

Journal of Inequalities in Pure and Applied Mathematics

NEW UPPER AND LOWER BOUNDS FOR THE ČEBYŠEV FUNCTIONAL

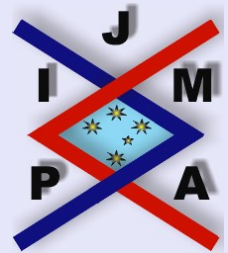
P. CERONE AND S.S. DRAGOMIR

School of Communications and Informatics
Victoria University of Technology
PO Box 14428
Melbourne City MC
8001 Victoria, Australia

*E*Mail: pc@matilda.vu.edu.au
*U*RL: <http://rgmia.vu.edu.au/cerone>

*E*Mail: sever@matilda.vu.edu.au
*U*RL: <http://rgmia.vu.edu.au/SSDragomirWeb.html>

©2000 Victoria University
ISSN (electronic): 1443-5756
048-02



volume 3, issue 5, article 77,
2002.

Received 8 May, 2002;
accepted 5 November, 2002.

Communicated by: F. Qi

Abstract

Contents



Home Page

Go Back

Close

Quit

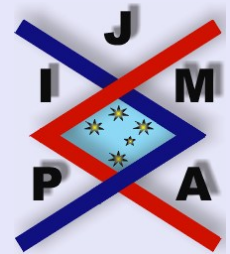
Abstract

New bounds are developed for the Čebyšev functional utilising an identity involving a Riemann-Stieltjes integral. A refinement of the classical Čebyšev inequality is produced for f monotonic non-decreasing, g continuous and $\mathcal{M}(g; t, b) - \mathcal{M}(g; a, t) \geq 0$, for $t \in [a, b]$ where $\mathcal{M}(g; c, d)$ is the integral mean over $[c, d]$.

2000 Mathematics Subject Classification: Primary 26D15; Secondary 26D10.
Key words: Čebyšev functional, Bounds, Refinement.

Contents

1	Introduction	3
2	Integral Inequalities	6
3	More on Čebyšev's Functional	13
	References	



New Upper and Lower Bounds for the Čebyšev Functional

P. Cerone and S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 2 of 26

1. Introduction

For two given integrable functions on $[a, b]$, define the Čebyšev functional ([2, 3, 4])

$$(1.1) \quad T(f, g) := \frac{1}{b-a} \int_a^b f(x) g(x) dx - \frac{1}{b-a} \int_a^b f(x) dx \cdot \frac{1}{b-a} \int_a^b g(x) dx.$$

In [1], P. Cerone has obtained the following identity that involves a Stieltjes integral (Lemma 2.1, p. 3):

Lemma 1.1. *Let $f, g : [a, b] \rightarrow \mathbb{R}$, where f is of bounded variation and g is continuous on $[a, b]$, then the $T(f, g)$ from (1.1) satisfies the identity,*

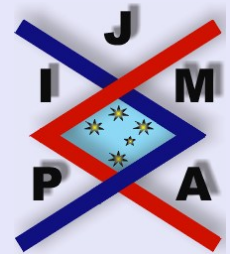
$$(1.2) \quad T(f, g) = \frac{1}{(b-a)^2} \int_a^b \Psi(t) df(t),$$

where

$$(1.3) \quad \Psi(t) := (t-a)A(t, b) - (b-t)A(a, t),$$

with

$$(1.4) \quad A(c, d) := \int_c^d g(x) dx.$$



New Upper and Lower Bounds
for the Čebyšev Functional

P. Cerone and S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 3 of 26

Using this representation and the properties of Stieltjes integrals he obtained the following result in bounding the functional $T(\cdot, \cdot)$ (Theorem 2.5, p. 4):

Theorem 1.2. *With the assumptions in Lemma 1.1, we have:*

$$(1.5) \quad |T(f, g)| \leq \frac{1}{(b-a)^2} \times \begin{cases} \sup_{t \in [a, b]} |\Psi(t)| V_a^b(f), \\ L \int_a^b |\Psi(t)| dt, & \text{for } L - \text{Lipschitzian}; \\ \int_a^b |\Psi(t)| df(t), & \text{for } f \text{ monotonic nondecreasing,} \end{cases}$$

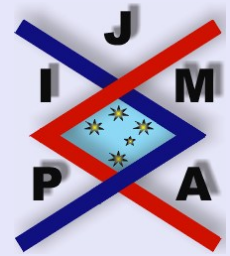
where $V_a^b(f)$ denotes the total variation of f on $[a, b]$.

Cerone [1] also proved the following theorem, which will be useful for the development of subsequent results, and is thus stated here for clarity. The notation $\mathcal{M}(g; c, d)$ is used to signify the integral mean of g over $[c, d]$. Namely,

$$(1.6) \quad \mathcal{M}(g; c, d) := \frac{A(c, d)}{d-c} = \frac{1}{d-c} \int_c^d f(t) dt.$$

Theorem 1.3. *Let $g : [a, b] \rightarrow \mathbb{R}$ be absolutely continuous on $[a, b]$, then for*

$$(1.7) \quad D(g; a, t, b) := \mathcal{M}(g; t, b) - \mathcal{M}(g; a, t),$$



New Upper and Lower Bounds
for the Čebyšev Functional

P. Cerone and S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 4 of 26

2. Integral Inequalities

Now, if we use the function $\varphi : (a, b) \rightarrow \mathbb{R}$,

$$(2.1) \quad \varphi(t) := D(g; a, t, b) = \frac{\int_t^b g(x) dx}{b-t} - \frac{\int_a^t g(x) dx}{t-a},$$

then by (1.2) we may obtain the identity:

$$(2.2) \quad T(f, g) = \frac{1}{(b-a)^2} \int_a^b (t-a)(b-t) \varphi(t) df(t).$$

We may prove the following lemma.

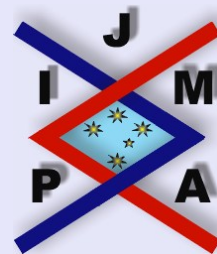
Lemma 2.1. *If $g : [a, b] \rightarrow \mathbb{R}$ is monotonic nondecreasing on $[a, b]$, then φ as defined by (2.1) is nonnegative on (a, b) .*

Proof. Since g is nondecreasing, we have $\int_t^b g(x) dx \geq (b-t)g(t)$ and thus from (2.1)

$$(2.3) \quad \varphi(t) \geq g(t) - \frac{\int_a^t g(x) dx}{t-a} = \frac{(t-a)g(t) - \int_a^t g(x) dx}{t-a} \geq 0,$$

by the monotonicity of g . □

The following result providing a refinement of the classical Čebyšev inequality holds.



New Upper and Lower Bounds
for the Čebyšev Functional

P. Cerone and S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

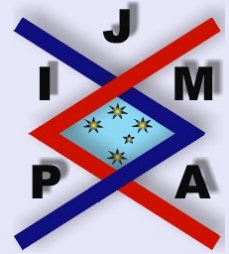
Page 6 of 26

Theorem 2.2. Let $f : [a, b] \rightarrow \mathbb{R}$ be a monotonic nondecreasing function on $[a, b]$ and $g : [a, b] \rightarrow \mathbb{R}$ a continuous function on $[a, b]$ so that $\varphi(t) \geq 0$ for each $t \in (a, b)$. Then one has the inequality:

$$(2.4) \quad T(f, g) \geq \frac{1}{(b-a)^2} \left| \int_a^b \left[(t-a) \left| \int_t^b g(x) dx \right| - (b-t) \left| \int_a^t g(x) dx \right| \right] df(t) \right| \geq 0.$$

Proof. Since $\varphi(t) \geq 0$ and f is monotonic nondecreasing, one has successively

$$\begin{aligned} T(f, g) &= \frac{1}{(b-a)^2} \int_a^b (t-a)(b-t) \left[\frac{\int_t^b g(x) dx}{b-t} - \frac{\int_a^t g(x) dx}{t-a} \right] df(t) \\ &= \frac{1}{(b-a)^2} \int_a^b (t-a)(b-t) \left| \frac{\int_t^b g(x) dx}{b-t} - \frac{\int_a^t g(x) dx}{t-a} \right| df(t) \\ &\geq \frac{1}{(b-a)^2} \int_a^b (t-a)(b-t) \left| \frac{\left| \int_t^b g(x) dx \right|}{b-t} - \frac{\left| \int_a^t g(x) dx \right|}{t-a} \right| df(t) \\ &\geq \frac{1}{(b-a)^2} \left| \int_a^b (t-a)(b-t) \left[\frac{\left| \int_t^b g(x) dx \right|}{b-t} - \frac{\left| \int_a^t g(x) dx \right|}{t-a} \right] df(t) \right| \\ &= \frac{1}{(b-a)^2} \left| \int_a^b \left[(t-a) \left| \int_t^b g(x) dx \right| - (b-t) \left| \int_a^t g(x) dx \right| \right] df(t) \right| \\ &\geq 0 \end{aligned}$$



New Upper and Lower Bounds
for the Čebyšev Functional

P. Cerone and S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 7 of 26

and the inequality (1.5) is proved. \square

Remark 2.1. By Lemma 2.1, we may observe that for any two monotonic non-decreasing functions $f, g : [a, b] \rightarrow \mathbb{R}$, one has the refinement of Čebyšev inequality provided by (2.4).

We are able now to prove the following inequality in terms of f and the function φ defined above in (2.1).

Theorem 2.3. Let $f : [a, b] \rightarrow \mathbb{R}$ be a function of bounded variation and $g : [a, b] \rightarrow \mathbb{R}$ an absolutely continuous function so that φ is bounded on (a, b) . Then one has the inequality:

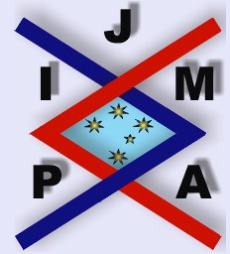
$$(2.5) \quad |T(f, g)| \leq \frac{1}{4} \|\varphi\|_{\infty} \bigvee_a^b(f),$$

where φ is as given by (2.1) and

$$\|\varphi\|_{\infty} := \sup_{t \in (a, b)} |\varphi(t)|.$$

Proof. Using the first inequality in Theorem 1.2, we have

$$\begin{aligned} |T(f, g)| &\leq \frac{1}{(b-a)^2} \sup_{t \in [a, b]} |\Psi(t)| \bigvee_a^b(f) \\ &= \frac{1}{(b-a)^2} \sup_{t \in [a, b]} |(t-a)(b-t)\varphi(t)| \bigvee_a^b(f) \end{aligned}$$



**New Upper and Lower Bounds
for the Čebyšev Functional**

P. Cerone and S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 8 of 26

$$\begin{aligned} &\leq \frac{1}{(b-a)^2} \sup_{t \in [a,b]} [(t-a)(b-t)] \sup_{t \in (a,b)} |\varphi(t)| \bigvee_a^b(f) \\ &\leq \frac{1}{4} \|\varphi\|_\infty \bigvee_a^b(f), \end{aligned}$$

since, obviously, $\sup_{t \in [a,b]} [(t-a)(b-t)] = \frac{(b-a)^2}{4}$. □

The case of Lipschitzian functions $f : [a, b] \rightarrow \mathbb{R}$ is embodied in the following theorem as well.

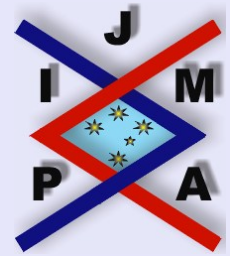
Theorem 2.4. *Let $f : [a, b] \rightarrow \mathbb{R}$ be an L -Lipschitzian function on $[a, b]$ and $g : [a, b] \rightarrow \mathbb{R}$ an absolutely continuous function on $[a, b]$. Then*

$$(2.6) \quad |T(f, g)| \leq \begin{cases} L \frac{(b-a)^3}{6} \|\varphi\|_\infty & \text{if } \varphi \in L_\infty[a, b]; \\ L (b-a)^{\frac{1}{q}} [B(q+1, q+1)]^{\frac{1}{q}} \|\varphi\|_p, & p > 1, \frac{1}{p} + \frac{1}{q} = 1 \\ & \text{if } \varphi \in L_p[a, b]; \\ \frac{L}{4} \|\varphi\|_1, & \text{if } \varphi \in L_1[a, b], \end{cases}$$

where $\|\cdot\|_p$ are the usual Lebesgue p -norms on $[a, b]$ and $B(\cdot, \cdot)$ is Euler's Beta function.

Proof. Using the second inequality in Theorem 1.2, we have

$$|T(f, g)| \leq \frac{L}{(b-a)^2} \int_a^b |\Psi(t)| dt = \frac{L}{(b-a)^2} \int_a^b (b-t)(t-a) |\varphi(t)| dt.$$



**New Upper and Lower Bounds
for the Čebyšev Functional**

P. Cerone and S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 9 of 26

Obviously

$$\begin{aligned} \int_a^b (b-t)(t-a) |\varphi(t)| dt &\leq \sup_{t \in [a,b]} |\varphi(t)| \int_a^b (b-t)(t-a) dt \\ &= \frac{(b-a)^3}{6} \|\varphi\|_\infty. \end{aligned}$$

giving the first result in (2.6).

By Hölder's integral inequality we have

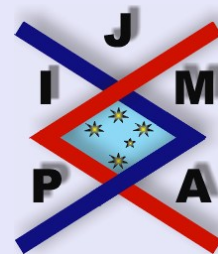
$$\begin{aligned} \int_a^b (b-t)(t-a) |\varphi(t)| dt &\leq \left(\int_a^b |\varphi(t)|^p dt \right)^{\frac{1}{p}} \left(\int_a^b [(b-t)(t-a)]^q dt \right)^{\frac{1}{q}} \\ &= \|\varphi\|_p (b-a)^{2+\frac{1}{q}} [B(q+1, q+1)]^{\frac{1}{q}}. \end{aligned}$$

Finally,

$$\begin{aligned} \int_a^b (b-t)(t-a) |\varphi(t)| dt &\leq \sup_{t \in [a,b]} [(b-t)(t-a)] \int_a^b |\varphi(t)| dt \\ &= \frac{(b-a)^2}{4} \|\varphi\|_1 \end{aligned}$$

and the inequality (2.6) is thus completely proved. \square

We will use the following inequality for the Stieltjes integral in the subse-



**New Upper and Lower Bounds
for the Čebyšev Functional**

P. Cerone and S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 10 of 26

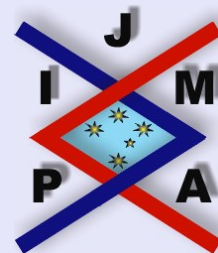
quent work, namely

$$(2.7) \quad \left| \int_a^b h(t) k(t) df(t) \right| \leq \begin{cases} \sup_{t \in [a,b]} |h(t)| \int_a^b |k(t)| df(t) \\ \left(\int_a^b |h(t)|^p df(t) \right)^{\frac{1}{p}} \left(\int_a^b |k(t)|^q df(t) \right)^{\frac{1}{q}}, \\ \quad \text{where } p > 1, \frac{1}{p} + \frac{1}{q} = 1 \\ \sup_{t \in [a,b]} |k(t)| \int_a^b |h(t)| df(t), \end{cases}$$

provided f is monotonic nondecreasing and h, k are continuous on $[a, b]$.

We note that a simple proof of these inequalities may be achieved by using the definition of the Stieltjes integral for monotonic functions. The following weighted inequalities for real numbers also hold,

$$(2.8) \quad \left| \sum_{i=1}^n a_i b_i w_i \right| \leq \begin{cases} \max_{i=1, n} |a_i| \sum_{i=1}^n |b_i| w_i \\ \left(\sum_{i=1}^n w_i |a_i|^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^n w_i |b_i|^q \right)^{\frac{1}{q}}, \quad p > 1, \frac{1}{p} + \frac{1}{q} = 1, \end{cases}$$



**New Upper and Lower Bounds
for the Čebyšev Functional**

P. Cerone and S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 11 of 26

where $a_i, b_i \in \mathbb{R}$ and $w_i \geq 0, i \in \{1, \dots, n\}$.

Using (2.7), we may state and prove the following theorem.

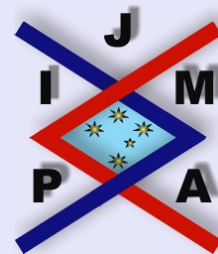
Theorem 2.5. *Let $f : [a, b] \rightarrow \mathbb{R}$ be a monotonic nondecreasing function on $[a, b]$. If g is continuous, then one has the inequality:*

$$(2.9) \quad |T(f, g)| \leq \begin{cases} \frac{1}{4} \int_a^b |\varphi(t)| df(t) \\ \frac{1}{(b-a)^2} \left(\int_a^b [(b-t)(t-a)]^q df(t) \right)^{\frac{1}{q}} \left(\int_a^b |\varphi(t)|^p df(t) \right)^{\frac{1}{p}}, \\ p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{1}{(b-a)^2} \sup_{t \in [a, b]} |\varphi(t)| \int_a^b (t-a)(b-t) df(t). \end{cases}$$

Proof. From the third inequality in (1.5), we have

$$(2.10) \quad |T(f, g)| \leq \frac{1}{(b-a)^2} \int_a^b |\Psi(t)| df(t) \\ = \frac{1}{(b-a)^2} \int_a^b (b-t)(t-a) |\varphi(t)| df(t).$$

Using (2.7), the inequality (2.9) is thus obtained. \square



**New Upper and Lower Bounds
for the Čebyšev Functional**

P. Cerone and S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 12 of 26

3. More on Čebyšev's Functional

Using the representation (1.2) and the integration by parts formula for the Stieltjes integral, we have (see also [4, p. 268], for a weighted version) the identity,

$$(3.1) \quad T(f, g) = \frac{1}{(b-a)^2} \left[\int_a^b (b-t) \left(\int_a^t (u-a) dg(u) \right) df(t) + \int_a^b (t-a) \left(\int_t^b (b-u) dg(u) \right) df(t) \right].$$

The following result holds.

Theorem 3.1. *Assume that $f : [a, b] \rightarrow \mathbb{R}$ is of bounded variation and $g : [a, b] \rightarrow \mathbb{R}$ is continuous and of bounded variation on $[a, b]$. Then one has the inequality:*

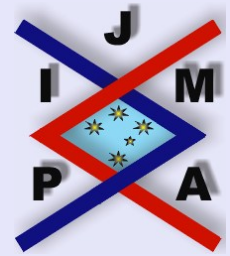
$$(3.2) \quad |T(f, g)| \leq \frac{1}{2} \bigvee_a^b(g) \bigvee_a^b(f).$$

If $g : [a, b] \rightarrow \mathbb{R}$ is Lipschitzian with the constant $L > 0$, then

$$(3.3) \quad |T(f, g)| \leq \frac{4}{27} (b-a) L \bigvee_a^b(f).$$

If $g : [a, b] \rightarrow \mathbb{R}$ is continuous and monotonic nondecreasing, then

$$(3.4) \quad |T(f, g)| \leq \frac{1}{(b-a)^2} \left\{ \sup_{t \in [a, b]} \left[(b-t) \left[(t-a)g(t) - \int_a^t g(u) du \right] \right] \right\}$$



New Upper and Lower Bounds
for the Čebyšev Functional

P. Cerone and S.S. Dragomir

Title Page

Contents

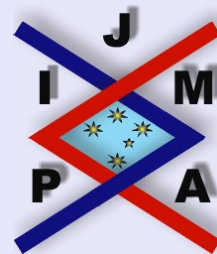


Go Back

Close

Quit

Page 13 of 26



**New Upper and Lower Bounds
for the Čebyšev Functional**

P. Cerone and S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 14 of 26

$$\begin{aligned}
 & + \sup_{t \in [a,b]} \left[(t-a) \left[\int_t^b g(u) du - g(t)(b-t) \right] \right] \bigg\} \bigvee_a^b(f) \\
 \leq & \left\{ \begin{aligned}
 & \frac{1}{b-a} \left\{ \sup_{t \in [a,b]} \left[(t-a)g(t) - \int_a^t g(u) du \right] \right. \\
 & \quad \left. + \sup_{t \in [a,b]} \left[\int_t^b g(u) du - g(t)(b-t) \right] \right\} \times \bigvee_a^b(f), \\
 & \frac{1}{4} \left\{ \sup_{t \in [a,b]} \left[g(t) - \frac{1}{t-a} \int_a^t g(u) du \right] \right. \\
 & \quad \left. + \sup_{t \in [a,b]} \left[\frac{1}{b-t} \int_t^b g(u) du - g(t) \right] \right\} \bigvee_a^b(f).
 \end{aligned} \right.
 \end{aligned}$$

Proof. Denote the two terms in (3.1) by

$$I_1 := \frac{1}{(b-a)^2} \int_a^b (b-t) \left(\int_a^t (u-a) dg(u) \right) df(t)$$

and by

$$I_2 := \frac{1}{(b-a)^2} \int_a^b (t-a) \left(\int_t^b (b-u) dg(u) \right) df(t).$$

Taking the modulus, we have

$$|I_1| \leq \frac{1}{(b-a)^2} \sup_{t \in [a,b]} \left[(b-t) \left| \int_a^t (u-a) dg(u) \right| \right] \bigvee_a^b(f)$$

and

$$|I_2| \leq \frac{1}{(b-a)^2} \sup_{t \in [a,b]} \left[(t-a) \left| \int_t^b (b-u) dg(u) \right| \right] \bigvee_a^b(f).$$

However,

$$\begin{aligned} \sup_{t \in [a,b]} \left[(b-t) \left| \int_a^t (u-a) dg(u) \right| \right] &\leq \sup_{t \in [a,b]} \left[(b-t)(t-a) \bigvee_a^t(g) \right] \\ &\leq \sup_{t \in [a,b]} [(b-t)(t-a)] \sup_{t \in [a,b]} \bigvee_a^t(g) \\ &= \frac{(b-a)^2}{4} \bigvee_a^b(g) \end{aligned}$$

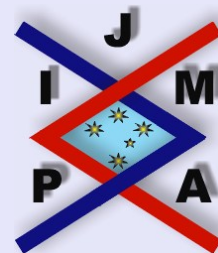
and, similarly,

$$\sup_{t \in [a,b]} \left[(t-a) \left| \int_t^b (b-u) dg(u) \right| \right] \leq \frac{(b-a)^2}{4} \bigvee_a^b(g).$$

Thus, from (3.1),

$$|T(f, g)| \leq |I_1| + |I_2| \leq \frac{1}{2} \bigvee_a^b(g) \bigvee_a^b(f)$$

and the inequality (3.2) is proved.



**New Upper and Lower Bounds
for the Čebyšev Functional**

P. Cerone and S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 15 of 26

If g is L -Lipschitzian, then we have

$$\left| \int_a^t (u - a) dg(u) \right| \leq L \int_a^t (u - a) du = \frac{L(t - a)^2}{2}$$

and

$$\left| \int_t^b (b - u) dg(u) \right| \leq L \int_t^b (b - u) du = \frac{L(b - t)^2}{2}$$

and thus

$$|I_1| \leq \frac{1}{2(b - a)^2} L \sup_{t \in [a, b]} [(b - t)(t - a)^2] \bigvee_a^b(f),$$

and

$$|I_2| \leq \frac{1}{2(b - a)^2} L \sup_{t \in [a, b]} [(t - a)(b - t)^2] \bigvee_a^b(f).$$

Since

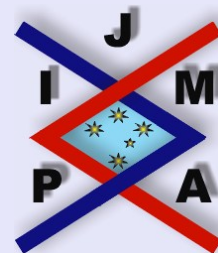
$$\sup_{t \in [a, b]} [(b - t)(t - a)^2] = \left(b - \frac{a + 2b}{3}\right) \left(\frac{a + 2b}{3} - a\right)^2 = \frac{4}{27} (b - a)^3,$$

then

$$|I_1| \leq \frac{2(b - a)}{27} L \bigvee_a^b(f)$$

and, similarly,

$$|I_2| \leq \frac{2(b - a)}{27} L \bigvee_a^b(f).$$



New Upper and Lower Bounds for the Čebyšev Functional

P. Cerone and S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 16 of 26

Consequently

$$|T(f, g)| \leq |I_1| + |I_2| \leq \frac{4(b-a)}{27} L \bigvee_a^b(f)$$

and the inequality (3.3) is also proved.

If g is monotonic nondecreasing, then

$$\left| \int_a^t (u-a) dg(u) \right| \leq \int_a^t (u-a) dg(u) = (t-a)g(t) - \int_a^t g(u) du$$

and

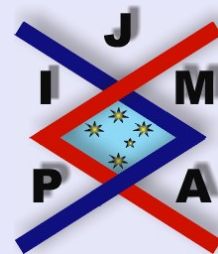
$$\left| \int_t^b (b-u) dg(u) \right| \leq \int_t^b (b-u) dg(u) = \int_t^b g(u) du - g(t)(b-t).$$

Consequently,

$$\begin{aligned} |I_1| &\leq \frac{1}{(b-a)^2} \sup_{t \in [a,b]} \left[(b-t) \left[(t-a)g(t) - \int_a^t g(u) du \right] \right] \bigvee_a^b(f) \\ &\leq \begin{cases} \frac{1}{b-a} \sup_{t \in [a,b]} \left[(t-a)g(t) - \int_a^t g(u) du \right] \bigvee_a^b(f), \\ \frac{1}{4} \sup_{t \in [a,b]} \left[g(t) - \frac{1}{t-a} \int_a^t g(u) du \right] \bigvee_a^b(f), \end{cases} \end{aligned}$$

and

$$|I_2| \leq \frac{1}{(b-a)^2} \sup_{t \in [a,b]} \left[(t-a) \left[\int_t^b g(u) du - g(t)(b-t) \right] \right] \bigvee_a^b(f)$$



**New Upper and Lower Bounds
for the Čebyšev Functional**

P. Cerone and S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 17 of 26

$$\leq \begin{cases} \frac{1}{b-a} \sup_{t \in [a,b]} \left[\int_t^b g(u) du - g(t)(b-t) \right] V_a^b(f), \\ \frac{1}{4} \sup_{t \in [a,b]} \left[\frac{1}{b-t} \int_t^b g(u) du - g(t) \right] V_a^b(f), \end{cases}$$

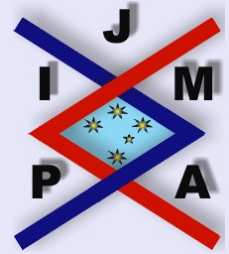
and the inequality (3.4) is also proved. \square

The following result concerning a differentiable function $g : [a, b] \rightarrow \mathbb{R}$ also holds.

Theorem 3.2. *Assume that $f : [a, b] \rightarrow \mathbb{R}$ is of bounded variation and $g : [a, b] \rightarrow \mathbb{R}$ is differentiable on (a, b) . Then,*

$$(3.5) \quad |T(f, g)| \leq \frac{1}{(b-a)^2} \bigvee_a^b(f)$$

$$\times \begin{cases} \sup_{t \in [a,b]} \left[(b-t)(t-a) \|g'\|_{[a,t],1} \right] \\ \quad + \sup_{t \in [a,b]} \left[(b-t)(t-a) \|g'\|_{[t,b],1} \right] \text{ if } g' \in L_1[a, b]; \\ \frac{1}{(q+1)^{\frac{1}{q}}} \left\{ \sup_{t \in [a,b]} \left[(b-t)(t-a)^{1+\frac{1}{q}} \|g'\|_{[a,t],p} \right] \right. \\ \quad \left. + \sup_{t \in [a,b]} \left[(t-a)(b-t)^{1+\frac{1}{q}} \|g'\|_{[t,b],p} \right] \right\} \\ \quad \text{if } g' \in L_p[a, b], p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{1}{2} \left\{ \sup_{t \in [a,b]} \left[(b-t)(t-a)^2 \|g'\|_{[a,t],\infty} \right] \right. \\ \quad \left. + \sup_{t \in [a,b]} \left[(t-a)(b-t)^2 \|g'\|_{[t,b],\infty} \right] \right\} \text{ if } g' \in L_\infty[a, b] \end{cases}$$



**New Upper and Lower Bounds
for the Čebyšev Functional**

P. Cerone and S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 18 of 26

$$\leq \bigvee_a^b (f) \times \begin{cases} \frac{1}{2} \|g'\|_{[a,b],1} & \text{if } g' \in L_1 [a, b]; \\ \frac{2q(q+1)(b-a)^{\frac{1}{q}}}{(2q+1)^{\frac{1}{q}+2}} \|g'\|_{[a,b],p} & \text{if } g' \in L_p [a, b], p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{4(b-a)}{27} \|g'\|_{[a,b],\infty} & \text{if } g' \in L_\infty [a, b], \end{cases}$$

where the Lebesgue norms over an interval $[c, d]$ are defined by

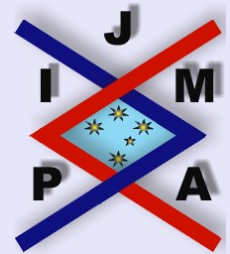
$$\|h\|_{[c,d],p} := \left(\int_c^d |h(t)|^p dt \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty$$

and

$$\|h\|_{[c,d],\infty} := \operatorname{ess\,sup}_{t \in [c,d]} |h(t)|.$$

Proof. Since g is differentiable on (a, b) , we have

$$(3.6) \quad \left| \int_a^t (u-a) dg(u) \right| = \left| \int_a^t (u-a) g'(u) du \right| \leq \begin{cases} (t-a) \|g'\|_{[a,t],1} \\ \left(\int_a^t (u-a)^q du \right)^{\frac{1}{q}} \|g'\|_{[a,t],p}, p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \int_a^t (u-a) du \|g'\|_{[a,t],\infty} \end{cases}$$



**New Upper and Lower Bounds
for the Čebyšev Functional**

P. Cerone and S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 19 of 26

$$= \begin{cases} (t-a) \|g'\|_{[a,t],1} \\ \frac{(t-a)^{1+\frac{1}{q}}}{(q+1)^{\frac{1}{q}}} \|g'\|_{[a,t],p}, \quad p > 1, \quad \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{(t-a)^2}{2} \|g'\|_{[a,t],\infty} \end{cases}$$

and, similarly,

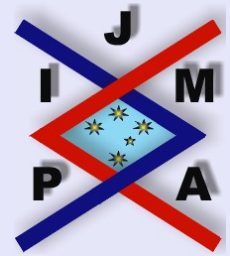
$$(3.7) \quad \left| \int_t^b (b-u) dg(u) \right| \leq \begin{cases} (b-t) \|g'\|_{[t,b],1} \\ \frac{(b-t)^{1+\frac{1}{q}}}{(q+1)^{\frac{1}{q}}} \|g'\|_{[t,b],p}, \quad p > 1, \quad \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{(b-t)^2}{2} \|g'\|_{[t,b],\infty}. \end{cases}$$

With the notation in Theorem 3.1, we have on using (3.6)

$$|I_1| \leq \frac{1}{(b-a)^2} \bigvee_a^b(f) \cdot \sup_{t \in [a,b]} \begin{cases} (b-t)(t-a) \|g'\|_{[a,t],1} \\ \frac{(b-t)(t-a)^{1+\frac{1}{q}}}{(q+1)^{\frac{1}{q}}} \|g'\|_{[a,t],p}, \quad p > 1, \quad \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{(b-t)(t-a)^2}{2} \|g'\|_{[a,t],\infty} \end{cases}$$

and from (3.7)

$$|I_2| \leq \frac{1}{(b-a)^2} \bigvee_a^b(f) \cdot \sup_{t \in [a,b]} \begin{cases} (t-a)(b-t) \|g'\|_{[t,b],1} \\ \frac{(t-a)(b-t)^{1+\frac{1}{q}}}{(q+1)^{\frac{1}{q}}} \|g'\|_{[t,b],p}, \quad p > 1, \quad \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{(t-a)(b-t)^2}{2} \|g'\|_{[t,b],\infty}. \end{cases}$$



New Upper and Lower Bounds for the Čebyšev Functional

P. Cerone and S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 20 of 26

Further, since

$$|T(f, g)| \leq |I_1| + |I_2|,$$

we deduce the first inequality in (3.5).

Now, observe that

$$\begin{aligned} \sup_{t \in [a,b]} \left[(b-t)(t-a) \|g'\|_{[a,t],1} \right] &\leq \sup_{t \in [a,b]} [(b-t)(t-a)] \sup_{t \in [a,b]} \|g'\|_{[a,t],1} \\ &= \frac{(b-a)^2}{4} \|g'\|_{[a,b],1}; \end{aligned}$$

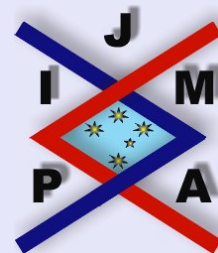
$$\begin{aligned} \sup_{t \in [a,b]} \left[\frac{(b-t)(t-a)^{1+\frac{1}{q}}}{(q+1)^{\frac{1}{q}}} \|g'\|_{[a,t],p} \right] \\ \leq \frac{1}{(q+1)^{\frac{1}{q}}} \sup_{t \in [a,b]} \left[(b-t)(t-a)^{1+\frac{1}{q}} \right] \sup_{t \in [a,b]} \|g'\|_{[a,t],p} \\ = M_q \|g'\|_{[a,b],p} \end{aligned}$$

where

$$M_q := \frac{1}{(q+1)^{\frac{1}{q}}} \sup_{t \in [a,b]} \left[(b-t)(t-a)^{1+\frac{1}{q}} \right].$$

Consider the arbitrary function $\rho(t) = (b-t)(t-a)^{r+1}$, $r > 0$. Then $\rho'(t) = (t-a)^r [(r+1)b + a - (r+2)t]$ showing that

$$\sup_{t \in [a,b]} \rho(t) = \rho \left[\frac{a + (r+1)b}{r+2} \right] = \frac{(b-a)^{r+2} (r+1)^{r+1}}{(r+2)^{r+2}}.$$



New Upper and Lower Bounds for the Čebyšev Functional

P. Cerone and S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 21 of 26

Consequently,

$$M_q = \frac{q}{(q+1)^{\frac{1}{q}}} \cdot \frac{(b-a)^{2+\frac{1}{q}} (q+1)^{1+\frac{1}{q}}}{(2q+1)^{2+\frac{1}{q}}} = \frac{q(q+1)(b-a)^{2+\frac{1}{q}}}{(2q+1)^{2+\frac{1}{q}}}.$$

Also,

$$\begin{aligned} \sup_{t \in [a,b]} \left[\frac{(b-t)(t-a)^2}{2} \|g'\|_{[a,t],\infty} \right] &\leq \frac{1}{2} \sup_{t \in [a,b]} [(b-t)(t-a)^2] \sup_{t \in [a,b]} \|g'\|_{[a,t],\infty} \\ &= \frac{2(b-a)^3}{27} \|g'\|_{[a,b],\infty}. \end{aligned}$$

In a similar fashion we have

$$\begin{aligned} \sup_{t \in [a,b]} \left[(t-a)(b-t) \|g'\|_{[t,b],1} \right] &\leq \frac{(b-a)^2}{4} \|g'\|_{[a,b],1}; \\ \sup_{t \in [a,b]} \left[\frac{(t-a)(b-t)^{1+\frac{1}{q}}}{(q+1)^{\frac{1}{q}}} \|g'\|_{[t,b],p} \right] &\leq \frac{q(q+1)(b-a)^{2+\frac{1}{q}}}{(2q+1)^{2+\frac{1}{q}}} \|g'\|_{[a,b],p}, \end{aligned}$$

and

$$\sup_{t \in [a,b]} \left[\frac{(t-a)(b-t)^2}{2} \|g'\|_{[t,b],\infty} \right] \leq \frac{2(b-a)^3}{27} \|g'\|_{[a,b],\infty}$$

and the last part of (3.5) is thus completely proved. \square



**New Upper and Lower Bounds
for the Čebyšev Functional**

P. Cerone and S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 22 of 26

Lemma 3.3. Let $g : [a, b] \rightarrow \mathbb{R}$ be absolutely continuous on $[a, b]$ then for

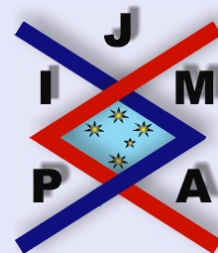
$$(3.8) \quad \varphi(t) = \mathcal{M}(g; t, b) - \mathcal{M}(g; a, t),$$

with $\mathcal{M}(g; c, d)$ defined by (1.6),

$$(3.9) \quad \|\varphi\|_{\infty} \leq \begin{cases} \left(\frac{b-a}{2}\right) \|g'\|_{\infty}, & g' \in L_{\infty}[a, b]; \\ \frac{b-a}{(\beta+1)^{\frac{1}{\beta}}} \|g'\|_{\alpha}, & g' \in L_{\alpha}[a, b], \quad \alpha > 1, \quad \frac{1}{\alpha} + \frac{1}{\beta} = 1; \\ \|g'\|_1, & g' \in L_1[a, b]; \\ V_a^b(g), & g \text{ of bounded variation}; \\ \left(\frac{b-a}{2}\right) L, & g \text{ is } L\text{-Lipschitzian}, \end{cases}$$

and for $p \geq 1$

$$(3.10) \quad \|\varphi\|_p \leq \begin{cases} \left(\frac{b-a}{2}\right)^{1+\frac{1}{p}} \|g'\|_{\infty}, & g' \in L_{\infty}[a, b]; \\ \left(\int_a^b \left[\frac{(t-a)^{\beta} + (b-t)^{\beta}}{\beta+1}\right]^{\frac{p}{\beta}} dt\right)^{\frac{1}{p}} \|g'\|_{\alpha}, & g' \in L_{\alpha}[a, b], \quad \alpha > 1, \quad \frac{1}{\alpha} + \frac{1}{\beta} = 1; \\ (b-a)^{\frac{1}{p}} \|g'\|_1, & g' \in L_1[a, b]; \\ (b-a)^{\frac{1}{p}} V_a^b(g), & g \text{ of bounded variation}; \\ \left(\frac{b-a}{2}\right)^{1+\frac{1}{p}} L, & g \text{ is } L\text{-Lipschitzian}. \end{cases}$$



**New Upper and Lower Bounds
for the Čebyšev Functional**

P. Cerone and S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 23 of 26

Proof. Identifying $\varphi(t)$ with $D(g; a, t, b)$ of (1.7) produces bounds for $|\varphi(t)|$ from (1.8). Taking the supremum over $t \in [a, b]$ readily gives (3.9), a bound for $\|\varphi\|_\infty$.

The bound for $\|\varphi\|_p$ is obtained from (1.8) using the definition of the Lebesgue p -norms over $[a, b]$. \square

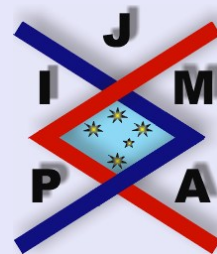
Remark 3.1. Utilising (3.9) of Lemma 3.3 in (2.5) produces a coarser upper bound for $|T(f, g)|$. Making use of the whole of Lemma 3.3 in (2.6) produces coarser bounds for (2.6) which may prove more amenable in practical situations.

Corollary 3.4. Let the conditions of Theorem 2.3 hold, then

$$(3.11) \quad |T(f, g)| \leq \frac{1}{4} \bigvee_a^b(f) \left\{ \begin{array}{ll} \left(\frac{b-a}{2}\right) \|g'\|_\infty, & g' \in L_\infty[a, b]; \\ \frac{b-a}{(\beta+1)^\beta} \|g'\|_\alpha, & g' \in L_\alpha[a, b], \\ & \alpha > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1; \\ \|g'\|_1, & g' \in L_1[a, b]; \\ \bigvee_a^b(g), & g \text{ of bounded variation}; \\ \left(\frac{b-a}{2}\right) L, & g \text{ is } L\text{-Lipschitzian.} \end{array} \right.$$

Proof. Using (3.9) in (2.5) produces (3.11). \square

Remark 3.2. We note from the last two inequalities of (3.11) that the bounds produced are sharper than those of Theorem 3.1, giving constants of $\frac{1}{4}$ and $\frac{1}{8}$



New Upper and Lower Bounds
for the Čebyšev Functional

P. Cerone and S.S. Dragomir

Title Page

Contents



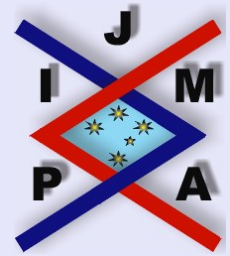
Go Back

Close

Quit

Page 24 of 26

compared with $\frac{1}{2}$ and $\frac{4}{27}$ of equations (3.2) and (3.3). For g differentiable then we notice that the first and third results of (3.11) are sharper than the first and third results in the second cluster of (3.5). The first cluster in (3.5) are sharper where the analysis is done over the two subintervals $[a, x]$ and $(x, b]$.



New Upper and Lower Bounds for the Čebyšev Functional

P. Cerone and S.S. Dragomir

Title Page

Contents



Go Back

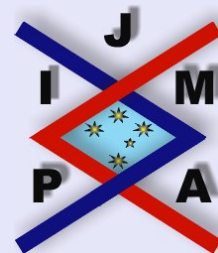
Close

Quit

Page 25 of 26

References

- [1] P. CERONE, On an identity for the Chebychev functional and some ramifications, *J. Ineq. Pure. & Appl. Math.*, **3**(1) (2002), Article 4. [ONLINE] http://jipam.vu.edu.au/v3n1/034_01.html
- [2] S.S. DRAGOMIR, Some integral inequalities of Grüss type, *Indian J. of Pure and Appl. Math.*, **31**(4) (2000), 397–415.
- [3] A.M. FINK, A treatise on Grüss' inequality, Th.M. Rassias and H.M. Srivastava (Ed.), Kluwer Academic Publishers, (1999), 93–114.
- [4] D.S. MITRINOVIĆ, J.E. PEČARIĆ AND A.M. FINK, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, Dordrecht, 1993.



**New Upper and Lower Bounds
for the Čebyšev Functional**

P. Cerone and S.S. Dragomir

Title Page

Contents



Go Back

Close

Quit

Page 26 of 26