0,\infty))$.
(3) *The function* $Q \in W_2(\varphi)$ *and* $Q(u) > 0$ *for* $u > 0$.
(4) *The function* $G \in W_1$.
(5) *The function* $u \in PC([-h,\infty), [0,\infty))$ *and it satisfies the following inequalities*

$$u(t) \le f_1(t) + f_2(t)G\left(c + \int_0^t p(s)Q(u(s))ds + \int_0^t g(t)Q(u(s-h))ds\right)$$

$$+ f_3(t) \sum_{0<t_k<t} \beta_k u(t_k) \quad \text{for } t \ge 0, \tag{3.1}$$

$$u(t) \le \psi(t) \quad \text{for } t \in [-h, 0], \tag{3.2}$$

where $c \ge 0, \beta_k \ge 0,\ (k = 1, 2, \dots)$.

Then for $t \in (t_k, t_{k+1}] \cap [0,\gamma),\ k = 0, 1, 2, \dots$ *we have the inequality*

$$u(t) \le \rho(t) \prod_{i=1}^{k} (1 + \beta_i \rho(t_i)) \tag{3.3}$$

$$\times \left(1 + G\left(H^{-1}\left\{H(A) + \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} p(s)\varphi\left[\rho(s) \prod_{j=1}^{i-1} (1 + \beta_j \rho(t_j))\right] ds\right.\right.\right.$$

$$+ \int_{t_k}^{t} p(s)\varphi\left[\rho(s) \prod_{j=1}^{k} (1 + \beta_j \rho(t_j))\right] ds$$

$$+\Lambda(t)\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i} g(s)\varphi\left[\rho(s-h))\prod_{j:0<t_j<s-h}(1+\beta_j\rho(t_k))\right]ds$$
$$\left.\left.\left.+\Lambda(t)\int_{t_k}^{t} g(s)\varphi\left[\rho(s-h))\prod_{j:0<t_j<s-h}(1+\beta_j\rho(t_k))\right]ds\right\}\right)\right),$$

where

$$\Lambda(t)=\begin{cases}0 & \textit{for} \quad t\in[0,h],\\ 1 & \textit{for} \quad t>h,\end{cases} \tag{3.4}$$

$$\rho(t)=\max\{f_i(t):i=1,2,3\},\qquad A=c+hB_1Q(B_2),$$
$$B_1=\max\{g(t):t\in[0,h]\},\qquad B_2=\max\{\psi(t):t\in[-h,0]\},$$

$$H(u)=\int_{u_0}^{u}\frac{ds}{Q(1+G(s))},\quad A\ge u_0\ge 0, \tag{3.5}$$

$$\gamma=\sup\left\{t\ge 0: H(A)+\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i} p(s)\varphi\left[\rho(s)\prod_{j=1}^{i-1}(1+\beta_j\rho(t_j))\right]ds\right.$$
$$+\int_{t_k}^{t} p(s)\varphi\left[\rho(s)\prod_{j=1}^{k}(1+\beta_j\rho(t_j))\right]ds$$
$$+\Lambda(t)\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i} g(s)\varphi\left[\rho(s-h))\prod_{0<t_j<s-h}(1+\beta_j\rho(t_k))\right]ds$$
$$+\Lambda(t)\int_{t_k}^{t} g(s)\varphi\left[\rho(s-h))\prod_{0<t_j<s-h}(1+\beta_j\rho(t_k))\right]ds\in\operatorname{dom}(H^{-1})$$
$$\left.\textit{for}\ \ \tau\in(t_k,t_{k+1}]\cap[0,t],\ k=0,1,\dots\right\},$$

and H^{-1} is the inverse function of $H(u)$.

Proof.
Case 1. Let $t_1\ge h$.

Let $t\in(0,h]\cap[0,\gamma)\ne\emptyset$.

It follows from the inequalities (3.1) and (3.2) that for $t\in(0,h]\cap[0,\gamma)$ the inequality

$$u(t)\le\rho(t)\left(1+G\left(A+\int_0^t p(s)Q(u(s))ds\right)\right) \tag{3.6}$$

holds.

Define the function $v_0^{(0)}:[0,h]\cap[0,\gamma)\to[0,\infty)$ by the equality

$$v_0^{(0)}(t)=A+\int_0^t p(s)Q(u(s))ds. \tag{3.7}$$

The function $v_0^{(0)}(t)$ is a nondecreasing differentiable function on $[0,h]\cap[0,\gamma)$ and it satisfies the inequality

$$u(t) \le \rho(t)\Big(1+G(v_0^{(0)}(t))\Big). \tag{3.8}$$

The inequality (3.8) and the definition (5) of the function $H(u)$ yield

$$\frac{d}{dt}H(v_0^{(0)}(t)) = \frac{(v_0^{(0)}(t))'}{Q\Big(1+G(v_0^{(0)}(t))\Big)} \le p(t)\varphi\Big(\rho(t)\Big). \tag{3.9}$$

We integrate the inequality (3.9) from 0 to t for $t\in[0,h]\cap[0,\gamma)$, and we use $v_0^{(0)}(0)=A$ in order to obtain

$$H(v_0^{(0)}(t)) \le H(A) + \int_0^t p(s)\varphi\Big(\rho(s)\Big)ds. \tag{3.10}$$

The inequalities (3.8) and (3.10) imply the validity of the inequality (3.3) for $t\in[0,h]\cap[0,\gamma)$.

Let $t\in(h,t_1]\cap[0,\gamma)\neq\emptyset$.

Then

$$\begin{aligned} u(t) &\le \rho(t)\Bigg(1+G\Big(v_0^{(0)}(h)+\int_h^t p(s)Q(u(s))ds \\ &\qquad + \int_h^t g(s)Q(u(s-h))ds\Big)\Bigg) \\ &= \rho(t)\Bigg(1+G\Big(v_0^{(1)}(t)\Big)\Bigg), \end{aligned} \tag{3.11}$$

where $v_0^{(1)}:[h,t_1]\cap[0,\gamma)\to[0,\infty)$ is defined by the equality

$$v_0^{(1)}(t) = v_0^{(0)}(h) + \int_h^t p(s)Q(u(s))ds + \int_h^t g(s)Q(u(s-h))ds. \tag{3.12}$$

Using the fact that the function $v_0^{(1)}(t)$ is nondecreasing continuous and $v_0^{(0)}(t-h)\le v_0^{(0)}(h)\le v_0^{(1)}(t)$ for $h<t\le\min\{2h,t_1\}$, we can prove as above that

$$\begin{aligned} H(v_0^{(1)}(t)) &\le H(v_0^{(0)}(h)) + \int_h^t p(s)\varphi\Big(\rho(s)\Big)ds \\ &\qquad + \int_h^t g(s)\varphi\Big(\rho(s-h)\Big)ds \\ &\le H(A) + \int_0^t p(s)\varphi\Big(\rho(s)\Big)ds \\ &\qquad + \int_h^t g(s)\varphi\Big(\rho(s-h)\Big)ds. \end{aligned} \tag{3.13}$$

The inequalities (3.11), (3.13) prove the validity of (3.3) on $t\in(h,t_1]\cap[0,\gamma)$.

Define the function

$$v_0(t) = \begin{cases} v_0^{(0)}(t) & \text{for } t\in[0,h], \\ v_0^{(1)}(t) & \text{for } t\in(h,t_1]. \end{cases}$$

Now let $t\in(t_1,t_2]\cap[0,\gamma)\neq\emptyset$.

Define the function $v_1 : [t_1, t_2] \cap [0, \gamma) \to [0, \infty)$ by the equality

$$v_1(t) = v_0(t_1) + \int_{t_1}^{t} p(s)Q(u(s))ds + \int_{t_1}^{t} g(s)Q(u(s-h))ds. \tag{3.14}$$

The function $v_1(t)$ is nondecreasing differentiable on $t \in (t_1, t_2] \cap [0, \gamma)$, $v_1(t) \geq v_0(t_1)$ and

$$\begin{aligned} u(t) &\leq \rho(t)\Big(1 + G\Big(v_1(t)\Big) + \beta_1 u(t_1)\Big) \\ &\leq \rho(t)\Big(1 + G\Big(v_1(t)\Big) + \beta_1 \rho(t_1)\Big(1 + G(v_0(t_1))\Big)\Big) \\ &\leq \rho(t)\Big(1 + G(v_1(t))\Big)\Big(1 + \beta_1\rho(t_1)\Big). \end{aligned} \tag{3.15}$$

Consider the following two possible cases:

Case 1.1. Let $h \leq t_2 - t_1$ and $t \in (t_1 + h, t_2] \cap [0, \gamma)$. Then from (3.15) we have

$$\begin{aligned} u(t-h) &\leq \rho(t-h)\Big(1 + G(v_1(t-h))\Big)\Big(1 + \beta_1\rho(t_1)\Big) \\ &\leq \rho(t-h)\Big(1 + G(v_1(t))\Big)\Big(1 + \beta_1\rho(t_1)\Big) \\ &= \rho(t-h)\Big(1 + G(v_1(t))\Big) \prod_{0<t_k<t-h} \Big(1 + \beta_k\rho(t_k)\Big). \end{aligned} \tag{3.16}$$

Case 1.2. Let $h > t_2 - t_1$ or $t \in (t_1, t_1 + h] \cap [0, \gamma)$. Then

$$u(t-h) \leq \rho(t-h)\Big(1 + G(v_0(t-h))\Big). \tag{3.17}$$

Using the inequality (3.17) and $v_0(t-h) \leq v_1(t)$ we obtain

$$\begin{aligned} u(t-h) &\leq \rho(t-h)\Big(1 + G(v_1(t))\Big) \\ &= \rho(t-h)\Big(1 + G(v_1(t))\Big) \prod_{0<t_k<t-h} \Big(1 + \beta_k\rho(t_k)\Big). \end{aligned} \tag{3.18}$$

The inequalities (3.15), (3.16), (3.18) and the properties of the function $Q(u)$ imply

$$\begin{aligned} v_1'(t) &= p(t)Q(u(t)) + g(t)Q(u(t-h)) \\ &\leq \Big\{p(t)\varphi\Big(\rho(t)(1 + \beta_1\rho(t_1))\Big) \\ &\quad + g(t)\varphi\Big(\rho(t-h) \prod_{0<t_k<t-h} \Big(1 + \beta_k\rho(t_k)\Big)\Big)\Big\} \\ &\quad \times Q\Big(1 + G(v_1(t))\Big). \end{aligned} \tag{3.19}$$

We obtain from the definition (5) and the inequality (3.19) that

$$\frac{d}{dt}H(v_1(t)) = \frac{(v_1(t))'}{Q\Big(1+G(v_1(t))\Big)}$$
$$\le p(t)\varphi\big(\rho(t)(1+\beta_1\rho(t_1))\big)$$
$$+ g(t)\varphi\left(\rho(t-h)\prod_{0<t_k<t-h}\Big(1+\beta_k\rho(t_k)\Big)\right). \tag{3.20}$$

We integrate the inequality (3.20) from t_1 to t, use the inequality (3.13) and obtain

$$H(v_1(t)) \le H(v_0(t_1)) + \int_{t_1}^{t} p(s)\varphi\Big(\rho(s)(1+\beta_1\rho(t_1))\Big)ds$$
$$+ \int_{t_1}^{t} g(s)\varphi\rho(s-h)\left(\prod_{0<t_k<t-h}(1+\beta_k\rho(t_k))\right)ds$$
$$\le H(A) + \int_{0}^{t_1} p(s)\varphi\Big(\rho(s)\Big)ds$$
$$+ \int_{t_1}^{t} p(s)\varphi\Big(\rho(s)(1+\beta_1\rho(t_1))\Big)ds$$
$$+ \int_{t_1}^{t} g(s)\varphi\rho(s-h)\left(\prod_{0<t_k<t-h}(1+\beta_k\rho(t_k))\right)ds. \tag{3.21}$$

The inequalities (3.15) and (3.21) imply the validity of the inequality (3.3) for $t \in (t_1, t_2] \cap [0,\gamma)$.

We define functions $v_k : [t_k, t_{k+1}] \cap [0,\gamma) \to [0,\infty)$ by the equalities

$$v_k(t) = v_{k-1}(t_k) + \int_{t_k}^{t} p(s)Q(u(s))ds + \int_{t_k}^{t} g(s)Q(u(s-h))ds. \tag{3.22}$$

The functions $v_k(t)$ are nondecreasing functions, $v_k(t) \ge v_{k-1}(t_k)$ and for $t \in (t_k, t_{k+1}] \cap [0,\gamma)$ the inequalities

$$u(t) \le \rho(t)\left(1 + G\Big(v_k(t)\Big) + \sum_{i=1}^{k}\beta_i u(t_i)\right)$$
$$\le \rho(t)\Bigg\{1 + G(v_k(t)) + \sum_{i=1}^{k-1}\beta_i u(t_i)$$
$$+\beta_k\rho(t_k)\left(1 + G(v_{k-1}(t_k)) + \sum_{i=1}^{k-1}\beta_i u(t_i)\right)\Bigg\}$$
$$\le \rho(t)\left(1 + G(v_k(t)) + \sum_{i=1}^{k-1}\beta_i u(t_i)\right)\Big(1+\beta_k\rho(t_k)\Big)$$
$$\le \cdots \le \rho(t)\left\{\prod_{i=1}^{k}\Big(1+\beta_i\rho(t_i)\Big)\right\}\Big(1+G(v_k(t))\Big) \tag{3.23}$$

hold.

Using mathematical induction we prove that inequality (3.3) is true for $t \in (t_k, t_{k+1}] \cap [0,\gamma), k = 1, 2, \dots$.

Case 2. Let there exist a natural number m such that $t_m \le h < t_{m+1}$. As in Case 1 we prove the validity of inequality (3.3) using the functions $v_k \in C\Big([t_k, t_{k+1}] \cap [0,\gamma), [0,\infty)\Big)$ defined by the equalities

$$v_k(t) = v_{k-1}(t_k) + \int_{t_k}^{t} p(s)Q(u(s))ds \ \text{ for } \ k = 0,1,\dots,m,$$

$$v_k(t) = v_{k-1}(t_k) + \int_{t_k}^{t} p(s)Q(u(s))ds + \int_{t_k}^{t} g(s)Q(u(s-h))ds, k > m. \tag{3.24}$$

□

In the partial case when the function $\varphi(s)$ in Definition 2.2 is multiplicative, the following result is true.

Corollary 3.2. *Let the conditions of Theorem 3.1 be satisfied and the function φ satisfy the inequality $\varphi(ts) \le \varphi(t)\varphi(s)$ for $t, s \ge 0$.*

Then for $t \in (t_k, t_{k+1}] \cap [0,\gamma_3)$ the inequality

$$u(t) \le \rho(t)\prod_{i=1}^{k}(1+\beta_i\rho(t_i)) \times \Bigg\{1 + G\Bigg(H^{-1}\Bigg(H(A) + \varphi\left(\prod_{i=1}^{k}(1+\beta_i\rho(t_i)\right) \times \int_0^t \big(p(s)\varphi(\rho(s)) + \Lambda(s)g(s)\varphi(\rho(s-h))\big)ds\Bigg)\Bigg)\Bigg\}, \tag{3.25}$$

holds, where the functions $\Lambda(t)$ and $H(u)$ are defined by the equalities (3.4) and (3.5), respectively, and

$$\gamma_3 = \sup\Bigg\{t \ge 0 : H(A) + \varphi\left(\prod_{i=1}^{k}(1+\beta_i\rho(t_i)\right) \times \int_0^t \big(p(s)\varphi(\rho(s)) + \Lambda(s)g(s)\varphi(\rho(s-h))\big)ds \in Dom(H^{-1}) \ \textit{for} \ \tau \in [0,t]\Bigg\}, \tag{3.26}$$

In the case when the function $f_1(t) = 0$ in inequality (3.3), we can obtain another bound in which the function $H(u)$ is different and in some cases easier to be used.

Theorem 3.3. *Let the following conditions be fulfilled:*

(1) *The functions $f_1, f_2, p, g \in C([0,\infty),[0,\infty))$.*
(2) *The function $\psi \in C([-h,0],[0,\infty))$.*
(3) *The function $Q \in W_2(\varphi)$ and $Q(u) > 0$ for $u > 0$.*
(4) *The function $G \in W_1$.*
(5) *The function $u \in PC([-h,\infty),[0,\infty))$ and it satisfies the inequalities*

$$u(t) \le f_1(t)G\left(c + \int_0^t p(s)Q(u(s))ds + \int_0^t g(t)Q(u(s-h))ds\right) + f_2(t)\sum_{0<t_k<t}\beta_k u(t_k) \quad \textit{for} \ \ t \ge 0, \tag{3.27}$$

$$u(t) \le \psi(t) \quad \textit{for} \ \ t \in [-h,0], \tag{3.28}$$

where $c \geq 0, \beta_k \geq 0, (k = 1, 2, \dots)$.

Then for $t \in (t_k, t_{k+1}] \cap [0, \gamma),\ (k = 1, 2, \dots)$ *we have the inequality*

$$\begin{aligned}(3.29)\quad u(t) \leq \rho(t) \prod_{i=1}^{k} (1 + \beta_i \rho(t_i)) \\ \times G\Bigg(H^{-1}\Bigg\{H(A) + \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} p(s)\varphi\left[\rho(s)\prod_{j=1}^{i-1}(1+\beta_j\rho(t_j))\right] ds \\ + \int_{t_k}^{t} p(s)\varphi\left[\rho(s)\prod_{j=1}^{k}(1+\beta_j\rho(t_j))\right] ds \\ + \Lambda(t)\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i} g(s)\varphi\left[\rho(s-h))\prod_{j:0<t_j<s-h}(1+\beta_j\rho(t_k))\right] ds \\ +\Lambda(t)\int_{t_k}^{t} g(s)\varphi\left[\rho(s-h))\prod_{j:0<t_j<s-h}(1+\beta_j\rho(t_k))\right] ds\Bigg\}\Bigg),\end{aligned}$$

where $\Lambda(t)$ *is defined by equality (3.4), the constants* A, B_1, B_2, γ *are the same as in Theorem 3.1,* $\rho(t) = \max\{f_i(t) : i = 1, 2\}$,

$$(3.30)\qquad H(u) = \int_{u_0}^{u} \frac{ds}{Q(G(s))}, \quad A \geq u_0 > 0.$$

The proof of Theorem 3.3 is similar to the proof of Theorem 3.1.

As a partial case of Theorem 3.1 we can obtain the following result about integral inequalities for piecewise continuous functions without delay.

Theorem 3.4. *Let the following conditions be satisfied:*

(1) *The functions* $f_1, f_2, f_3, p \in C([0, \infty), [0, \infty))$.
(2) *The function* $Q \in W_2(\varphi)$ *and* $Q(u) > 0$ *for* $u > 0$.
(3) *The function* $G \in W_1$.
(4) *The function* $u \in PC([0, \infty), [0, \infty))$ *and it satisfies the inequalities*

$$(3.31)\quad u(t) \leq f_1(t) + f_2(t) G\left\{c + \int_0^t p(s)Q(u(s))ds\right\} + f_3(t)\sum_{0<t_k<t}\beta_k u(t_k), \text{for} \quad t \geq 0.$$

where $c \geq 0, \beta_k \geq 0, (k = 1, 2, \dots)$.

Then for $t \in (t_k, t_{k+1}] \cap [0, \gamma_1),\ k = 0, 1, 2, \dots$ *we have the inequality*

$$\begin{aligned}(3.32)\quad u(t) \leq \rho(t) \prod_{i=1}^{k} (1 + \beta_i \rho(t_i)) \\ \times \Bigg(1 + G\Bigg(H^{-1}\Bigg\{H(c) + \sum_{i=1}^{k}\int_{t_{i-1}}^{t_i} p(s)\varphi\left[\rho(s)\prod_{j=1}^{i-1}(1+\beta_j\rho(t_j))\right] ds \\ + \int_{t_k}^{t} p(s)\varphi\left[\rho(s)\prod_{j=1}^{k}(1+\beta_j\rho(t_j))\right] ds\Bigg\}\Bigg)\Bigg),\end{aligned}$$

where the function $H(u)$ *is defined by equality (3.5),* $\rho(t) = \max\{f_i(t); i = 1, 2\}$*,*

$$\gamma_1 = \sup\Bigg\{t \geq 0 : H(c) + \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} p(s)\varphi\left[\rho(s)\prod_{j=1}^{i-1}(1+\beta_j\rho(t_j))\right]ds \\ + \int_{t_k}^{t} p(s)\varphi\left[\rho(s)\prod_{j=1}^{k}(1+\beta_j\rho(t_j))\right]ds \in \operatorname{dom}(H^{-1}) \text{ for } \tau \in [0,t]\Bigg\}.$$

Remark 3.5. We note that the obtained inequalities are generalizations of many known results. For example, in the case when $f_1(t) = 0$, $f_2(t) = 0$, $\beta_k = 0$, $G(u) = u$, $Q(u) = u$, $h = 0$, $g(t) = 0$ the result in Theorem 3.3 reduces to the classical Gronwall inequality.

Now we will consider different types of nonlinear integral inequalities in which the unknown function is powered.

Theorem 3.6. *Let the following conditions be fulfilled:*

(1) *The functions* $f, g, h, r \in C([0,\infty), [0,\infty))$*.*
(2) *The function* $\psi \in C([-h, 0], [0,\infty))$ *and* $\psi(t) \leq c$ *for* $t \in [-h, 0]$ *where* $c \geq 0$*. The constants* $p > 1$, $0 \leq q \leq p$*.*
(3) *The function* $u \in PC([0,\infty), [0,\infty))$ *and satisfies the inequalities*

$$u^p(t) \leq c + \int_0^t [f(s)u^p(t) + g(s)u^q(s)u^{p-q}(s-h) \\ + h(s)u(s) + r(s)u(s-h)]ds \\ + \sum_{0<t_k<t} \beta_k u^p(t_k), \quad \text{for } t \geq 0, \tag{3.33}$$

$$u(t) \leq \psi(t) \quad \text{for } t \in [-h, 0]. \tag{3.34}$$

Then for $t \in (t_k, t_{k+1}],\ k = 0, 1, 2, \ldots$ *the inequality*

$$u(t) \leq \sqrt[p]{\prod_{i=1}^{k}(1+\beta_i)} \times \sqrt[p]{\left(c + \frac{p-1}{p}\int_0^t (h(s)+r(s))ds\right)} \\ \times \sqrt[p]{\exp\left(\int_0^t (f(s)+g(s)+\frac{h(s)+r(s)}{p})ds\right)} \tag{3.35}$$

holds.

Proof.
Case 1. Let $t_1 \geq h$.

Let $t \in (0, h]$. We define the function $v_0^{(0)} : [-h, h] \to [0,\infty)$ by the equalities

$$v_0^{(0)}(t) = \begin{cases} c + \int_0^t [f(s)u^p(t) + g(s)u^q(s)u^{p-q}(s-h) + h(s)u(s) \\ \qquad + r(s)u(s-h)]ds, \quad t \in [0,h], \\ \psi^p(t) \quad \text{for } t \in [-h, 0). \end{cases}$$

The function $v_0^{(0)}(t)$ is a nondecreasing differentiable function on $[0, h]$, $u^p(t) \leq v_0^{(0)}(t)$ and using the inequality $x^m y^n \leq \frac{x}{m} + \frac{y}{n},\ n + m = 1$ we obtain

$$u(t) \leq \sqrt[p]{v_0^{(0)}(t)} \leq \frac{v_0^{(0)}(t)}{p} + \frac{p-1}{p}, \quad t \in [0,h] \tag{3.36}$$

and

$$u(t-h) \le \frac{\psi(t-h)}{p} + \frac{p-1}{p} \le \frac{c}{p} + \frac{p-1}{p} \le \frac{v_0^{(0)}(t)}{p} + \frac{p-1}{p}, \quad t \in [0,h]. \tag{3.37}$$

Therefore the inequality

$$\begin{aligned}\left(v_0^{(0)}(t)\right)' &= f(t)u^p(t) + g(t)u^q(t)u^{p-q}(t-h) + h(t)u(t) + r(t)u(t-h)\\ &\le f(t)v_0^{(0)}(t) + g(t)v_0^{(0)}(t)^{q/p} v_0^{(0)}(t-h)^{(p-q)/p}\\ &\qquad + h(t)\left(\frac{v_0^{(0)}(t)}{p} + \frac{p-1}{p}\right) + r(t)\left(\frac{v_0^{(0)}(t)}{p} + \frac{p-1}{p}\right)\\ &\le \left(f(t) + g(t) + \frac{h(t)+r(t))}{p}\right) v_0^{(0)}(t) + (h(t)+r(t))\frac{p-1}{p} \qquad (3.38)\end{aligned}$$

holds.

According to Corollary 1.2 ([1]) from the inequality (3.38) we obtain the validity of the inequality

$$\left(v_0^{(0)}(t)\right) \le \left(c + \frac{p-1}{p}\int_0^t (h(s)+r(s))ds\right) \times \exp\left(\int_0^t \left(f(t) + g(t) + \frac{h(s)+r(s)}{p}\right) ds\right). \tag{3.39}$$

From inequality (3.39) follows the validity of (3.35) for $t \in [0,h]$.

Let $t \in (h, t_1]$.

Define the function $v_0^{(1)} : [h, t_1] \to [0, \infty)$ by the equation

$$v_0^{(1)}(t) = v_0^{(0)}(h) + \int_h^t [f(s)u^p(t) + g(s)u^q(s)u^{p-q}(s-h) + h(s)u(s) + r(s)u(s-h)]ds.$$

From the definition of the function $v_0^{(1)}(t)$ and the inequality (3.33), the validity of the inequality

$$u^p(t) \le v_0^{(1)}(t), \quad t \in (h, t_1]. \tag{3.40}$$

follows.

Case 1.1. Let $h < t \le \min\{t_1, 2h\}$. Then $t - h \in (0, h]$ and

$$u(t-h) \le \sqrt[p]{v_0^{(0)}(t-h)} \le \sqrt[p]{v_0^{(0)}(h)} \le \sqrt[p]{v_0^{(1)}(t)} \le \frac{v_0^{(1)}(t)}{p} + \frac{p-1}{p}.$$

Case 1.2. Let $t_1 > 2h$ or $t \in (2h, t_1]$. Then

$$u(t-h) \le \sqrt[p]{v_0^{(1)}(t-h)} \le \sqrt[p]{v_0^{(1)}(t)} \le \frac{v_0^{(1)}(t)}{p} + \frac{p-1}{p}$$

and

$$\left(v_0^{(1)}(t)\right)' \le \left(f(t) + g(t) + \frac{h(t)+r(t))}{p}\right) v_0^{(1)}(t) + (h(t)+r(t))\frac{p-1}{p}. \tag{3.41}$$

From the inequalities (3.40), (3.41) and applying Corollary 1.2 ([1]) we obtain

$$(v_0^{(1)}(t)) \le \left(v_0^{(0)}(h) + \frac{p-1}{p}\int_h^t (h(s)+r(s))ds\right) \times \exp\left(\int_h^t \left(f(t)+g(t)+\frac{h(s)+r(s)}{p}\right)ds\right)$$
$$\le \left(c + \frac{p-1}{p}\int_0^t (h(s)+r(s))ds\right) \times \exp\left(\int_0^t \left(f(t)+g(t)+\frac{h(s)+r(s)}{p}\right)ds\right). \tag{3.42}$$

The inequalities (3.40) and (3.43) prove the validity of the inequality (3.35) on $(h, t_1]$.

Define a function

$$v_0(t) = \begin{cases} v_0^{(0)}(t) & \text{for } t \in [0,h], \\ v_0^{(1)}(t) & \text{for } t \in [h,t_1]. \end{cases}$$

Now let $t \in (t_1, t_2]$.

Define the function $v_1 : [t_1, t_2] \to [0,\infty)$ by the equation

$$v_1(t) = v_0(t_1) + \int_h^t [f(s)u^p(t) + g(s)u^q(s)u^{p-q}(s-h) + h(s)u(s) + r(s)u(s-h)]ds + \beta_1 u^p(t_1). \tag{3.43}$$

We note that $v_0(t) \le v_1(t)$, $u^p(t) \le v_1(t)$, $u^p(t_1) \le v_0(t_1)$, $u(t-h) \le \sqrt[p]{v_1(t)}$, $u(t-h) \le \sqrt[p]{v_1(t)}$ and $\sqrt[p]{v_1(t)} \le \frac{v_1(t)}{p} + \frac{p-1}{p}$ for $t \in (t_1, t_2]$.

The function $v_1(t)$ satisfies the inequality

$$v_1(t) \le \left((1+\beta_1)v_0(t_1) + \frac{p-1}{p}\int_{t_1}^t (h(s)+r(s))ds\right) \times \exp\left(\int_{t_1}^t \left(f(t)+g(t)+\frac{h(s)+r(s)}{p}\right)ds\right)$$
$$\le (1+\beta_1)\left(c + \frac{p-1}{p}\int_0^t (h(s)+r(s))ds\right) \times \exp\left(\int_0^t \left(f(t)+g(t)+\frac{h(s)+r(s)}{p}\right)ds\right). \tag{3.44}$$

From the inequalities $u(t) \le \sqrt[p]{v_1(t)}$ and (3.44) it follows (3.35) for $t \in (t_1, t_2]$.

Using mathematical induction we can prove the validity of (3.35) for $t \ge 0$.

Case 2. Let there exist a natural number m such that $t_m \le h < t_{m+1}$. As in Case 1 we can prove the validity of the inequality (3.35), using functions $v_k(t)$, $k = 1, 2, \dots$ defined by the inequality

$$v_k(t) = v_{k-1}(t_k) + \int_{t_k}^t [f(s)u^p(t) + g(s)u^q(s)u^{p-q}(s-h) + h(s)u(s) + r(s)u(s-h)]ds + \beta_k u^p(t_k). \tag{3.45}$$

□

Remark 3.7. Some of the inequalities proved by B.G. Pachpatte in [5], [6], [7] are partial cases of Theorem 3.1 and Theorem 3.3.

4. APPLICATIONS

We will use above results to obtain bounds for the solutions of different type of equations.

Example 4.1. Consider the nonlinear impulsive integral equation with delay

$$u(t) = f(t) + \left[\int_0^t p(s)\sqrt{u(s)}ds + \int_0^t g(s)\sqrt{u(s-h)}ds\right]^2 \tag{4.1}$$
$$+ \sum_{0<t_k<t} \beta_k u(t_k), \qquad \text{for} \quad t \geq 0,$$

$$u(t) = 0 \quad t \in [-h, 0], \tag{4.2}$$

where $\beta_k \geq 0,\ k = 1, 2, \dots,\ p, g \in C([0,\infty), [0,\infty)),\ f \in C([0,\infty), [0,1]), h > 0.$

We note the solutions of the problem (4.1), (4.2) are nonnegative.

Define the functions $G(u) = u^2, Q(u) = \sqrt{u}$. Then $Q \in W_2(\phi)$, where $\phi(u) = \sqrt{u}$.

Consider the function

$$H(u) = \int_0^u \frac{ds}{Q(1+G(s))} = \int_0^u \frac{ds}{\sqrt{1+s^2}} = ln(u + \sqrt{1+u^2}). \tag{4.3}$$

Then the inverse function of $H(u)$ will be defined by

$$H^{-1}(u) = \sinh(u) = \frac{1}{2}(e^u - e^{-u}). \tag{4.4}$$

According to Corollary 3.2 we obtain the upper bound for the solution

$$u(t) \leq \prod_{0<t_k<t}(1+\beta_k)\left\{1 + \left[\sinh\left(\sqrt{\prod_{0<t_k<t}(1+\beta_k)\int_0^t (p(s)+\Lambda g(s))ds}\right)\right]^2\right\}.$$

Example 4.2. Consider the initial value problem for the nonlinear impulsive integro-differential equation

$$u'(t) = 2f(t)\sqrt{u(t)}\int_0^t f(s)\sqrt{u(s)}ds, \quad t > 0, t \neq t_k, \tag{4.5}$$

$$u(t_k + 0) = \beta_k u(t_k), \tag{4.6}$$

$$u(0) = c, \tag{4.7}$$

where $c \geq 0, \beta_k \geq 0, k = 1, 2, \dots, f \in C([0,\infty), [0,\infty))$.

The solutions of the given problem satisfy the inequality

$$u(t) \leq c + \left[\int_0^t f(s)\sqrt{u(s)}ds\right]^2 + \sum_{0<t_k<t}\beta_k u(t_k), \quad t_k \geq 0. \tag{4.8}$$

Define the functions $G(u) = u^2, Q(u) = \sqrt{u}$. Then the functions $H(u)$ and $H^{-1}(u)$ are defined by the equations (4.3) and (4.4).

We note that the solutions of the problem (4.5) – (4.7) are nonnegative. According to Theorem 3.4 the solutions satisfy the estimate

$$u(t) \leq A\prod_{0<t_k<t}(1+A\beta_k)\left\{1 + \left[sh\left(\sqrt{A\prod_{0<t_k<t}(1+A\beta_k)}\int_0^t f(s)ds\right)\right]^2\right\}, \tag{4.9}$$

where $A = \max\{1, c\}$.

Example 4.3. Consider the initial value problem for the nonlinear impulsive differential-difference equation

$$u'(t) = F(t, u(t), u(t-h)) + r(t), \quad t > 0, t \neq t_k, \tag{4.10}$$

$$u(t_k + 0) = \beta_k u(t_k), \tag{4.11}$$

$$u(t) = \psi(t), \quad t \in [-h, 0], \tag{4.12}$$

where $\beta_k =$const, $k = 1, 2, \dots,$ $r \in C([0,\infty), \mathbb{R})$, $\psi \in C([-h, 0], \mathbb{R})$, $F \in C([0,\infty) \times \mathbb{R}^2, \mathbb{R})$ and there exist functions $p, g \in C([0,\infty), [0,\infty))$ such that $|F(t,u,v)| \leq p(t)\sqrt{|u|} + g(t)\sqrt{|v|}$. Let $|\psi(0) + \int_0^t r(s)ds| \leq 1, \ \ t \geq 0$.

We assume that the solutions of the initial value problem (4.10) – (4.12) exist on $[0,\infty)$.

The solution satisfies the inequalities

$$\begin{aligned} |u(t)| \leq \left|\psi(0) + \int_0^t r(s)ds\right| + \int_0^t (p(s)\sqrt{|u(s)|} + g(s)\sqrt{|u(s-h)|})ds \\ + \sum_{0<t_k<t} |\beta_k|.|u(t_k)|, \quad t > 0, \end{aligned} \tag{4.13}$$

$$|u(t)| = |\psi(t)|, \quad t \in [-h, 0]. \tag{4.14}$$

Consider functions $f_1(t) = |\psi(0) + \int_0^t r(s)ds|$, $G(u) = u$, $Q(u) = \sqrt{u}$, $f_2(t) = 1$, $f_3(t) = 1$. Then $\varphi(u) = \sqrt{u}$ and according to the equality (3.5) the function $H(u) = 2(\sqrt{1+u} - 1)$ and its inverse is $H^{-1}(u) = (\frac{u}{2} + 1)^2 - 1$.

According to Corollary 3.2 the following bound for the solution of (4.10) – (4.12)

$$\begin{aligned} |u(t)| \leq \prod_{0<t_k<t} (1 + |\beta_k|) \Bigg(\sqrt{1 + hB_1\sqrt{B_2}} \\ + \frac{1}{2}\sqrt{\prod_{0<t_k<t}(1+|\beta_k|)} \int_0^t (p(s) + \Lambda g(s) ds \Bigg)^2, \end{aligned} \tag{4.15}$$

is satisfied, where $B_1 = \max\{|g(s)| : s \in [0, h]\}$, $B_2 = \max\{\psi(s) : s \in [-h, 0]\}$.

Example 4.4. Consider the initial value problem for the nonlinear impulsive differential-difference equation

$$u(t)u'(t) = F(t, u(t), u(t-h)) + q(t)u(t) + r(t)u(t-h), \quad t > 0, t \neq t_k, \tag{4.16}$$

$$u(t_k + 0) = \beta_k u(t_k), \tag{4.17}$$

$$u(t) = \psi(t), \quad t \in [-h, 0], \tag{4.18}$$

where $\beta_k = const, k = 1, 2, \dots, r, q \in C([0,\infty), \mathbb{R})$, $\psi \in C([-h, 0], \mathbb{R})$, $F \in C([0,\infty) \times \mathbb{R}^2, \mathbb{R})$ and there exist functions $p, g \in C([0,\infty), [0,\infty))$ such that $|F(t,u,v)| \leq p(t)u^2 + g(t)v^2$. Let $|\psi(0) + \int_0^t r(s)ds| \leq 1, \ \ t \geq 0$. We assume that the solutions of the initial value problem (4.10) – (4.12) exist on $[0,\infty)$.

The solution satisfies the inequalities

$$\begin{aligned} u(t)^2 \leq \psi^2(0) + \int_0^t \Big(p(s)u^2(s) + g(s)u^2(s-h) + q(s)u(s) + r(s)u(s-h) \Big) ds \\ + \sum_{0<t_k<t} \beta_k^2 u^2(t_k), \quad t > 0, \end{aligned} \tag{4.19}$$

$$u^2(t) = \psi^2(t), \quad t \in [-h, 0]. \tag{4.20}$$

Apply Theorem 3.6 to the inequalities (4.18), (4.19) and obtain the following upper bound for the solution of the initial value problem (4.15) – (4.17)

$$u(t)^2 \le \psi^2(0) + \int_0^t \Big(p(s)u^2(s) + g(s)u^2(s-h) + q(s)u(s) + r(s)u(s-h) \Big) ds + \sum_{0<t_k<t} \beta_k^2 u^2(t_k), \quad t > 0, \tag{4.21}$$

$$u^2(t) = \psi^2(t), \quad t \in [-h, 0]. \tag{4.22}$$

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