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NEW UPPER AND LOWER BOUNDS FOR THE ČEBYŠEV FUNCTIONAL

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Abstract
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Abstract

New bounds are developed for the Čebyšev functional utilising an identity involving a Riemann-Stieltjes integral. A refinement of the classical Čebyšev inequality is produced for f monotonic non-decreasing, g continuous and $\mathcal{M}(g;t,b)-\mathcal{M}(g;a,t)\geq 0$, for $t\in[a,b]$ where $\mathcal{M}(g;c,d)$ is the integral mean over $[c,d]$.

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1. Introduction

For two given integrable functions on $[a,b]$, define the Čebyšev functional ([2, 3, 4])

$$T(f,g) := \frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx - \frac{1}{b-a}\int_a^b f(x)\,dx \cdot \frac{1}{b-a}\int_a^b g(x)\,dx. \tag{1.1}$$

In [1], P. Cerone has obtained the following identity that involves a Stieltjes integral (Lemma 2.1, p. 3):

Lemma 1.1. *Let $f,g : [a,b] \to \mathbb{R}$, where f is of bounded variation and g is continuous on $[a,b]$, then the $T(f,g)$ from (1.1) satisfies the identity,*

$$T(f,g) = \frac{1}{(b-a)^2}\int_a^b \Psi(t)\,df(t), \tag{1.2}$$

where

$$\Psi(t) := (t-a)\,A(t,b) - (b-t)\,A(a,t), \tag{1.3}$$

with

$$A(c,d) := \int_c^d g(x)\,dx. \tag{1.4}$$

Using this representation and the properties of Stieltjes integrals he obtained the following result in bounding the functional $T(\cdot,\cdot)$ (Theorem 2.5, p. 4):

Theorem 1.2. *With the assumptions in Lemma 1.1, we have:*

$$|T(f,g)| \le \frac{1}{(b-a)^2} \times \begin{cases} \sup_{t\in[a,b]} |\Psi(t)| \bigvee_a^b (f), & \\ L\int_a^b |\Psi(t)|\, dt, & \text{for } L-\text{Lipschitzian;} \\ \int_a^b |\Psi(t)|\, df(t), & \text{for } f \text{ monotonic nondecreasing,} \end{cases} \tag{1.5}$$

where $\bigvee_a^b (f)$ *denotes the total variation of* f *on* $[a,b]$.

Cerone [1] also proved the following theorem, which will be useful for the development of subsequent results, and is thus stated here for clarity. The notation $\mathcal{M}(g;c,d)$ is used to signify the integral mean of g over $[c,d]$. Namely,

$$\mathcal{M}(g;c,d) := \frac{A(c,d)}{d-c} = \frac{1}{d-c}\int_c^d f(t)\, dt. \tag{1.6}$$

Theorem 1.3. *Let* $g:[a,b]\to\mathbb{R}$ *be absolutely continuous on* $[a,b]$, *then for*

$$D(g;a,t,b) := \mathcal{M}(g;t,b) - \mathcal{M}(g;a,t), \tag{1.7}$$

$$
(1.8)\quad |D(g;a,t,b)| \leq \begin{cases} \left(\frac{b-a}{2}\right)\|g'\|_{\infty}, & g' \in L_{\infty}[a,b]; \\ \left[\frac{(t-a)^q+(b-t)^q}{q+1}\right]^{\frac{1}{q}} \|g'\|_p, & g' \in L_p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \|g'\|_1, & g' \in L_1[a,b]; \\ \bigvee_a^b(g), & g \ \textit{of bounded variation;} \\ \left(\frac{b-a}{2}\right)L, & g \ \textit{is } L-\textit{Lipschitzian.} \end{cases}
$$

Although the possibility of utilising Theorem 1.3 to obtain bounds on $\psi(t)$, as given by (1.3), was mentioned in [1], it was not capitalised upon. This aspect will be investigated here since even though this will provide coarser bounds, they may be more useful in practice.

A lower bound for the Čebyšev functional improving the classical result due to Čebyšev is also developed and thus providing a refinement.

2. Integral Inequalities

Now, if we use the function $\varphi : (a, b) \to \mathbb{R}$,

$$\varphi(t) := D(g; a, t, b) = \frac{\int_t^b g(x)\,dx}{b-t} - \frac{\int_a^t g(x)\,dx}{t-a}, \tag{2.1}$$

then by (1.2) we may obtain the identity:

$$T(f, g) = \frac{1}{(b-a)^2} \int_a^b (t-a)(b-t)\,\varphi(t)\,df(t). \tag{2.2}$$

We may prove the following lemma.

Lemma 2.1. *If $g : [a, b] \to \mathbb{R}$ is monotonic nondecreasing on $[a, b]$, then φ as defined by (2.1) is nonnegative on (a, b).*

Proof. Since g is nondecreasing, we have $\int_t^b g(x)\,dx \geq (b-t)\,g(t)$ and thus from (2.1)

$$\varphi(t) \geq g(t) - \frac{\int_a^t g(x)\,dx}{t-a} = \frac{(t-a)\,g(t) - \int_a^t g(x)\,dx}{t-a} \geq 0, \tag{2.3}$$

by the monotonicity of g. □

The following result providing a refinement of the classical Čebyšev inequality holds.

Theorem 2.2. *Let $f:[a,b]\to\mathbb{R}$ be a monotonic nondecreasing function on $[a,b]$ and $g:[a,b]\to\mathbb{R}$ a continuous function on $[a,b]$ so that $\varphi(t)\geq 0$ for each $t\in(a,b)$. Then one has the inequality:*

$$T(f,g)\geq\frac{1}{(b-a)^2}\left|\int_a^b\left[(t-a)\left|\int_t^b g(x)\,dx\right| - (b-t)\left|\int_a^t g(x)\,dx\right|\right]df(t)\right|\geq 0. \tag{2.4}$$

Proof. Since $\varphi(t)\geq 0$ and f is monotonic nondecreasing, one has successively

$$\begin{aligned}T(f,g)&=\frac{1}{(b-a)^2}\int_a^b(t-a)(b-t)\left[\frac{\int_t^b g(x)\,dx}{b-t}-\frac{\int_a^t g(x)\,dx}{t-a}\right]df(t)\\&=\frac{1}{(b-a)^2}\int_a^b(t-a)(b-t)\left|\frac{\int_t^b g(x)\,dx}{b-t}-\frac{\int_a^t g(x)\,dx}{t-a}\right|df(t)\\&\geq\frac{1}{(b-a)^2}\int_a^b(t-a)(b-t)\left|\frac{\left|\int_t^b g(x)\,dx\right|}{b-t}-\frac{\left|\int_a^t g(x)\,dx\right|}{t-a}\right|df(t)\\&\geq\frac{1}{(b-a)^2}\left|\int_a^b(t-a)(b-t)\left[\frac{\left|\int_t^b g(x)\,dx\right|}{b-t}-\frac{\left|\int_a^t g(x)\,dx\right|}{t-a}\right]df(t)\right|\\&=\frac{1}{(b-a)^2}\left|\int_a^b\left[(t-a)\left|\int_t^b g(x)\,dx\right|-(b-t)\left|\int_a^t g(x)\,dx\right|\right]df(t)\right|\\&\geq 0\end{aligned}$$

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and the inequality (1.5) is proved. $\square$

Remark 2.1. *By Lemma 2.1, we may observe that for any two monotonic non-decreasing functions $f, g : [a, b] \to \mathbb{R}$, one has the refinement of Čebyšev inequality provided by (2.4).*

We are able now to prove the following inequality in terms of f and the function φ defined above in (2.1).

Theorem 2.3. *Let $f : [a, b] \to \mathbb{R}$ be a function of bounded variation and $g : [a, b] \to \mathbb{R}$ an absolutely continuous function so that φ is bounded on (a, b). Then one has the inequality:*

$$|T(f, g)| \leq \frac{1}{4} \|\varphi\|_\infty \bigvee_a^b (f), \tag{2.5}$$

where φ is as given by (2.1) and

$$\|\varphi\|_\infty := \sup_{t \in (a,b)} |\varphi(t)|.$$

Proof. Using the first inequality in Theorem 1.2, we have

$$\begin{aligned} |T(f, g)| &\leq \frac{1}{(b-a)^2} \sup_{t \in [a,b]} |\Psi(t)| \bigvee_a^b (f) \\ &= \frac{1}{(b-a)^2} \sup_{t \in [a,b]} |(t-a)(b-t)\varphi(t)| \bigvee_a^b (f) \end{aligned}$$

$$\leq \frac{1}{(b-a)^2} \sup_{t\in[a,b]} [(t-a)(b-t)] \sup_{t\in(a,b)} |\varphi(t)| \bigvee_a^b (f)$$
$$\leq \frac{1}{4} \|\varphi\|_\infty \bigvee_a^b (f),$$

since, obviously, $\sup_{t\in[a,b]}[(t-a)(b-t)] = \frac{(b-a)^2}{4}$. □

The case of Lipschitzian functions $f:[a,b]\to\mathbb{R}$ is embodied in the following theorem as well.

Theorem 2.4. *Let $f:[a,b]\to\mathbb{R}$ be an $L-$Lipschitzian function on $[a,b]$ and $g:[a,b]\to\mathbb{R}$ an absolutely continuous function on $[a,b]$. Then*

$$(2.6)\quad |T(f,g)| \leq \begin{cases} L\frac{(b-a)^3}{6}\|\varphi\|_\infty & \text{if } \varphi\in L_\infty[a,b]; \\ L(b-a)^{\frac{1}{q}}[B(q+1,q+1)]^{\frac{1}{q}}\|\varphi\|_p, & p>1,\ \frac{1}{p}+\frac{1}{q}=1 \\ & \text{if } \varphi\in L_p[a,b]; \\ \frac{L}{4}\|\varphi\|_1, & \text{if } \varphi\in L_1[a,b], \end{cases}$$

where $\|\cdot\|_p$ are the usual Lebesgue $p-$norms on $[a,b]$ and $B(\cdot,\cdot)$ is Euler's Beta function.

Proof. Using the second inequality in Theorem 1.2, we have

$$|T(f,g)| \leq \frac{L}{(b-a)^2}\int_a^b |\Psi(t)|\,dt = \frac{L}{(b-a)^2}\int_a^b (b-t)(t-a)|\varphi(t)|\,dt.$$

Obviously

$$\int_a^b (b-t)(t-a)|\varphi(t)|\,dt \le \sup_{t\in[a,b]} |\varphi(t)| \int_a^b (t-a)(b-t)\,dt$$
$$= \frac{(b-a)^3}{6}\|\varphi\|_\infty .$$

giving the first result in (2.6).

By Hölder's integral inequality we have

$$\int_a^b (b-t)(t-a)|\varphi(t)|\,dt \le \left(\int_a^b |\varphi(t)|^p\,dt\right)^{\frac{1}{p}} \left(\int_a^b [(b-t)(t-a)]^q\,dt\right)^{\frac{1}{q}}$$
$$= \|\varphi\|_p (b-a)^{2+\frac{1}{q}} [B(q+1,q+1)]^{\frac{1}{q}} .$$

Finally,

$$\int_a^b (b-t)(t-a)|\varphi(t)|\,dt \le \sup_{t\in[a,b]} [(b-t)(t-a)] \int_a^b |\varphi(t)|\,dt$$
$$= \frac{(b-a)^2}{4}\|\varphi\|_1$$

and the inequality (2.6) is thus completely proved. □

We will use the following inequality for the Stieltjes integral in the subsequent work, namely

$$\left|\int_a^b h(t)\,k(t)\,df(t)\right| \leq \begin{cases} \sup\limits_{t\in[a,b]} |h(t)| \int_a^b |k(t)|\,df(t) \\ \left(\int_a^b |h(t)|^p\,df(t)\right)^{\frac{1}{p}} \left(\int_a^b |k(t)|^q\,df(t)\right)^{\frac{1}{q}}, \\ \qquad\qquad \text{where } p>1,\ \frac{1}{p}+\frac{1}{q}=1 \\ \sup\limits_{t\in[a,b]} |k(t)| \int_a^b |h(t)|\,df(t), \end{cases} \tag{2.7}$$

provided f is monotonic nondecreasing and h, k are continuous on $[a,b]$.

We note that a simple proof of these inequalities may be achieved by using the definition of the Stieltjes integral for monotonic functions. The following weighted inequalities for real numbers also hold,

$$\left|\sum_{i=1}^n a_i b_i w_i\right| \leq \begin{cases} \max\limits_{i=\overline{1,n}} |a_i| \sum\limits_{i=1}^n |b_i|\,w_i \\ \left(\sum\limits_{i=1}^n w_i |a_i|^p\right)^{\frac{1}{p}} \left(\sum\limits_{i=1}^n w_i |b_i|^q\right)^{\frac{1}{q}},\ p>1,\ \frac{1}{p}+\frac{1}{q}=1, \end{cases} \tag{2.8}$$

where $a_i, b_i \in \mathbb{R}$ and $w_i \geq 0$, $i \in \{1, \dots, n\}$.

Using (2.7), we may state and prove the following theorem.

Theorem 2.5. *Let $f : [a,b] \to \mathbb{R}$ be a monotonic nondecreasing function on $[a,b]$. If g is continuous, then one has the inequality:*

$$(2.9)\quad |T(f,g)| \leq \begin{cases} \frac{1}{4}\int_a^b |\varphi(t)|\, df(t) \\ \frac{1}{(b-a)^2}\left(\int_a^b [(b-t)(t-a)]^q\, df(t)\right)^{\frac{1}{q}} \left(\int_a^b |\varphi(t)|^p\, df(t)\right)^{\frac{1}{p}}, \\ \qquad\qquad p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{(b-a)^2}\sup_{t\in[a,b]} |\varphi(t)| \int_a^b (t-a)(b-t)\, df(t). \end{cases}$$

Proof. From the third inequality in (1.5), we have

$$(2.10)\quad |T(f,g)| \leq \frac{1}{(b-a)^2}\int_a^b |\Psi(t)|\, df(t) = \frac{1}{(b-a)^2}\int_a^b (b-t)(t-a)|\varphi(t)|\, df(t).$$

Using (2.7), the inequality (2.9) is thus obtained. □

3. More on Čebyšev's Functional

Using the representation (1.2) and the integration by parts formula for the Stieltjes integral, we have (see also [4, p. 268], for a weighted version) the identity,

$$T(f,g)=\frac{1}{(b-a)^2}\left[\int_a^b (b-t)\left(\int_a^t (u-a)\,dg(u)\right)df(t)\right.$$
$$\left.+\int_a^b (t-a)\left(\int_t^b (b-u)\,dg(u)\right)df(t)\right]. \tag{3.1}$$

The following result holds.

Theorem 3.1. *Assume that* $f:[a,b]\to\mathbb{R}$ *is of bounded variation and* $g:[a,b]\to\mathbb{R}$ *is continuous and of bounded variation on* $[a,b]$. *Then one has the inequality:*

$$|T(f,g)|\le\frac{1}{2}\bigvee_a^b(g)\bigvee_a^b(f). \tag{3.2}$$

If $g:[a,b]\to\mathbb{R}$ *is Lipschitzian with the constant* $L>0$, *then*

$$|T(f,g)|\le\frac{4}{27}(b-a)\,L\bigvee_a^b(f). \tag{3.3}$$

If $g:[a,b]\to\mathbb{R}$ *is continuous and monotonic nondecreasing, then*

$$|T(f,g)|$$
$$\le\frac{1}{(b-a)^2}\left\{\sup_{t\in[a,b]}\left[(b-t)\left[(t-a)\,g(t)-\int_a^t g(u)\,du\right]\right]\right. \tag{3.4}$$

$$+\sup_{t\in[a,b]}\left[(t-a)\left[\int_t^b g(u)\,du-g(t)(b-t)\right]\right]\right\}\bigvee_a^b(f)$$

$$\leq\begin{cases}\dfrac{1}{b-a}\left\{\sup_{t\in[a,b]}\left[(t-a)g(t)-\int_a^t g(u)\,du\right]\right.\\ \qquad\left.+\sup_{t\in[a,b]}\left[\int_t^b g(u)\,du-g(t)(b-t)\right]\right\}\times\bigvee_a^b(f), \\ \\ \dfrac{1}{4}\left\{\sup_{t\in[a,b]}\left[g(t)-\dfrac{1}{t-a}\int_a^t g(u)\,du\right]\right.\\ \qquad\left.+\sup_{t\in[a,b]}\left[\dfrac{1}{b-t}\int_t^b g(u)\,du-g(t)\right]\right\}\bigvee_a^b(f).\end{cases}$$

Proof. Denote the two terms in (3.1) by

$$I_1:=\frac{1}{(b-a)^2}\int_a^b(b-t)\left(\int_a^t(u-a)\,dg(u)\right)df(t)$$

and by

$$I_2:=\frac{1}{(b-a)^2}\int_a^b(t-a)\left(\int_t^b(b-u)\,dg(u)\right)df(t).$$

Taking the modulus, we have

$$|I_1|\leq\frac{1}{(b-a)^2}\sup_{t\in[a,b]}\left[(b-t)\left|\int_a^t(u-a)\,dg(u)\right|\right]\bigvee_a^b(f)$$

and

$$|I_2| \leq \frac{1}{(b-a)^2} \sup_{t\in[a,b]} \left[(t-a) \left| \int_t^b (b-u)\, dg(u) \right| \right] \bigvee_a^b (f) \, .$$

However,

$$\begin{aligned} \sup_{t\in[a,b]} \left[(b-t) \left| \int_a^t (u-a)\, dg(u) \right| \right] &\leq \sup_{t\in[a,b]} \left[(b-t)(t-a) \bigvee_a^t (g) \right] \\ &\leq \sup_{t\in[a,b]} \left[(b-t)(t-a) \right] \sup_{t\in[a,b]} \bigvee_a^t (g) \\ &= \frac{(b-a)^2}{4} \bigvee_a^b (g) \end{aligned}$$

and, similarly,

$$\sup_{t\in[a,b]} \left[(t-a) \left| \int_t^b (b-u)\, dg(u) \right| \right] \leq \frac{(b-a)^2}{4} \bigvee_a^b (g) \, .$$

Thus, from (3.1),

$$|T(f,g)| \leq |I_1| + |I_2| \leq \frac{1}{2} \bigvee_a^b (g) \bigvee_a^b (f)$$

and the inequality (3.2) is proved.

If g is $L-$Lipschitzian, then we have

$$\left|\int_a^t (u-a)\,dg(u)\right| \le L\int_a^t (u-a)\,du = \frac{L(t-a)^2}{2}$$

and

$$\left|\int_t^b (b-u)\,dg(u)\right| \le L\int_t^b (b-u)\,du = \frac{L(b-t)^2}{2}$$

and thus

$$|I_1| \le \frac{1}{2(b-a)^2} L \sup_{t\in[a,b]}\left[(b-t)(t-a)^2\right]\bigvee_a^b(f),$$

and

$$|I_2| \le \frac{1}{2(b-a)^2} L \sup_{t\in[a,b]}\left[(t-a)(b-t)^2\right]\bigvee_a^b(f).$$

Since

$$\sup_{t\in[a,b]}\left[(b-t)(t-a)^2\right] = \left(b-\frac{a+2b}{3}\right)\left(\frac{a+2b}{3}-a\right)^2 = \frac{4}{27}(b-a)^3,$$

then

$$|I_1| \le \frac{2(b-a)}{27} L\bigvee_a^b(f)$$

and, similarly,

$$|I_2| \le \frac{2(b-a)}{27} L\bigvee_a^b(f).$$

Consequently

$$|T(f,g)| \leq |I_1| + |I_2| \leq \frac{4(b-a)}{27} L \bigvee_a^b (f)$$

and the inequality (3.3) is also proved.

If g is monotonic nondecreasing, then

$$\left| \int_a^t (u-a)\, dg(u) \right| \leq \int_a^t (u-a)\, dg(u) = (t-a)\, g(t) - \int_a^t g(u)\, du$$

and

$$\left| \int_t^b (b-u)\, dg(u) \right| \leq \int_t^b (b-u)\, dg(u) = \int_t^b g(u)\, du - g(t)(b-t).$$

Consequently,

$$|I_1| \leq \frac{1}{(b-a)^2} \sup_{t\in[a,b]} \left[(b-t) \left[(t-a)\, g(t) - \int_a^t g(u)\, du \right] \right] \bigvee_a^b (f)$$

$$\leq \begin{cases} \frac{1}{b-a} \sup_{t\in[a,b]} \left[(t-a)\, g(t) - \int_a^t g(u)\, du \right] \bigvee_a^b (f), \\ \frac{1}{4} \sup_{t\in[a,b]} \left[g(t) - \frac{1}{t-a} \int_a^t g(u)\, du \right] \bigvee_a^b (f), \end{cases}$$

and

$$|I_2| \leq \frac{1}{(b-a)^2} \sup_{t\in[a,b]} \left[(t-a) \left[\int_t^b g(u)\, du - g(t)(b-t) \right] \right] \bigvee_a^b (f)$$

$$\leq \begin{cases} \frac{1}{b-a}\sup_{t\in[a,b]}\left[\int_t^b g(u)\,du - g(t)(b-t)\right]\bigvee_a^b(f), \\ \frac{1}{4}\sup_{t\in[a,b]}\left[\frac{1}{b-t}\int_t^b g(u)\,du - g(t)\right]\bigvee_a^b(f), \end{cases}$$

and the inequality (3.4) is also proved. □

The following result concerning a differentiable function $g : [a, b] \to \mathbb{R}$ also holds.

Theorem 3.2. *Assume that* $f : [a, b] \to \mathbb{R}$ *is of bounded variation and* $g : [a, b] \to \mathbb{R}$ *is differentiable on* (a, b). *Then,*

$$|T(f,g)| \leq \frac{1}{(b-a)^2}\bigvee_a^b(f) \tag{3.5}$$

$$\times \begin{cases} \sup_{t\in[a,b]}\left[(b-t)(t-a)\|g'\|_{[a,t],1}\right] \\ \qquad + \sup_{t\in[a,b]}\left[(b-t)(t-a)\|g'\|_{[t,b],1}\right] \quad \text{if } g' \in L_1[a,b]; \\ \frac{1}{(q+1)^{\frac{1}{q}}}\left\{\sup_{t\in[a,b]}\left[(b-t)(t-a)^{1+\frac{1}{q}}\|g'\|_{[a,t],p}\right]\right. \\ \qquad \left. + \sup_{t\in[a,b]}\left[(t-a)(b-t)^{1+\frac{1}{q}}\|g'\|_{[t,b],p}\right]\right\} \\ \qquad\qquad \text{if } g' \in L_p[a,b],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{2}\left\{\sup_{t\in[a,b]}\left[(b-t)(t-a)^2\|g'\|_{[a,t],\infty}\right]\right. \\ \qquad \left. + \sup_{t\in[a,b]}\left[(t-a)(b-t)^2\|g'\|_{[t,b],\infty}\right]\right\} \quad \text{if } g' \in L_\infty[a,b] \end{cases}$$

$$\leq \bigvee_a^b (f) \times \begin{cases} \frac{1}{2}\|g'\|_{[a,b],1} & \textit{if } g' \in L_1[a,b]; \\ \frac{2q(q+1)(b-a)^{\frac{1}{q}}}{(2q+1)^{\frac{1}{q}+2}}\|g'\|_{[a,b],p} & \textit{if } g' \in L_p[a,b],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{4(b-a)}{27}\|g'\|_{[a,b],\infty} & \textit{if } g' \in L_\infty[a,b], \end{cases}$$

where the Lebesgue norms over an interval $[c,d]$ *are defined by*

$$\|h\|_{[c,d],p} := \left(\int_c^d |h(t)|^p\, dt\right)^{\frac{1}{p}},\quad 1 \leq p < \infty$$

and

$$\|h\|_{[c,d],\infty} := ess \sup_{t\in[c,d]} |h(t)|.$$

Proof. Since g is differentiable on (a,b), we have

$$\text{(3.6)}\qquad \left|\int_a^t (u-a)\, dg(u)\right| = \left|\int_a^t (u-a)\, g'(u)\, du\right| \leq \begin{cases} (t-a)\|g'\|_{[a,t],1} \\ \left(\int_a^t (u-a)^q du\right)^{\frac{1}{q}}\|g'\|_{[a,t],p},\ p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \int_a^t (u-a)\, du\, \|g'\|_{[a,t],\infty} \end{cases}$$

$$= \begin{cases} (t-a)\left\|g'\right\|_{[a,t],1} \\ \frac{(t-a)^{1+\frac{1}{q}}}{(q+1)^{\frac{1}{q}}}\left\|g'\right\|_{[a,t],p}, \ p>1, \ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{(t-a)^2}{2}\left\|g'\right\|_{[a,t],\infty} \end{cases}$$

and, similarly,

$$\left|\int_t^b (b-u)\,dg(u)\right| \leq \begin{cases} (b-t)\left\|g'\right\|_{[t,b],1} \\ \frac{(b-t)^{1+\frac{1}{q}}}{(q+1)^{\frac{1}{q}}}\left\|g'\right\|_{[t,b],p}, \ p>1, \ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{(b-t)^2}{2}\left\|g'\right\|_{[t,b],\infty}. \end{cases} \tag{3.7}$$

With the notation in Theorem 3.1, we have on using (3.6)

$$|I_1| \leq \frac{1}{(b-a)^2}\bigvee_a^b(f)\cdot\sup_{t\in[a,b]} \begin{cases} (b-t)(t-a)\left\|g'\right\|_{[a,t],1} \\ \frac{(b-t)(t-a)^{1+\frac{1}{q}}}{(q+1)^{\frac{1}{q}}}\left\|g'\right\|_{[a,t],p}, \ \ p>1, \ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{(b-t)(t-a)^2}{2}\left\|g'\right\|_{[a,t],\infty} \end{cases}$$

and from (3.7)

$$|I_2| \leq \frac{1}{(b-a)^2}\bigvee_a^b(f)\cdot\sup_{t\in[a,b]} \begin{cases} (t-a)(b-t)\left\|g'\right\|_{[t,b],1} \\ \frac{(t-a)(b-t)^{1+\frac{1}{q}}}{(q+1)^{\frac{1}{q}}}\left\|g'\right\|_{[t,b],p}, \ \ p>1, \ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{(t-a)(b-t)^2}{2}\left\|g'\right\|_{[t,b],\infty}. \end{cases}$$

Further, since

$$|T(f,g)| \leq |I_1| + |I_2|,$$

we deduce the first inequality in (3.5).

Now, observe that

$$\begin{aligned}\sup_{t\in[a,b]}\left[(b-t)(t-a)\|g'\|_{[a,t],1}\right] &\leq \sup_{t\in[a,b]}[(b-t)(t-a)]\sup_{t\in[a,b]}\|g'\|_{[a,t],1} \\ &= \frac{(b-a)^2}{4}\|g'\|_{[a,b],1};\end{aligned}$$

$$\begin{aligned}&\sup_{t\in[a,b]}\left[\frac{(b-t)(t-a)^{1+\frac{1}{q}}}{(q+1)^{\frac{1}{q}}}\|g'\|_{[a,t],p}\right] \\ &\qquad\leq \frac{1}{(q+1)^{\frac{1}{q}}}\sup_{t\in[a,b]}\left[(b-t)(t-a)^{1+\frac{1}{q}}\right]\sup_{t\in[a,b]}\|g'\|_{[a,t],p} \\ &\qquad= M_q\|g'\|_{[a,b],p}\end{aligned}$$

where

$$M_q := \frac{1}{(q+1)^{\frac{1}{q}}}\sup_{t\in[a,b]}\left[(b-t)(t-a)^{1+\frac{1}{q}}\right].$$

Consider the arbitrary function $\rho(t) = (b-t)(t-a)^{r+1}$, $r > 0$. Then $\rho'(t) = (t-a)^r[(r+1)b + a - (r+2)t]$ showing that

$$\sup_{t\in[a,b]}\rho(t) = \rho\left[\frac{a+(r+1)b}{r+2}\right] = \frac{(b-a)^{r+2}(r+1)^{r+1}}{(r+2)^{r+2}}.$$

Consequently,

$$M_q = \frac{q}{(q+1)^{\frac{1}{q}}} \cdot \frac{(b-a)^{2+\frac{1}{q}} (q+1)^{1+\frac{1}{q}}}{(2q+1)^{2+\frac{1}{q}}} = \frac{q(q+1)(b-a)^{2+\frac{1}{q}}}{(2q+1)^{2+\frac{1}{q}}}.$$

Also,

$$\sup_{t\in[a,b]} \left[\frac{(b-t)(t-a)^2}{2} \|g'\|_{[a,t],\infty} \right] \le \frac{1}{2} \sup_{t\in[a,b]} \left[(b-t)(t-a)^2 \right] \sup_{t\in[a,b]} \|g'\|_{[a,t],\infty} = \frac{2(b-a)^3}{27} \|g'\|_{[a,b],\infty}.$$

In a similar fashion we have

$$\sup_{t\in[a,b]} \left[(t-a)(b-t) \|g'\|_{[t,b],1} \right] \le \frac{(b-a)^2}{4} \|g'\|_{[a,b],1};$$

$$\sup_{t\in[a,b]} \left[\frac{(t-a)(b-t)^{1+\frac{1}{q}}}{(q+1)^{\frac{1}{q}}} \|g'\|_{[t,b],p} \right] \le \frac{q(q+1)(b-a)^{2+\frac{1}{q}}}{(2q+1)^{2+\frac{1}{q}}} \|g'\|_{[a,b],p},$$

and

$$\sup_{t\in[a,b]} \left[\frac{(t-a)(b-t)^2}{2} \|g'\|_{[t,b],\infty} \right] \le \frac{2(b-a)^3}{27} \|g'\|_{[a,b],\infty}$$

and the last part of (3.5) is thus completely proved. □

Lemma 3.3. *Let $g : [a,b] \to \mathbb{R}$ be absolutely continuous on $[a,b]$ then for*

$$\varphi(t) = \mathcal{M}(g;t,b) - \mathcal{M}(g;a,t), \tag{3.8}$$

with $\mathcal{M}(g;c,d)$ defined by (1.6),

$$\|\varphi\|_\infty \le \begin{cases} \left(\frac{b-a}{2}\right)\|g'\|_\infty, & g' \in L_\infty[a,b]; \\ \frac{b-a}{(\beta+1)^{\frac{1}{\beta}}}\|g'\|_\alpha, & g' \in L_\alpha[a,b], \quad \alpha > 1, \ \frac{1}{\alpha}+\frac{1}{\beta} = 1; \\ \|g'\|_1, & g' \in L_1[a,b]; \\ \bigvee_a^b(g), & g \text{ of bounded variation;} \\ \left(\frac{b-a}{2}\right)L, & g \text{ is } L-\text{Lipschitzian,} \end{cases} \tag{3.9}$$

and for $p \ge 1$

$$\|\varphi\|_p \le \begin{cases} \left(\frac{b-a}{2}\right)^{1+\frac{1}{p}}\|g'\|_\infty, \ g' \in L_\infty[a,b]; \\ \left(\int_a^b \left[\frac{(t-a)^\beta+(b-t)^\beta}{\beta+1}\right]^{\frac{p}{\beta}} dt\right)^{\frac{1}{p}}\|g'\|_\alpha, \\ \qquad g' \in L_\alpha[a,b], \quad \alpha > 1, \ \frac{1}{\alpha}+\frac{1}{\beta} = 1; \\ (b-a)^{\frac{1}{p}}\|g'\|_1, \quad g' \in L_1[a,b]; \\ (b-a)^{\frac{1}{p}}\bigvee_a^b(g), \ g \text{ of bounded variation;} \\ \left(\frac{b-a}{2}\right)^{1+\frac{1}{p}}L, \quad g \text{ is } L-\text{Lipschitzian.} \end{cases} \tag{3.10}$$

Proof. Identifying $\varphi(t)$ with $D(g;a,t,b)$ of (1.7) produces bounds for $|\varphi(t)|$ from (1.8). Taking the supremum over $t \in [a,b]$ readily gives (3.9), a bound for $\|\varphi\|_\infty$.

The bound for $\|\varphi\|_p$ is obtained from (1.8) using the definition of the Lebesque $p-$norms over $[a,b]$. □

Remark 3.1. *Utilising (3.9) of Lemma 3.3 in (2.5) produces a coarser upper bound for $|T(f,g)|$. Making use of the whole of Lemma 3.3 in (2.6) produces coarser bounds for (2.6) which may prove more amenable in practical situations.*

Corollary 3.4. *Let the conditions of Theorem 2.3 hold, then*

$$|T(f,g)| \le \frac{1}{4}\bigvee_a^b(f)\begin{cases}\left(\frac{b-a}{2}\right)\|g'\|_\infty, & g' \in L_\infty[a,b];\\ \frac{b-a}{(\beta+1)^{\frac{1}{\beta}}}\|g'\|_\alpha, & g' \in L_\alpha[a,b],\\ & \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1;\\ \|g'\|_1, & g' \in L_1[a,b];\\ \bigvee_a^b(g), & g \text{ of bounded variation;}\\ \left(\frac{b-a}{2}\right)L, & g \text{ is } L-\text{Lipschitzian.}\end{cases} \tag{3.11}$$

Proof. Using (3.9) in (2.5) produces (3.11). □

Remark 3.2. *We note from the last two inequalities of (3.11) that the bounds produced are sharper than those of Theorem 3.1, giving constants of $\frac{1}{4}$ and $\frac{1}{8}$*

compared with $\frac{1}{2}$ *and* $\frac{4}{27}$ *of equations (3.2) and (3.3). For* g *differentiable then we notice that the first and third results of (3.11) are sharper than the first and third results in the second cluster of (3.5). The first cluster in (3.5) are sharper where the analysis is done over the two subintervals* $[a, x]$ *and* $(x, b]$.

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