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A NON LOCAL QUANTITATIVE CHARACTERIZATION OF ELLIPSES LEADING TO A SOLVABLE DIFFERENTIAL RELATION

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ABSTRACT. In this paper we prove that there are no domains $\mathcal{E} \subset \mathbb{R}^2$, other than the ellipses, such that the Lebesgue measure of the intersection of $\mathcal{E}$ and its homothetic image $\varepsilon\mathcal{E}$ translated to a boundary point $q \in \partial\mathcal{E}$ is independent of q, provided that $\mathcal{E}$ is "centered" at a certain interior point $O \in \mathcal{E}$ (the center of homothety).
Similar problems arise in various fields of mathematics, including, for example, the study of stationary isothermal surfaces and rearrangements.

Key words and phrases: Convex sets, asymptotic expansion, ordinary differential equations.

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1. INTRODUCTION

In this paper we devote ourselves to the investigation, in two dimensions, of the following problem, which was originally proposed in a more general N-dimensional setting by one of the authors in [4] and up to this moment has remained an open problem.

Problem 1.1. Determine all the open bounded convex sets $\mathcal{E}$ in $\mathbb{R}^2$ for which there exists a point $O \in \mathcal{E}$ such that, for every $\varepsilon > 0$, the measure of the intersection of $\mathcal{E}$ with its homothetic image $\varepsilon\mathcal{E}$ with respect to O, translated to a boundary point q, is independent of q, for every chosen q belonging to the boundary of $\mathcal{E}$.

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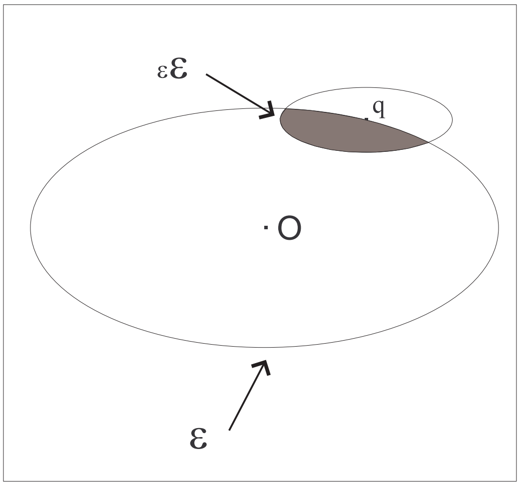

Figure 1.1: The area of the shaded region $\varepsilon\mathcal{E}$ is independent of q.

In other words, we are interested in determining all the open bounded convex sets $\mathcal{E}$ in $\mathbb{R}^2$ satisfying the following property:

$$\forall\varepsilon > 0 \quad \exists C = C(\varepsilon) > 0 \qquad s.t. \qquad |\mathcal{E} \cap [\varepsilon\mathcal{E} + (q - O)]| = C \qquad \forall q \in \partial\mathcal{E}\,, \tag{1.1}$$

with C independent of q (see Fig. 1.1).

In fact, we will answer this question by solving a more general problem:

Problem 1.2. Determine all those open bounded convex sets $E \subset \mathbb{R}^2$ such that there exists an open bounded convex set $\mathcal{E} \subset \mathbb{R}^2$, with the property that the measure of the intersection $\mathcal{E} \cap [\varepsilon E + (q - O)]$ is independent of q, for any $q \in \partial\mathcal{E}$, i.e.

$$\forall\varepsilon > 0 \quad \exists C = C(\varepsilon) > 0 \qquad s.t. \qquad |\mathcal{E} \cap [\varepsilon E + (q - O)]| = C \qquad \forall q \in \partial\mathcal{E}\,, \tag{1.2}$$

where C is independent of q and O is a suitable interior point of E.

We will prove that, assuming sufficient regularity for the sets E and $\mathcal{E}$, the only sets E for which property (1.2) is satisfied are the ellipses. Hence, if a solution to Problem 1.1 exists, it must be an ellipse (thus giving uniqueness). On the other hand, homothetic ellipses clearly satisfy (1.1). Indeed, if $\mathcal{E}$ and E are two discs, (1.1) is obviously satisfied, and the homothetic ellipses case can be reduced to this last one, by means of a proper dilatation, under which our problem is invariant.

Actually, we will show that, in Problem 1.2, $\mathcal{E}$ must be an ellipse as well (see Corollary 2.4). This result is not trivial for $N > 2$ and it is obtained in [7].

The result proved here strongly suggests that also in $\mathbb{R}^N$ the only admissible convex sets E should be the ellipsoidal domains. This multidimensional version of our result will be the object of future investigations.

It is worthwhile to point out that the assumption that $\mathcal{E}$ is bounded is crucial since, otherwise, many more cases appear. For example, in $\mathbb{R}^2$, when $\mathcal{E}$ is the half plane, E can be any bounded set, or in $\mathbb{R}^3$, when E is a sphere, many classes of unbounded domains $\mathcal{E}$ are admissible (see [7]).

The problem treated in this paper, though interesting in itself, is strangely related to some other problems appearing in different contexts. For example, in [7] the authors show that the domains $\mathcal{E}$ satisfying (1.2), where E is a sphere, are related to the determination of stationary isothermic surfaces. They prove that, in the bounded case, the only admissible sets $\mathcal{E}$ must be the spheres, while, in the unbounded case, the admissible sets $\mathcal{E}$ are classified, as recalled above.

The result obtained in [7] suggests many possible extensions, among which the one studied in this paper is definitely the most general, at least in the two dimensional setting.

Another possible application of the result obtained in this paper is in connection with rearrangements (see [3]), with the aim of deriving a generalized version of the Riesz-Sobolev type inequality making use of the Hardy-Littlewood inequality (see [5]).

An abstract version of the Riesz-Sobolev inequality can be written in the form

$$\int_{\mathbb{R}^N} (f \star g)(x) h(x)\, dx \leq \int_{\mathbb{R}^N} \left(f^{\#\mathcal{B}} \star g^{\#\mathcal{B}}\right)(x) h^{\#\mathcal{B}}(x)\, dx\,, \tag{1.3}$$

where $\mathcal{B} = \{B_r : r \in \mathbb{R}^+\}$ is the family of all the homothetic sets of a given *open convex neighborhood* of the origin with compact closure, and, for any measurable function ϕ with level sets of finite measure, $\phi^{\#\mathcal{B}}$ is its *$\mathcal{B}$-rearrangement*, i.e

$$\phi^{\#\mathcal{B}}(x) = \sup\left\{\lambda > 0 : x \in (\phi_\lambda)^{\#\mathcal{B}}\right\}, \tag{1.4}$$

where $(\phi_\lambda)^{\#\mathcal{B}}$ is the *$\mathcal{B}$-rearrangement* of the sublevel $\phi_\lambda := \{x \in \mathbb{R}^N \;:\; \phi(x) < \lambda\}$ (see, for instance, [2], [6], [8]). Here, f, g, h are measurable functions on $\mathbb{R}^N$ and $\star$ denotes the convolution products.

In the first place, an argument based on linearity reduces the task of proving inequality (1.3) for $\mathcal{B}$-rearrangements to the proof of its validity in the case of positive step functions. In particular, we have to prove such an inequality for the case

$$I := \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \chi_{B_r}(x-y) f(y) \chi_B(x)\, dxdy,$$

(see the beginning of pg. 24 in [6]). A simple calculation shows that, in this case, inequality (1.3) is clearly satisfied when, for example,

$$|(B_r + y) \cap B| = \left|[(B_r + y) \cap B]^{\#\mathcal{B}}\right|\,. \tag{1.5}$$

But (1.5) holds if and only if, for every chosen $r > 0$, there exists $C > 0$ such that

$$|B \cap (x + B_r)| = C \qquad \text{when} \qquad x \in \partial B_r\,.$$

In this paper, it is proved that this last property holds only for ellipsoidal domains.

We conclude by observing that the proof of our main theorem strongly relies on the McLaurin expansion, with respect to ε, of the function $\varepsilon \mapsto A(\varepsilon, q) := |\mathcal{E} \cap (\varepsilon E + q)|$, which allows us to obtain a particular differential equation, satisfied by any E having property (1.2). This particular technique connects our problem to other related ones, already studied by the authors (see, e.g., [1]).

The paper is organized as follows: in Section 2 we give the definition of a "proper testing set" and state our main result (see Theorem 2.1), with its consequences. In Section 3 we give the McLaurin expansion, with respect to ε, up to the fifth order, of the area function $\varepsilon \mapsto A(\varepsilon, q)$, defined above (see Propositions 3.1 and 3.2). Finally, in Section 4 we prove the main theorem. A Section 5, with the conclusions and some final remarks, is added.

2. POSITION OF THE PROBLEM

Let $\mathcal{E}$ and E be two bounded convex subsets of $\mathbb{R}^N$, with $|E| = 1$. Let O be a point in the interior of E and εE be the set

$$\varepsilon E := \{y \in \mathbb{R}^N \;:\; y = \varepsilon(x - O) \text{ with } x \in E\}\,.$$

Finally, for every point $q \in \partial\mathcal{E}$, we denote with $A(\varepsilon, q)$ the Lebesgue measure of the region $\mathcal{E} \cap \varepsilon E_q$, where $\varepsilon E_q = \varepsilon E + (q - O)$. From now on, we will call the set $\mathcal{E}$ the "tested convex set" and the set E the "testing convex set".

In agreement with the notations introduced in [7], we will make use of the following definitions:

Definition 2.1. Given two sets $\mathcal{E}$ and E, we will say that $\mathcal{E}$ is uniformly E-dense on its boundary if $A(\varepsilon, q)$ does not depend on $q \in \partial\mathcal{E}$. In this case, E will be called a "proper testing set".

In this regard, the question arises of whether it is possible to characterize the convex sets E, together with the point O (which will be later chosen as the origin of both the cartesian axis and the polar coordinates), for which a convex set $\mathcal{E}$, uniformly E-dense on its boundary, exists.

In the N-dimensional setting, the problem has been treated by Magnanini, Prajapat and Sakaguchi in [7], where it is proved that, if E is a sphere, then it is a proper testing set and, in this case, $\mathcal{E}$ must be a sphere, too. In the 2-dimensional case this property is a consequence of Proposition 3.2, as it is stated in Corollary 3.3 (see Section 3).

Remark 1. In general, it is possible to prove that any ellipsoid is a proper testing set. This can be easily obtained observing that the problem is invariant under dilatation of the axes under which any ellipsoid can be reduced to a sphere. Clearly, in this case the point O must be the center of the testing ellipsoid and the tested convex set is, up to a translation, homothetic to the testing one.

Nevertheless, the problem of determining all the proper testing sets remains open. In this paper, this problem will be solved for the case $N = 2$, for tested convex sets of class $\mathcal{C}^4$ and testing convex sets of class $\mathcal{C}^2$, as stated in Theorem 2.1 below.

From now on, we assume $N = 2$.

Theorem 2.1. *Let $\mathcal{E}$ and E be a tested set and a testing convex set of class $\mathcal{C}^4$ and $\mathcal{C}^2$, respectively. If the McLaurin expansion up to the fifth order, with respect to ε, of the function $A(\varepsilon, q) = |\mathcal{E} \cap [\varepsilon E + (q - O)]|$ has coefficients which do not depend on $q \in \partial\mathcal{E}$, then E must be an ellipse and O must be its center.*

Corollary 2.2. *The only proper testing sets of class $\mathcal{C}^2$ are the ellipses.*

Proof. It is a direct consequence of the previous theorem since, if E is a proper testing set, by definition the function $A(\varepsilon, q)$ does not depend on q, so that its fifth order power expansion also does not depend on q. □

Corollary 2.3. *The ellipses Ω are the only sets which are uniformly $\lambda\Omega$-dense on their boundary, where $\lambda = 1/|\Omega|$ (see Definition 2.1 with $\mathcal{E} = \Omega$ and $E = \lambda\Omega$).*

Proof. It is a direct consequence of Corollary 2.2. □

Corollary 2.4. *Let $\mathcal{E}$ and E be a tested set and a testing convex set of class $\mathcal{C}^4$ and $\mathcal{C}^2$, respectively. If $\mathcal{E}$ is uniformly E-dense on its boundary, then E is an ellipse and $\mathcal{E} \equiv \lambda E$, for a suitable $\lambda > 0$.*

Proof. From Corollary 2.2, we get that E is an ellipse. Since the problem is invariant under dilatation of the axes, we can perform a proper dilatation Λ in such a way that E is transformed in a circle $\Lambda(E)$. Using the forthcoming Corollary 3.3, we have that $\Lambda(\mathcal{E})$ is a circle, too. Hence, $\mathcal{E}$ is an ellipse homothetic to E. □

3. PRELIMINARY RESULTS

Let us now fix a system (x, y) of cartesian coordinates and let (θ, ρ) be the associated polar coordinates (in which $\theta = 0$ corresponds to the positive x-axis), centered in the point O belonging to the interior of E. In the following, we will use a local cartesian representation for the tested convex set $\mathcal{E}$, while for the testing convex set E we will use a global polar representation $\rho = \rho(\theta)$, $0 \leq \theta \leq 2\pi$. Moreover, $\mathcal{E}$ and E are always assumed to be of class $\mathcal{C}^4$ and $\mathcal{C}^2$, respectively.

Given a unit vector $\nu \in \mathbb{S}^1$, we set $C(\nu)$ as the area of the portion of the plane, not containing the vector ν, bounded by E and by the straight line orthogonal to ν passing through the origin.

Proposition 3.1. *The second order McLaurin expansion of the function $A(\varepsilon, q)$ with respect to ε is given by*

$$A(\varepsilon, q) = C(\nu(q))\varepsilon^2 + o(\varepsilon^2)\,, \tag{3.1}$$

where $\nu(q)$ is the outward unit normal vector to $\mathcal{E}$ in q.

Moreover, such a power expansion does not depend on q if and only if the testing convex set E is centrally symmetric with respect to O; i.e., $\rho(\theta) = \rho(\theta + \pi)$ for every $\theta \in \mathbb{R}$. Obviously, in this case, $C(\nu(q)) = 1/2$.

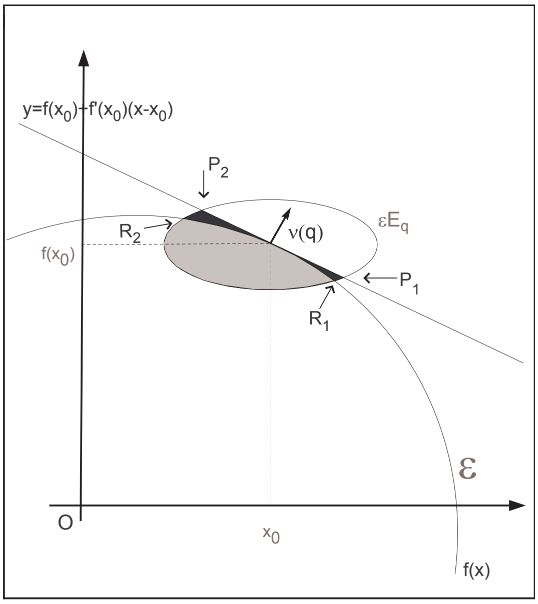

Figure 3.1: $q = \big(x_0, f(x_0)\big)$, $A(\varepsilon, q)$ is the area of the grey region and $\mathcal{D}(\varepsilon, q)$ is the area of the black region.

Proof. Since $A(\varepsilon, q) = |\mathcal{E} \cap \varepsilon E_q|$ and the diameter of εE_q is of the order ε, the first term in the expansion of $A(\varepsilon, q)$ is of order ε^2. Moreover, keeping account of this fact, it is clear that we can locally approximate the arc $\widehat{R_2 q R_1}$ with the segment $\overline{P_2 P_1}$, up to an error of order ε^2 (see Figure 3.1); thus producing in the computation of $A(\varepsilon, q)$ an error of order ε^3, which does not affect the second order McLaurin expansion.

This implies that $A(\varepsilon, q) = C(\nu(q))\varepsilon^2 + o(\varepsilon^2)$. Clearly, if the second order power expansion of $A(\varepsilon, q)$ does not depend on q, the function $C(\nu(q))$ also does not depend on q. Rewriting

$C(\nu(q))$ in terms of the angle ϕ, between the normal $\nu(q)$ and the positive x-axis, and calling this new function $\tilde{C}(\phi)$, we have that it is constant if and only if

$$\begin{aligned} 0 = \tilde{C}'(\phi) &= \frac{\tilde{C}(\phi + d\phi) - \tilde{C}(\phi)}{d\phi} \\ &= \frac{\rho^2(\phi + 3\pi/2)\, d\phi - \rho^2(\phi + \pi/2)\, d\phi}{d\phi} \end{aligned}$$

which implies $\rho(\phi + 3\pi/2) = \rho(\phi/2)$. Since the boundary of $\mathcal{E}$ is a closed connected simple curve, ϕ attains any value in $[0, 2\pi)$, as q varies on $\partial\mathcal{E}$. Consequently, $\rho(\theta + \pi) = \rho(\theta)$; i.e., E is centrally symmetric with respect to O. Clearly, in this case, $C(\nu(q)) = \tilde{C}(\phi) = \frac{1}{2}|E| = 1/2$. □

Having found the second order expansion of $A(\varepsilon, q)$, we will now devote our attention to determining its fifth order expansion. To this purpose, given the convex set $\mathcal{E}$, let us assume that $y = f(x)$ is a local parametrization of class $\mathcal{C}^4$ of $\partial\mathcal{E}$, in a neighborhood of q, such that $q = (x_0, f(x_0))$.

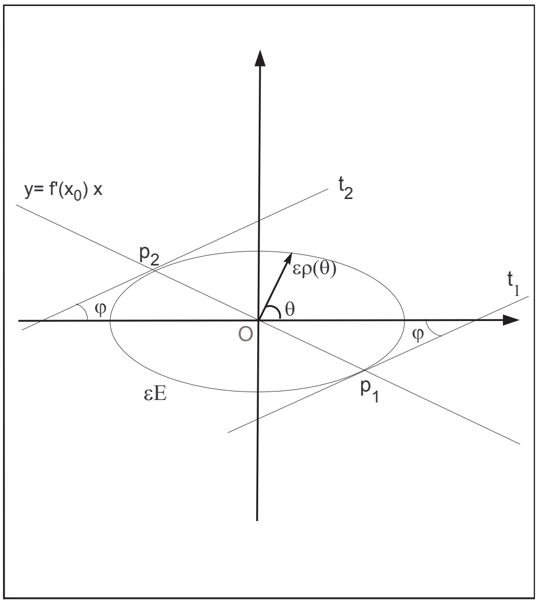

Figure 3.2: $\alpha = \tan\varphi$.

Let t_1 and t_2 be the tangent lines (in their cartesian representation) to εE at the points (expressed in polar coordinates)

$$\begin{aligned} p_1 &= (\arctan f'(x_0), \varepsilon\rho\big(\arctan f'(x_0)\big)) \\ p_2 &= (\arctan f'(x_0) + \pi, \varepsilon\rho\big(\arctan f'(x_0) + \pi\big)) \,. \end{aligned}$$

Because of the central symmetry we have

$$\rho\big(\arctan f'(x_0) + \pi\big) = \rho\big(\arctan f'(x_0)\big)$$

and $t_1 \parallel t_2$.

We denote by α the angular coefficient of the tangent line t_1 to εE at the point p_1. Straightforward computations give the following expression for α:

$$\alpha = \frac{\rho'(\theta_0)\sin\theta_0 + \rho(\theta_0)\cos\theta_0}{\rho'(\theta_0)\cos\theta_0 - \rho(\theta_0)\sin\theta_0}\,, \tag{3.2}$$

where $\theta_0 = \arctan f'(x_0)$ (see Figure 3.2).

Let $P_1, P_2 \in \varepsilon E_q$ be the corresponding points of $p_1, p_2 \in \varepsilon E$ and S_1 and S_2 be the intersection points of the tangent lines to εE_q at P_1 and P_2 with the curve whose equation is

$$\begin{aligned} y &= T_{(x_0,4)}(x) \\ &:= f(x_0) + f'(x_0)(x-x_0) + f''(x_0)\frac{(x-x_0)^2}{2} + \frac{f'''(x_0)(x-x_0)^3}{3!} + \frac{f^{(iv)}(x_0)(x-x_0)^4}{4!} \end{aligned}$$

(i.e. the fourth order expansion of $\mathcal{E}$).

Finally, $t_1 + q$ and $t_2 + q$ are the tangent lines, obtained translating the lines t_1 and t_2 by adding the vector $(q - O)$ (see Figure 3.3).

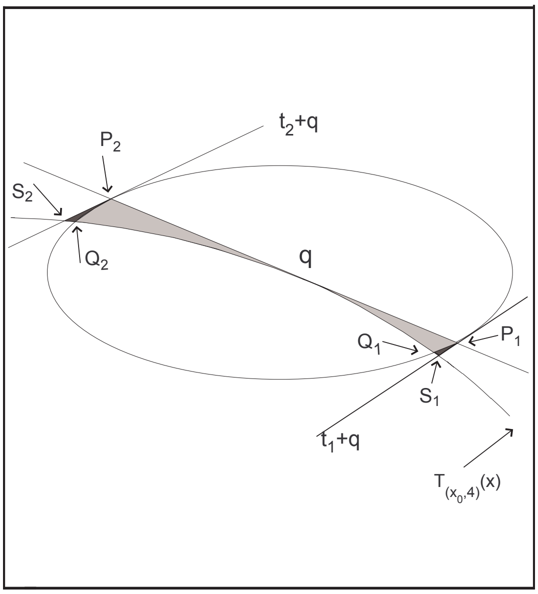

Figure 3.3: $\mathcal{C}_1(\varepsilon, q)$ is the area of the grey region, while $\mathcal{C}_2(\varepsilon, q) - \mathcal{C}_1(\varepsilon, q)$ is the area of the black region.

Proposition 3.2. *Let us assume that E is centrally symmetric with respect to O. Then the fifth order McLaurin expansion of the function $A(\varepsilon, q)$ with respect to ε is given by*

$$A(\varepsilon, q) = \frac{1}{2}\varepsilon^2 + C_3(q)\varepsilon^3 + C_5(q)\varepsilon^5 + o(\varepsilon^5)\,, \tag{3.3}$$

where

$$C_3(q) = \frac{f''(x_0)}{3\left[1+\big(f'(x_0)\big)^2\right]^{3/2}}\,\rho^3\big(\arctan f'(x_0)\big)\,; \tag{3.4}$$

$$C_5(q) = \left[\frac{\big(f''(x_0)\big)^3}{4\big(\alpha - f'(x_0)\big)^2} + \frac{f''(x_0)f'''(x_0)}{6\big(\alpha - f'(x_0)\big)} + \frac{f^{(iv)}(x_0)}{60}\right]\,\frac{\rho^5\big(\arctan f'(x_0)\big)}{\left[1+\big(f'(x_0)\big)^2\right]^{5/2}}\,; \tag{3.5}$$

and the term of fourth order is zero.

Remark 2. It is a straightforward computation to prove that the ellipses $C_3(q)$ and $C_5(q)$ given in (3.4) and (3.5) are actually constants independent of q.

Proof. It is clear that, if ε is sufficiently small, the difference $\mathcal{D}(\varepsilon, q)$ between the area $A(\varepsilon, q)$ of $\mathcal{E} \cap \varepsilon E_q$ and its second order expansion is given by the area (with the minus sign) of that portion of εE_q in between $f(x)$ and the line $y = f(x_0) + f'(x_0)(x - x_0)$ (i.e. the black region $P_2P_1R_1R_2$ in Figure 3.1).

Since we are looking for the fifth order expansion of $A(\varepsilon, q)$, we can locally (i.e. in a neighborhood of q) replace the cartesian representation $(x, f(x))$ of $\mathcal{E}$ by means of its fourth order Taylor expansion $T_{(x_0,4)}(x)$, centered in x_0 (in this regard, we use the fact that the length $|P_1P_2|$ is of order ε).

Henceforth, $\mathcal{D}(\varepsilon, q) = -\mathcal{C}_1(\varepsilon, q) + o(\varepsilon^5)$, where $\mathcal{C}_1(\varepsilon, q)$ is the area of that portion of εE_q in between $y = T_{(x_0,4)}(x)$ and the line $y = f(x_0) + f'(x_0)(x - x_0)$ (i.e. the grey region $P_2P_1Q_1Q_2$ in Figure 3.3).

Nevertheless, $\mathcal{C}_1(\varepsilon, q)$ cannot be easily computed; for this reason we need a further approximation which, however, does not affect the fifth order of the McLaurin expansion of $\mathcal{C}_1(\varepsilon, q)$. To this purpose, we replace the boundary of εE_q with the tangent lines $t_1 + q$ and $t_2 + q$. Accordingly, we denote by $\mathcal{C}_2(\varepsilon, q)$ the area of the region thus obtained, which is bounded by the graph of the function $y = T_{(x_0,4)}(x)$ and by the lines $y = f(x_0) + f'(x_0)(x - x_0)$, $t_1 + q$ and $t_2 + q$, i.e. the grey region together with the black one in Figure 3.3.

We claim that $\mathcal{C}_1(\varepsilon, q) = \mathcal{C}_2(\varepsilon, q) + o(\varepsilon^5)$. This is mainly due to the following facts:

(1) Firstly, $|(P_1 - q) \wedge (P_1 - S_1)| \geq \eta > 0$, for every $q \in \partial\mathcal{E}$, with η independent of q. Indeed, if this is not the case, due to the compactness of E, there will be a point q for which the tangent line $t_1 + q$ to εE_q at the corresponding point P_1 will coincide with the tangent line $\overline{P_1P_2}$ to $\mathcal{E}$. Consequently, all E should stay either on the left or on the right side of the line $\overline{P_1P_2}$, in contrast with the central symmetry of εE_q with respect to q, proved in Proposition 3.1.

(2) Secondly, the length $|P_1S_1|$ is of order ε^2. This is a consequence of the fact that the difference between the abscissae of P_1 and S_1 is of order ε^2, as it can be seen using (3.16) below (with δ replaced by δ_0 as given in (3.9)), provided that $|\alpha - f'(x_0)| \geq \tilde{\eta} > 0$. This final inequality is guaranteed by (1).

(3) Using (1) and (2), it is easy to realize that the area of the black region $P_1S_1Q_1$ in Figure 3.3 can be bounded from above by the integral (with respect to a cartesian reference frame attached to the line $t_1 + q$) of the function whose graphs gives the profile of εE_q (which, in the cartesian representation, is clearly a function of second order) along the interval $|P_1S_1| \sim \varepsilon^2$. Henceforth, such area is $O(\varepsilon^6)$. Obviously the same holds for the black region $P_2S_2Q_2$.

Having proved the claim, we now evaluate the area $\mathcal{C}_2(\varepsilon, q)$.

To this purpose, let us consider the line $y = \alpha(x - x_0) - (\delta - f(x_0))$ which is parallel to $t_1 + q$ and $t_2 + q$. Moreover, let us call $P(\delta)$ the intersection point between the two lines $y = f(x_0) + f'(x_0)(x - x_0)$ and $y = \alpha(x - x_0) - (\delta - f(x_0))$ and $S(\delta)$ the intersection point between the line $y = \alpha(x - x_0) - (\delta - f(x_0)))$ and $y = T_{(x_0,4)}(x)$ (see Figure 3.4).

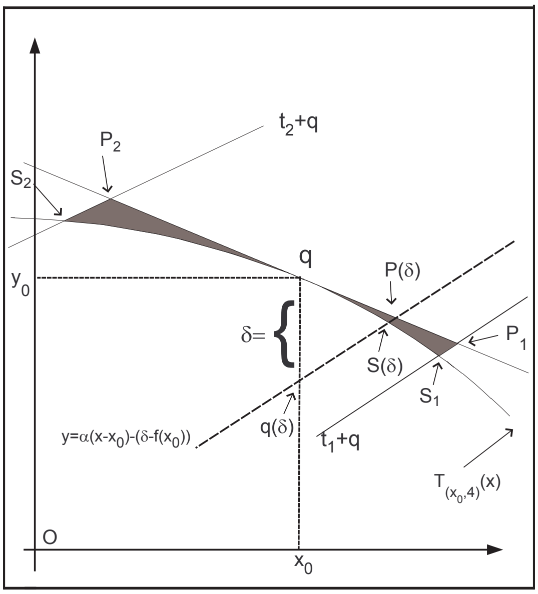

Figure 3.4: $\mathcal{C}_2(\varepsilon, q)$ is the area of the shaded region.

Clearly, the x-coordinate X_P of the point $P(\delta)$ is given by

$$\alpha(X_P - x_0) - (\delta - f(x_0)) = f(x_0) + f'(x_0)(X_P - x_0) \implies X_P - x_0 = \frac{\delta}{\alpha - f'(x_0)}. \tag{3.6}$$

In particular, we set δ_0 to be the value of the parameter δ for which $P(\delta_0) = P_1$ and $P(-\delta_0) = P_2$; consequently, $S(\delta_0) = S_1$ and $S(-\delta_0) = S_2$.

Keeping in mind that the angular coefficient of the line $\overline{P_1P_2}$ is $f'(x_0)$, by (3.6) we get

$$|P(\delta_0) - q| = \frac{\delta_0}{\alpha - f'(x_0)}\sqrt{1 + \left(f'(x_0)\right)^2}. \tag{3.7}$$

On the other hand,

$$|P(\delta_0) - q| = |P_1 - q| = |p_1| = \varepsilon\rho\big(\arctan f'(x_0)\big) \qquad \text{(see Figure 3.4)}, \tag{3.8}$$

hence, by (3.7) and (3.8), it follows that

$$\delta_0 = \frac{\left(\alpha - f'(x_0)\right)\rho\big(\arctan f'(x_0)\big)}{\sqrt{1 + \left(f'(x_0)\right)^2}}\,\varepsilon. \tag{3.9}$$

Moreover, the x-coordinate X_S of the point $S(\delta)$ is obtained by solving the following algebraic equation:

$$\alpha(X_S - x_0) - (\delta - f(x_0)) = f(x_0) + f'(x_0)(X_S - x_0) + \frac{f''(x_0)(X_S - x_0)^2}{2} + \frac{f'''(x_0)(X_S - x_0)^3}{3!} + \frac{f^{(iv)}(x_0)(X_S - x_0)^4}{4!}, \tag{3.10}$$

which gives

$$X_S - x_0 = \frac{\delta}{\alpha - f'(x_0)} + \frac{f''(x_0)}{2[\alpha - f'(x_0)]}(X_S - x_0)^2 + \frac{f'''(x_0)}{3![\alpha - f'(x_0)]}(X_S - x_0)^3 + \frac{f^{(iv)}(x_0)}{4![\alpha - f'(x_0)]}(X_S - x_0)^4 . \tag{3.11}$$

This is a non trivial computation. For this reason, we confine ourselves to finding the fourth order McLaurin expansion with respect to δ of $X_S - x_0$, i.e.:

$$X_S - x_0 = D_1(x_0)\delta + D_2(x_0)\delta^2 + D_3(x_0)\delta + D_4(x_0)\delta^4 + o(\delta^4) ,$$

which is, however, enough to carry on all the other computations of this paper.

Firstly, let us observe that $X_S - x_0 = O(\delta)$ and hence,

$$\text{(at the 1st order)} \qquad X_S - x_0 = \left[\frac{1}{\alpha - f'(x_0)}\right] \delta + o(\delta) . \tag{3.12}$$

Replacing (3.12) in the right hand side of (3.11), we get

$$\text{(at the 2nd order)} \qquad D_2(x_0) = \left[\frac{f''(x_0)}{2(\alpha - f'(x_0))^3}\right] . \tag{3.13}$$

Finally, by means of a standard bootstraps argument, we have

$$\text{(at the 3rd order)} \quad D_3(x_0) = \left[\frac{f'''(x_0)}{3!(\alpha - f'(x_0))^4} + \frac{2\big(f''(x_0)\big)^2}{4(\alpha - f'(x_0))^5}\right] , \tag{3.14}$$

$$\text{(at the 4th order)} \quad D_4(x_0) = \left[\frac{5\big(f''(x_0)\big)^3}{8(\alpha - f'(x_0))^7} + \frac{5f''(x_0)f'''(x_0)}{12(\alpha - f'(x_0))^6} + \frac{f^{(iv)}(x_0)}{4!(\alpha - f'(x_0))^5}\right] . \tag{3.15}$$

Hence,

$$X_P - X_S = -\left[\frac{f''(x_0)}{2(\alpha - f'(x_0))^3}\right] \delta^2 - \left[\frac{f'''(x_0)}{3!(\alpha - f'(x_0))^4} + \frac{2\big(f''(x_0)\big)^2}{4(\alpha - f'(x_0))^5}\right] \delta^3 - \left[\frac{5\big(f''(x_0)\big)^3}{8(\alpha - f'(x_0))^7} + \frac{5f''(x_0)f'''(x_0)}{12(\alpha - f'(x_0))^6} + \frac{f^{(iv)}(x_0)}{4!(\alpha - f'(x_0))^5}\right] \delta^4 + o(\delta^4) . \tag{3.16}$$

This implies, in accordance with Figure 3.5, that $\mathcal{C}_2(\varepsilon, q)$ is obtained by integrating with respect to δ, from $-\delta_0$ to δ_0, the infinitesimal area $d\mathcal{A}(\delta)$ of the shaded region in Fig. 3.5, found by multiplying the base $|P(\delta)S(\delta)| = |X_P - X_S|\sqrt{1 + \alpha^2}$ by the corresponding height, whose

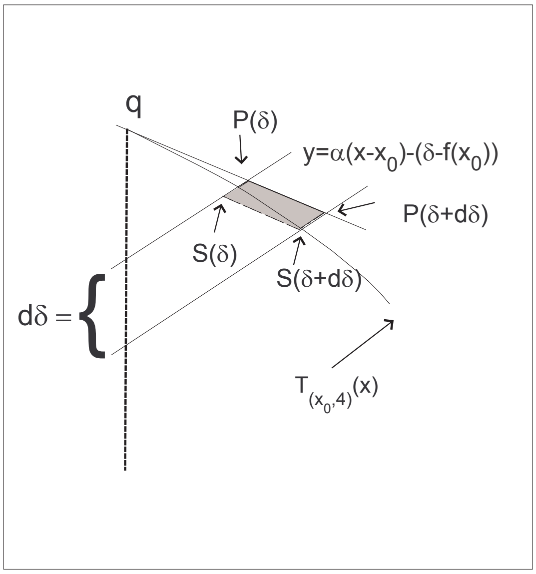

Figure 3.5: The shaded region is the infinitesimal area $d\mathcal{A}(\delta)$.

value is $d\delta/\sqrt{1+\alpha^2}$. Hence, we have

$$\begin{aligned}
\mathcal{C}_2(\varepsilon,q) &= \int_{-\delta_0}^{\delta_0} |X_P - X_S|\, d\delta \\
&= -\left[\frac{f''(x_0)}{3(\alpha - f'(x_0))^3}\right]\delta_0^3 \\
&\quad - \left[\frac{\big(f''(x_0)\big)^3}{4(\alpha - f'(x_0))^7} + \frac{f''(x_0)f'''(x_0)}{6(\alpha - f'(x_0))^6} + \frac{f^{(iv)}(x_0)}{60(\alpha - f'(x_0))^5}\right]\delta_0^5,
\end{aligned} \tag{3.17}$$

and, replacing δ_0 as given by (3.9), it follows that

$$\begin{aligned}
\mathcal{C}_2(\varepsilon,q) &= -\left[\frac{f''(x_0)\rho^3\big(\arctan f'(x_0)\big)}{3\Big(1+\big(f'(x_0)\big)^2\Big)^{3/2}}\right]\varepsilon^3 \\
&\quad - \frac{\rho^5\big(\arctan f'(x_0)\big)}{\Big(1+\big(f'(x_0)\big)^2\Big)^{5/2}}\left[\frac{\big(f''(x_0)\big)^3}{4(\alpha - f'(x_0))^2} + \frac{f''(x_0)f'''(x_0)}{6(\alpha - f'(x_0))} + \frac{f^{(iv)}(x_0)}{60}\right]\varepsilon^5.
\end{aligned} \tag{3.18}$$

Recalling that

$$A(\varepsilon,q) = \frac{1}{2}|\varepsilon E_q| + \mathcal{D}(\varepsilon,q) = \frac{1}{2}\varepsilon^2 - \mathcal{C}_1(\varepsilon,q) + o(\varepsilon^5) = \frac{1}{2}\varepsilon^2 - \mathcal{C}_2(\varepsilon,q) + o(\varepsilon^5)$$

and using (3.18), we finally get the required result. □

Corollary 3.3. *If the proper testing convex set $E \in \mathcal{C}^2$ is a circle, then the tested convex set $\mathcal{E} \in \mathcal{C}^4$ must also be a circle.*

Proof. Since E is a proper testing set, by Definition 2.1 $A(\varepsilon,q)$ is constant. Hence, Proposition 3.2 applied to this particular case, implies

$$\frac{f''(x_0)}{\Big[1+\big(f'(x_0)\big)^2\Big]^{3/2}} = \text{cost}.$$

It follows that the boundary of the tested convex set $\mathcal{E}$ has a positive constant curvature, which, as far as bounded sets are concerned, implies that it is a circle. □

In the case $N \geq 2$, the same result stated in Corollary 3.3 was previously proven in [7, Theorem 1.2].

4. PROOF OF THE MAIN THEOREM

Proof. By (3.4) and (3.5) in Proposition 3.2 and the fact that, by assumption, the McLaurin expansion of the function $A(\varepsilon, q)$ up to the fifth order does not depend on the point $q \in \mathcal{E}$, we obtain

$$C_3 = \frac{f''(x_0)}{3\left[1 + \big(f'(x_0)\big)^2\right]^{3/2}}\, \rho^3\big(\arctan f'(x_0)\big) \, ; \tag{4.1}$$

$$C_5 = \left[\frac{\big(f''(x_0)\big)^3}{4\big(\alpha - f'(x_0)\big)^2} + \frac{f''(x_0)f'''(x_0)}{6\big(\alpha - f'(x_0)\big)} + \frac{f^{(iv)}(x_0)}{60}\right] \frac{\rho^5\big(\arctan f'(x_0)\big)}{\left[1 + \big(f'(x_0)\big)^2\right]^{5/2}} \, ; \tag{4.2}$$

where C_3 and C_5 are now constants independent of q. The next step is to eliminate the function f putting together (4.1) and (4.2), thus obtaining an ordinary differential equation for a new function w defined by

$$w(f') = \frac{\big(1 + (f')^2\big)^{1/2}}{\rho\big(\arctan(f')\big)} \, . \tag{4.3}$$

Note that, now, w is regarded as a function of the new variable f'.

By (4.1), we obtain

$$f''(x) = \frac{C\left[1 + \big(f'(x)\big)^2\right]^{3/2}}{\rho^3\big(\arctan f'(x)\big)} \, , \tag{4.4}$$

which gives

$$f''(x) = Cw^3\big(f'(x)\big) \, . \tag{4.5}$$

Hence, differentiating iteratively the previous equation with respect to x, we get

$$f'''(x) = 3C^2 w^5\big(f'(x)\big) w'\big(f'(x)\big) \, ; \tag{4.6}$$

$$f^{(iv)}(x) = 3C^3 w^7\big(f'(x)\big)\left[5(w')^2\big(f'(x)\big) + w\big(f'(x)\big)w''\big(f'(x)\big)\right] \, . \tag{4.7}$$

Recalling that $f'(x) = \tan\theta$, (4.3) implies

$$\rho(\theta) = \frac{1}{w(\tan\theta)\cos\theta} \, ,$$

$$\rho'(\theta) = \frac{w(\tan\theta)\tan\theta - (1 + \tan^2\theta)w'(\tan\theta)}{w^2(\tan\theta)\cos\theta} \, ,$$

and, by (3.2),

$$\alpha(\theta) = \tan\theta - \frac{w(\theta)}{w'(\theta)} \qquad \Longrightarrow \qquad \alpha(\theta) - f'(x) = -\frac{w(\theta)}{w'(\theta)} \, . \tag{4.8}$$

Replacing (4.3) and (4.5)–(4.8) in (4.2), we get

$$C_5 = \left[\frac{C^3 w^9}{4w^2/(w')^2} - \frac{C^3 w^8 w'}{2w/w'} + \frac{C^3 w^7}{20}\left(5(w')^2 + ww''\right)\right] \cdot \frac{1}{w^5} \, , \tag{4.9}$$

which, after a simplification, gives

$$w''\big(f'(x)\big)w^3\big(f'(x)\big) = \widetilde{C}\,, \tag{4.10}$$

where $\widetilde{C}$ is a proper constant.

From equation (4.10), it easily follows that E has a boundary of class $\mathcal{C}^\infty$.

At this point, using Lemma 4.1 below, with $y(\xi) = w(\xi)$ and $\xi = f'(x) = \tan\theta$, together with (4.3) and (4.14), we get

$$w^2(\tan\theta) = \frac{1+\tan^2\theta}{\rho^2(\theta)} = \frac{\widetilde{C} + (B + 2A\tan\theta)^2}{2A}\,. \tag{4.11}$$

Hence,

$$\rho(\theta) = \sqrt{\frac{2A}{(\widetilde{C} + B^2)\cos^2\theta + 4A^2\sin^2\theta + 4AB\sin\theta\cos\theta}}\,. \tag{4.12}$$

It is well known that equation (4.12) is the polar representation of a conic curve whose center is the origin of the polar coordinates. On the other hand, the testing convex E is a closed curve and hence it must be an ellipse. □

Lemma 4.1. *Let $y(\xi)$ be a $\mathcal{C}^2$-function satisfying the equation*

$$y''(\xi)y^3(\xi) = \widetilde{C}\,. \tag{4.13}$$

Then,

$$y(\xi) = \pm\sqrt{\frac{\widetilde{C} + (B + 2A\xi)^2}{2A}}\,, \tag{4.14}$$

where A and B are two arbitrary constants.

Proof. Introducing the auxiliary function $v(p) = y'\left(y^-1(p)\right)$, with $p = y(\xi)$, the equation (4.13) reduces to

$$\frac{dv(p)}{dp}\cdot v(p) = \frac{\widetilde{C}}{p^3} \qquad \Longrightarrow \qquad v^2(p) = -\frac{\widetilde{C}}{p^2} + 2A,$$

where A is an arbitrary constant. This implies

$$y'(\xi) = \pm\frac{\sqrt{2Ay^2(\xi) - \widetilde{C}}}{y(\xi)}\,.$$

This is a standard ordinary differential equation, whose solution is given by

$$y^2(\xi) = \frac{\widetilde{C} + (B + 2A\xi)^2}{2A}\,.$$

□

5. CONCLUSIONS AND FINAL REMARKS

We want to stress the fact that, though applied to the case in which E and $\mathcal{E}$ are convex sets, the technique used in this paper should work equally well in the case in which E is star-shaped with respect to a point O and its boundary is a simple closed curve such that in any point P, the vector $(P - O)$ and the unit tangent vector $\vec{t}$ in P satisfy the condition $|(P - O) \wedge \vec{t}\,| \geq \delta$, for some $\delta > 0$, while $\mathcal{E}$ has a curvature $k(s)$ (where s is the arc-length) which does not change sign infinitely many times.

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