journal of inequalities in pure and applied mathematics
http://jipam.vu.edu.au
issn: 1443-5756
Volume 8 (2007), Issue 3, Article 73, 11 pp.
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CHARACTERIZATION OF BESOV SPACES FOR THE DUNKL OPERATOR ON THE REAL LINE

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Received 07 February, 2007; accepted 28 August, 2007
Communicated by S.S. Dragomir

ABSTRACT. In this paper, we define subspaces of L^p by differences using the Dunkl translation operators that we call Besov-Dunkl spaces. We provide characterization of these spaces by the Dunkl convolution.

Key words and phrases: Dunkl operators, Dunkl transform, Dunkl translation operators, Dunkl convolution, Besov-Dunkl spaces.

2000 *Mathematics Subject Classification.* 46E30, 44A15, 44A35.

1. INTRODUCTION

On the real line, Dunkl operators are differential-difference operators introduced in 1989, by C. Dunkl in [5] and are denoted by Λ_α, where α is a real parameter $> -\frac{1}{2}$. These operators, are associated with the reflection group $\mathbb{Z}_2$ on $\mathbb{R}$. The Dunkl kernel E_α is used to define the Dunkl transform $\mathcal{F}_\alpha$ which was introduced by C. Dunkl in [6]. Rösler in [13] showed that the Dunkl kernel satisfies a product formula. This allows us to define the Dunkl translation τ_x, $x \in \mathbb{R}$. As a result, we have the Dunkl convolution $*_\alpha$.

There are many ways to define Besov spaces (see [4, 12, 16]). This paper deals with Besov-Dunkl spaces (see [1, 2, 8]). Let $\beta > 0$, $1 \leq p < +\infty$ and $1 \leq q \leq +\infty$, the Besov-Dunkl space denoted by $\mathcal{BD}^{p,q}_{\beta,\alpha}$ is the subspace of functions $f \in L^p(\mu_\alpha)$ satisfying

$$\int_0^{+\infty} \left(\frac{\|\tau_x(f) + \tau_{-x}(f) - 2f\|_{p,\alpha}}{x^\beta} \right)^q \frac{dx}{x} < +\infty \qquad \text{if} \quad q < +\infty$$

The authors thank the referee for his remarks and suggestions.
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and

$$\sup_{x\in(0,+\infty)} \frac{\|\tau_x(f)+\tau_{-x}(f)-2f\|_{p,\alpha}}{x^\beta} < +\infty \qquad \text{if} \quad q=+\infty,$$

where μ_α is a weighted Lebesgue measure on $\mathbb{R}$ (see next section).

Put

$$\mathcal{A} = \left\{ \phi \in \mathcal{S}_*(\mathbb{R}) \ : \ \int_0^{+\infty} \phi(x) d\mu_\alpha(x) = 0 \right\},$$

where $\mathcal{S}_*(\mathbb{R})$ is the space of even Schwartz functions on $\mathbb{R}$. Given $\phi \in \mathcal{A}$, we shall denote by $\mathcal{C}^{p,q}_{\phi,\beta,\alpha}$ the subspace of functions $f \in L^p(\mu_\alpha)$ satisfying

$$\int_0^{+\infty} \left(\frac{\|f *_\alpha \phi_t\|_{p,\alpha}}{t^\beta} \right)^q \frac{dt}{t} < +\infty \qquad \text{if} \quad q<+\infty$$

and

$$\sup_{t\in(0,+\infty)} \frac{\|f *_\alpha \phi_t\|_{p,\alpha}}{t^\beta} < +\infty \qquad \text{if} \quad q=+\infty,$$

where $\phi_t(x) = \frac{1}{t^{2(\alpha+1)}}\phi(\frac{x}{t})$, for all $t \in (0,+\infty)$ and $x \in \mathbb{R}$.

Our objective will be to prove that $\mathcal{BD}^{p,q}_{\beta,\alpha} \subset \mathcal{C}^{p,q}_{\phi,\beta,\alpha}$ and when $1 < p < +\infty$, $0 < \beta < 1$ then $\mathcal{BD}^{p,q}_{\beta,\alpha} = \mathcal{C}^{p,q}_{\phi,\beta,\alpha}$.

Observe that the Besov-Dunkl spaces are independent of the specific selection of ϕ in $\mathcal{A}$ and for $1 < p < +\infty$, $0 < \beta < 1$, we have $\mathcal{BD}^{p,q}_{\beta,\alpha} \subset \widetilde{\mathcal{BD}}^{p,q}_{\beta,\alpha}$, where $\widetilde{\mathcal{BD}}^{p,q}_{\beta,\alpha}$ is the subspace of functions $f \in L^p(\mu_\alpha)$ satisfying

$$\int_0^{+\infty} \left(\frac{\|\tau_x(f)-f\|_{p,\alpha}}{x^\beta} \right)^q \frac{dx}{x} < +\infty \qquad \text{if} \quad q<+\infty$$

and

$$\sup_{x\in(0,+\infty)} \frac{\|\tau_x(f)-f\|_{p,\alpha}}{x^\beta} < +\infty \qquad \text{if} \quad q=+\infty,$$

(see Remark 3.7 in Section 3, below).

Analogous results have been obtained for the weighted Besov spaces (see [3]).

The contents of this paper are as follows. In Section 2, we collect some basic definitions and results about harmonic analysis associated with Dunkl operators .In Section 3, we prove the results about inclusion and coincidence between the spaces $\mathcal{BD}^{p,q}_{\beta,\alpha}$ and $\mathcal{C}^{p,q}_{\phi,\beta,\alpha}$.

In what follows, c represents a suitable positive constant which is not necessarily the same in each occurence.

2. PRELIMINARIES

On the real line, we consider the first-order differential-difference operator defined by

$$\Lambda_\alpha(f)(x) = \frac{df}{dx}(x) + \frac{2\alpha+1}{x}\left[\frac{f(x)-f(-x)}{2}\right], \quad f \in \mathcal{E}(\mathbb{R}), \quad \alpha > -\frac{1}{2},$$

which is called the Dunkl operator. For $\lambda \in \mathbb{C}$, the Dunkl kernel $E_\alpha(\lambda\ .)$ on $\mathbb{R}$ was introduced by C. Dunkl in [5] and is given by

$$E_\alpha(\lambda x) = j_\alpha(i\lambda x) + \frac{\lambda x}{2(\alpha+1)} j_{\alpha+1}(i\lambda x), \quad x \in \mathbb{R},$$

where j_α is the normalized Bessel function of the first kind of order α (see [17]). The Dunkl kernel $E_\alpha(\lambda\ .)$ is the unique solution on $\mathbb{R}$ of the initial problem for the Dunkl operator (see [5]).

Let μ_α be the weighted Lebesgue measure on $\mathbb{R}$ given by

$$d\mu_\alpha(x) = \frac{|x|^{2\alpha+1}}{2^{\alpha+1}\Gamma(\alpha+1)}dx.$$

For every $1 \leq p \leq +\infty$, we denote by $L^p(\mu_\alpha)$ the space $L^p(\mathbb{R}, d\mu_\alpha)$ and we use $\|\cdot\|_{p,\alpha}$ as a shorthand for $\|\cdot\|_{L^p(\mu_\alpha)}$.

The Dunkl transform $\mathcal{F}_\alpha$ which was introduced by C. Dunkl in [6], is defined for $f \in L^1(\mu_\alpha)$ by

$$\mathcal{F}_\alpha(f)(x) = \int_{\mathbb{R}} E_\alpha(-ixy)f(y)d\mu_\alpha(y), \quad x \in \mathbb{R}.$$

For all $x, y, z \in \mathbb{R}$, consider

$$W_\alpha(x,y,z) = \frac{(\Gamma(\alpha+1)^2)}{2^{\alpha-1}\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}(1 - b_{x,y,z} + b_{z,x,y} + b_{z,y,x})\Delta_\alpha(x,y,z), \tag{2.1}$$

where

$$b_{x,y,z} = \begin{cases} \frac{x^2+y^2-z^2}{2xy} & \text{if } x,y \in \mathbb{R}\backslash\{0\},\ z \in \mathbb{R} \\ 0 & \text{otherwise} \end{cases}$$

and

$$\Delta_\alpha(x,y,z) = \begin{cases} \frac{([(|x|+|y|)^2-z^2][z^2-(|x|-|y|)^2])^{\alpha-\frac{1}{2}}}{|xyz|^{2\alpha}} & \text{if } |z| \in S_{x,y} \\ 0 & \text{otherwise} \end{cases}$$

where

$$S_{x,y} = \Big[||x|-|y||\ ,\ |x|+|y|\Big].$$

The kernel W_α (see [13]) is even and we have

$$W_\alpha(x,,y,z) = W_\alpha(y,x,z) = W_\alpha(-x,z,y) = W_\alpha(-z,y,-x)$$

and

$$\int_{\mathbb{R}} |W_\alpha(x,y,z)|d\mu_\alpha(z) \leq 4.$$

In the sequel we consider the signed measure $\gamma_{x,y}$, on $\mathbb{R}$, given by

$$d\gamma_{x,y}(z) = \begin{cases} W_\alpha(x,y,z)d\mu_\alpha(z) & \text{if } x,y \in \mathbb{R}\backslash\{0\} \\ d\delta_x(z) & \text{if } y = 0 \\ d\delta_y(z) & \text{if } x = 0. \end{cases} \tag{2.2}$$

For $x, y \in \mathbb{R}$ and f a continuous function on $\mathbb{R}$, the Dunkl translation operator τ_x is given by

$$\tau_x(f)(y) = \int_{\mathbb{R}} f(z)d\gamma_{x,y}(z).$$

According to [9], for $x \in \mathbb{R}$, τ_x is a continuous linear operator from $\mathcal{E}(\mathbb{R})$ into itself and for all $f \in \mathcal{E}(\mathbb{R})$, we have

$$\tau_x(f)(y) = \tau_y(f)(x), \quad \tau_0(f)(x) = f(x)\,, \quad \text{for } x,\ y \in \mathbb{R},$$

where $\mathcal{E}(\mathbb{R})$ denotes the space of C^∞-functions on $\mathbb{R}$.

According to [14, 15], the operator τ_x can be extended to $L^p(\mu_\alpha)$, $1 \leq p \leq +\infty$ and for $f \in L^p(\mu_\alpha)$ we have

$$\|\tau_x(f)\|_{p,\alpha} \leq 4\|f\|_{p,\alpha}. \tag{2.3}$$

Using the change of variable $z = \Psi(x, y, \theta) = \sqrt{x^2 + y^2 - 2xy\cos\theta}$, we have also

$$\tau_x(f)(y) = c_\alpha \int_0^\pi \left[f(\Psi) + f(-\Psi) + \frac{x+y}{\Psi}(f(\Psi) - f(-\Psi)) \right] d\nu_\alpha(\theta), \tag{2.4}$$

where $d\nu_\alpha(\theta) = (1 - \cos\theta)\sin^{2\alpha}\theta d\theta$ and $c_\alpha = \frac{\Gamma(\alpha+1)}{2\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}$.

From the generalized Taylor formula with integral remainder (see [11, Theorem 2, p. 349]), we have for $f \in \mathcal{E}(\mathbb{R})$ and $x,\, y \in \mathbb{R}$

$$(\tau_x(f) - f)(y) = \int_{-|x|}^{|x|} \left(\frac{\operatorname{sgn}(x)}{2|x|^{2\alpha+1}} + \frac{\operatorname{sgn}(z)}{2|z|^{2\alpha+1}} \right) \tau_z(\Lambda_\alpha f)(y)\, d\mu_\alpha(z).$$

The Dunkl convolution $f *_\alpha g$, of two continuous functions f and g on $\mathbb{R}$ with compact support, is defined by

$$(f *_\alpha g)(x) = \int_{\mathbb{R}} \tau_x(f)(-y) g(y) d\mu_\alpha(y), \quad x \in \mathbb{R}.$$

The convolution $*_\alpha$ is associative and commutative (see [13]).

We have the following results (see [14]).

i) Assume that $p, q, r \in [1, +\infty[$ satisfying $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$ (the Young condition). Then the map $(f, g) \to f *_\alpha g$ defined on $C_c(\mathbb{R}) \times C_c(\mathbb{R})$, extends to a continuous map from $L^p(\mu_\alpha) \times L^q(\mu_\alpha)$ to $L^r(\mu_\alpha)$ and we have

$$\|f *_\alpha g\|_{r,\alpha} \leq 4\|f\|_{p,\alpha}\|g\|_{q,\alpha}. \tag{2.5}$$

ii) For all $f \in L^1(\mu_\alpha)$ and $g \in L^2(\mu_\alpha)$, we have

$$\mathcal{F}_\alpha(f *_\alpha g) = \mathcal{F}_\alpha(f)\mathcal{F}_\alpha(g) \tag{2.6}$$

and for $f \in L^1(\mu_\alpha)$, $g \in L^p(\mu_\alpha)$ and $1 \leq p < \infty$, we get

$$\tau_t(f *_\alpha g) = \tau_t(f) *_\alpha g = f *_\alpha \tau_t(g), \quad t \in \mathbb{R}. \tag{2.7}$$

3. CHARACTERIZATION OF BESOV-DUNKL SPACES

Let $\beta > 0$, $1 \leq p < +\infty$ and $1 \leq q \leq +\infty$. We say that a measurable function f on $\mathbb{R}$ is in the Besov-Dunkl space $\mathcal{BD}^{p,q}_{\beta,\alpha}$ if $f \in L^p(\mu_\alpha)$ with

$$\int_0^{+\infty} \left(\frac{\|\tau_x(f) + \tau_{-x}(f) - 2f\|_{p,\alpha}}{x^\beta} \right)^q \frac{dx}{x} < +\infty \qquad \text{if} \quad q < +\infty$$

and

$$\sup_{x \in (0,+\infty)} \frac{\|\tau_x(f) + \tau_{-x}(f) - 2f\|_{p,\alpha}}{x^\beta} < +\infty \qquad \text{if} \quad q = +\infty.$$

Remark 3.1. Note that for $f \in L^p(\mu_\alpha)$ the function $\mathbb{R} \to \mathbb{R}^+$, $x \mapsto \|\tau_x(f) + \tau_{-x}(f) - 2f\|_{p,\alpha}$ is measurable (see [10, Lemma 1, (ii)]).

Lemma 3.2. *Let $0 < \beta < 1$, $1 \leq p < +\infty$, $1 \leq q \leq +\infty$ and $f \in L^p(\mu_\alpha)$. If $\Lambda_\alpha(f) \in L^p(\mu_\alpha)$ then $f \in \mathcal{BD}^{p,q}_{\beta,\alpha}$.*

Proof. Using the generalized Taylor formula, Minkowski's inequality for integrals and (2.3), we can write

$$\begin{aligned} \|\tau_x(f) + \tau_{-x}(f) - 2f\|_{p,\alpha} &\leq \|\tau_x(f) - f\|_{p,\alpha} + \|\tau_{-x}(f) - f\|_{p,\alpha} \\ &\leq c\,\|\Lambda_\alpha(f)\|_{p,\alpha} \int_{-x}^{x} \left(\frac{1}{2|x|^{2\alpha+1}} + \frac{1}{2|z|^{2\alpha+1}} \right) d\mu_\alpha(z)\,, \end{aligned}$$

hence we obtain for $x > 0$,

$$\|\tau_x(f) + \tau_{-x}(f) - 2f\|_{p,\alpha} \le c\, x\, \|\Lambda_\alpha(f)\|_{p,\alpha}\,.$$

Then it follows that for $A > 0$

$$\int_0^{+\infty} \left(\frac{\|\tau_x(f) + \tau_{-x}(f) - 2f\|_{p,\alpha}}{x^\beta}\right)^q \frac{dx}{x} \le c \int_0^A \left(\frac{x\|\Lambda_\alpha(f)\|_{p,\alpha}}{x^\beta}\right)^q \frac{dx}{x} + c \int_A^{+\infty} \left(\frac{\|f\|_{p,\alpha}}{x^\beta}\right)^q \frac{dx}{x} < +\infty.$$

Here when $q = +\infty$, we make the usual modification. This completes the proof. □

Example 3.1. Let $0 < \beta < 1$, $1 \le p < +\infty$ and $1 \le q \le +\infty$. By Lemma 3.2, we can assert that

(1) $\mathcal{S}(\mathbb{R}),\ C_c^1(\mathbb{R}) \subset \mathcal{BD}^{p,q}_{\beta,\alpha}$.
(2) The functions g, h_n defined on $\mathbb{R}$, by $g(x) = e^{-|x|}$ and $h_n(x) = \frac{x^n}{\cosh x}$, $n \in \mathbb{N}$ are in $\mathcal{BD}^{p,q}_{\beta,\alpha}$.

Now, in order to establish that for all $\phi \in \mathcal{A}$, $\mathcal{BD}^{p,q}_{\beta,\alpha} \subset \mathcal{C}^{p,q}_{\phi,\beta,\alpha}$ and for $1 < p < +\infty$, $0 < \beta < 1$, $\mathcal{BD}^{p,q}_{\beta,\alpha} = \mathcal{C}^{p,q}_{\phi,\beta,\alpha}$, we need to show some useful lemmas.

Lemma 3.3. *Let $\phi \in \mathcal{A}$, $1 \le p < +\infty$ and $r > 0$, then there exists a constant $c > 0$ such that for all $f \in L^p(\mu_\alpha)$ and $t > 0$, we have*

$$\|\phi_t *_\alpha f\|_{p,\alpha} \le c \int_0^{+\infty} \min\left\{\left(\frac{x}{t}\right)^{2(\alpha+1)}, \left(\frac{t}{x}\right)^r\right\} \|\tau_x(f) + \tau_{-x}(f) - 2f\|_{p,\alpha} \frac{dx}{x}. \tag{3.1}$$

Proof. Let $t > 0$, we have

$$\int_0^{+\infty} \phi_t(x) d\mu_\alpha(x) = \int_0^{+\infty} \phi(x) d\mu_\alpha(x) = 0$$

and

$$\begin{aligned}(\phi_t *_\alpha f)(y) &= \int_{\mathbb{R}} \phi_t(x)\tau_y(f)(-x) d\mu_\alpha(x)\\ &= \int_{\mathbb{R}} \phi_t(x)\tau_y(f)(x) d\mu_\alpha(x),\end{aligned}$$

then we can write for $y \in \mathbb{R}$

$$\begin{aligned}2(\phi_t *_\alpha f)(y) &= \int_{\mathbb{R}} \phi_t(x)\left[\tau_y(f)(x) + \tau_y(f)(-x) - 2f(y)\right] d\mu_\alpha(x)\\ &= 2\int_0^{+\infty} \phi_t(x)\left[\tau_x(f)(y) + \tau_{-x}(f)(y) - 2f(y)\right] d\mu_\alpha(x).\end{aligned}$$

Using Minkowski's inequality for integrals, we obtain

$$\|\phi_t *_\alpha f\|_{p,\alpha} \le \int_0^{+\infty} |\phi_t(x)|\ \|\tau_x(f) + \tau_{-x}(f) - 2f\|_{p,\alpha} d\mu_\alpha(x)$$

$$\le c \int_0^{+\infty} \left(\frac{x}{t}\right)^{2(\alpha+1)} \left|\phi\left(\frac{x}{t}\right)\right| \|\tau_x(f) + \tau_{-x}(f) - 2f\|_{p,\alpha} \frac{dx}{x} \tag{3.2}$$

$$\le c \int_0^{+\infty} \left(\frac{x}{t}\right)^{2(\alpha+1)} \|\tau_x(f) + \tau_{-x}(f) - 2f\|_{p,\alpha} \frac{dx}{x}. \tag{3.3}$$

On the other hand, since the function ϕ belongs to $\mathcal{S}_*(\mathbb{R})$, then for $r>0$ there exists a constant c such that

$$\left(\frac{x}{t}\right)^{2(\alpha+1)+r}\left|\phi\left(\frac{x}{t}\right)\right|\leq c.$$

By (3.2), we obtain

$$\|\phi_t *_\alpha f\|_{p,\alpha}\leq c\int_0^{+\infty}\left(\frac{t}{x}\right)^r\|\tau_x(f)+\tau_{-x}(f)-2f\|_{p,\alpha}\frac{dx}{x}. \tag{3.4}$$

From (3.3) and (3.4), we deduce (3.1). □

Lemma 3.4. *Let $\phi\in\mathcal{A}$ and $1<p<+\infty$, then there exists a constant $c>0$ such that for all $f\in L^p(\mu_\alpha)$ and $x>0$, we have*

$$\|\tau_x(f)+\tau_{-x}(f)-2f\|_{p,\alpha}\leq c\int_0^{+\infty}\min\left\{1,\frac{x}{t}\right\}\|\phi_t *_\alpha f\|_{p,\alpha}\frac{dt}{t}. \tag{3.5}$$

Proof. Put for $0<\varepsilon<\delta<+\infty$

$$f_{\varepsilon,\delta}(y)=\int_\varepsilon^\delta(\phi_t *_\alpha \phi_t *_\alpha f)(y)\frac{dt}{t},\quad y\in\mathbb{R}.$$

By interchanging the orders of integration and (2.7), we obtain

$$\begin{aligned}\tau_x(f_{\varepsilon,\delta})(y)&=\int_\varepsilon^\delta\tau_x(\phi_t *_\alpha \phi_t *_\alpha f)(y)\frac{dt}{t}\\&=\int_\varepsilon^\delta(\tau_x(\phi_t) *_\alpha \phi_t *_\alpha f)(y)\frac{dt}{t},\quad y\in\mathbb{R},\ x\in(0,+\infty),\end{aligned}$$

so we can write for $x\in(0,+\infty)$ and $y\in\mathbb{R}$,

$$(\tau_x(f_{\varepsilon,\delta})+\tau_{-x}(f_{\varepsilon,\delta})-2f_{\varepsilon,\delta})(y)=\int_\varepsilon^\delta[(\tau_x(\phi_t)+\tau_{-x}(\phi_t)-2\phi_t) *_\alpha \phi_t *_\alpha f](y)\frac{dt}{t}.$$

Using Minkowski's inequality for integrals and (2.5), we get

$$\begin{aligned}\|(\tau_x(f_{\varepsilon,\delta})+\tau_{-x}(f_{\varepsilon,\delta})-2f_{\varepsilon,\delta}\|_{p,\alpha}&\leq\int_\varepsilon^\delta\|(\tau_x(\phi_t)+\tau_{-x}(\phi_t)-2\phi_t) *_\alpha \phi_t *_\alpha f\|_{p,\alpha}\frac{dt}{t}\\&\leq c\int_\varepsilon^\delta\|\tau_x(\phi_t)+\tau_{-x}(\phi_t)-2\phi_t\|_{1,\alpha}\|\phi_t *_\alpha f\|_{p,\alpha}\frac{dt}{t}.\end{aligned} \tag{3.6}$$

For $x,\ t\in(0,+\infty)$, we have

$$\begin{aligned}&\|\tau_x(\phi_t)+\tau_{-x}(\phi_t)-2\phi_t\|_{1,\alpha}\\&=\int_\mathbb{R}\left|\left[\int_\mathbb{R}\phi_t(z)(d\gamma_{x,y}(z)+d\gamma_{-x,y}(z))\right]-2\phi_t(y)\right|d\mu_\alpha(y)\\&=\int_\mathbb{R}\left|\left[\int_\mathbb{R}\phi\left(\frac{z}{t}\right)(d\gamma_{x,y}(z)+d\gamma_{-x,y}(z))\right]-2\phi\left(\frac{y}{t}\right)\right|t^{-2(\alpha+1)}d\mu_\alpha(y).\end{aligned}$$

By (2.1) and the change of variable $z'=\frac{z}{t}$, we have

$$W_\alpha(x,y,z't)t^{2(\alpha+1)}=W_\alpha\left(\frac{x}{t},\frac{y}{t},z'\right),$$

then from (2.2), we get

$$d\gamma_{x,y}(z)=d\gamma_{\frac{x}{t},\frac{y}{t}}(z')\quad\text{and}\quad d\gamma_{-x,y}(z)=d\gamma_{\frac{-x}{t},\frac{y}{t}}(z')\ ,$$

hence

$$\begin{aligned}(3.7)\qquad &\|\tau_x(\phi_t)+\tau_{-x}(\phi_t)-2\phi_t\|_{1,\alpha}\\ &=\int_{\mathbb{R}}\left|\left[\int_{\mathbb{R}}\phi(z')\left(d\gamma_{\frac{x}{t},\frac{y}{t}}(z')+d\gamma_{\frac{-x}{t},\frac{y}{t}}(z')\right)\right]-2\phi\left(\frac{y}{t}\right)\right|t^{-2(\alpha+1)}d\mu_\alpha(y)\\ &=\int_{\mathbb{R}}\left|\left[\tau_{\frac{x}{t}}(\phi)\left(\frac{y}{t}\right)+\tau_{\frac{-x}{t}}(\phi)\left(\frac{y}{t}\right)\right]t^{-2(\alpha+1)}-2\phi_t(y)\right|d\mu_\alpha(y)\\ &=\left\|\left(\tau_{\frac{x}{t}}(\phi)+\tau_{\frac{-x}{t}}(\phi)-2\phi\right)_t\right\|_{1,\alpha}\\ &=\left\|\tau_{\frac{x}{t}}(\phi)+\tau_{\frac{-x}{t}}(\phi)-2\phi\right\|_{1,\alpha}.\end{aligned}$$

Since $\phi\in\mathcal{S}_*(\mathbb{R})$, then using (2.4) and [7, Theorem 2.1] (see also [11, Theorem 2, p. 349]), we can assert that

$$\left\|\tau_{\frac{x}{t}}(\phi)+\tau_{\frac{-x}{t}}(\phi)-2\phi\right\|_{1,\alpha}\le c\frac{x}{t}\|\phi'\|_{1,\alpha}\le c\frac{x}{t}.$$

On the other hand, by (2.3) we have

$$\left\|\tau_{\frac{x}{t}}(\phi)+\tau_{\frac{-x}{t}}(\phi)-2\phi\right\|_{1,\alpha}\le 10\|\phi\|_{1,\alpha}\le c,$$

then we get,

$$(3.8)\qquad \left\|\tau_{\frac{x}{t}}(\phi)+\tau_{\frac{-x}{t}}(\phi)-2\phi\right\|_{1,\alpha}\le c\min\left\{1,\frac{x}{t}\right\}.$$

From (3.6), (3.7) and (3.8), we obtain

$$(3.9)\qquad \|\tau_x(f_{\varepsilon,\delta})+\tau_{-x}(f_{\varepsilon,\delta})-2f_{\varepsilon,\delta}\|_{p,\alpha}\le c\int_\varepsilon^\delta \min\left\{1,\frac{x}{t}\right\}\|\phi_t*_\alpha f\|_{p,\alpha}\frac{dt}{t}.$$

Using (2.6), observe that

$$\begin{aligned}\int_{\mathbb{R}}(\phi*_\alpha\phi)(x)|x|^{2\alpha+1}dx&=2^{\alpha+1}\Gamma(\alpha+1)\mathcal{F}_\alpha(\phi*_\alpha\phi)(0)\\ &=2^{\alpha+1}\Gamma(\alpha+1)(\mathcal{F}_\alpha(\phi)(0))^2\\ &=2^{\alpha+1}\Gamma(\alpha+1)\left(\int_{\mathbb{R}}\phi(z)d\mu_\alpha(z)\right)^2=0,\end{aligned}$$

and since $\phi*_\alpha\phi$ is in the Schwarz space $\mathcal{S}(\mathbb{R})$, we have

$$\int_{\mathbb{R}}|\log|x||\;|\phi*_\alpha\phi(x)|\;|x|^{2\alpha+1}dx<+\infty.$$

Then, by the Calderón reproducing formula related to the Dunkl operators (see [10, Theorem 3]), we have

$$\lim_{\varepsilon\to0,\,\delta\to+\infty} f_{\varepsilon,\delta}=c\,f,\quad \text{in } L^p(\mu_\alpha).$$

From (2.3) and (3.9), we deduce (3.5). □

Lemma 3.5. *Let* $0\le\varepsilon,\ r<+\infty$ *and* $r>\beta>0$*, then there exists constants* $c_1,c_2>0$ *such that we have*

$$(3.10)\qquad \int_0^{+\infty}\left[\left(\frac{y}{z}\right)^\beta\min\left\{\left(\frac{y}{z}\right)^\varepsilon,\left(\frac{z}{y}\right)^r\right\}\right]\frac{dy}{y}\le c_1,\quad z\in(0,+\infty)$$

and

$$\int_0^{+\infty}\left[\left(\frac{y}{z}\right)^{\beta}\min\left\{\left(\frac{y}{z}\right)^{\varepsilon},\left(\frac{z}{y}\right)^{r}\right\}\right]\frac{dz}{z}\leq c_2,\quad y\in(0,+\infty). \tag{3.11}$$

Proof. We can write

$$\int_0^{+\infty}\left[\left(\frac{y}{z}\right)^{\beta}\min\left\{\left(\frac{y}{z}\right)^{\varepsilon},\left(\frac{z}{y}\right)^{r}\right\}\right]\frac{dy}{y}$$
$$=z^{-(\beta+\varepsilon)}\int_0^{z}y^{\beta+\varepsilon-1}dy+z^{r-\beta}\int_z^{+\infty}y^{\beta-r-1}dy\leq c_1,\quad z\in(0,+\infty)$$

and

$$\int_0^{+\infty}\left[\left(\frac{y}{z}\right)^{\beta}\min\left\{\left(\frac{y}{z}\right)^{\varepsilon},\left(\frac{z}{y}\right)^{r}\right\}\right]\frac{dz}{z}$$
$$=y^{\beta-r}\int_0^{y}z^{-\beta+r-1}dz+y^{\beta+\varepsilon}\int_y^{+\infty}z^{-\beta-\varepsilon-1}dz\leq c_2,\quad y\in(0,+\infty),$$

which proves the results. □

Theorem 3.6.

(1) *Let* $1\leq p<+\infty$, $1\leq q\leq+\infty$ *and* $\beta>0$, *then we have for all* $\phi\in\mathcal{A}$

$$\mathcal{BD}^{p,q}_{\beta,\alpha}\subset\mathcal{C}^{p,q}_{\phi,\beta,\alpha}. \tag{3.12}$$

(2) *Let* $1<p<+\infty$, $1\leq q\leq+\infty$ *and* $0<\beta<1$, *then we have for all* $\phi\in\mathcal{A}$

$$\mathcal{BD}^{p,q}_{\beta,\alpha}=\mathcal{C}^{p,q}_{\phi,\beta,\alpha}. \tag{3.13}$$

Proof. Put $\omega_p^{\alpha}(f)(x)=\|\tau_x(f)+\tau_{-x}(f)-2f\|_{p,\alpha}$ for $f\in L^p(\mu_\alpha)$ and $q'=\frac{q}{q-1}$ the conjugate of q when $1<q<+\infty$.

- We start with the proof of the inclusion (3.12). Suppose that $1\leq p<+\infty$, $1\leq q\leq+\infty$, $\phi\in\mathcal{A}$, $r>\beta$ and $f\in\mathcal{BD}^{p,q}_{\beta,\alpha}$.

Case when $q=1$. By (3.1) and Fubini's theorem, we have

$$\begin{aligned}\int_0^{+\infty}\frac{\|f*_\alpha\phi_t\|_{p,\alpha}}{t^{\beta}}\frac{dt}{t}&\leq c\int_0^{+\infty}\int_0^{+\infty}\min\left\{\left(\frac{x}{t}\right)^{2(\alpha+1)},\left(\frac{t}{x}\right)^{r}\right\}\omega_p^{\alpha}(f)(x)t^{-\beta-1}dt\frac{dx}{x}\\&\leq c\int_0^{+\infty}\omega_p^{\alpha}(f)(x)\left(\int_0^{+\infty}\min\left\{\left(\frac{x}{t}\right)^{2(\alpha+1)},\left(\frac{t}{x}\right)^{r}\right\}t^{-\beta-1}dt\right)\frac{dx}{x}\\&\leq c\int_0^{+\infty}\omega_p^{\alpha}(f)(x)\left(x^{-r}\int_0^{x}t^{r-\beta-1}dt+x^{2(\alpha+1)}\int_x^{+\infty}t^{-\beta-2\alpha-3}dt\right)\frac{dx}{x}\\&\leq c\int_0^{+\infty}\frac{\omega_p^{\alpha}(f)(x)}{x^{\beta}}\frac{dx}{x}<+\infty,\end{aligned}$$

hence $f\in\mathcal{C}^{p,1}_{\phi,\beta,\alpha}$.

Case when $q = +\infty$. By (3.1), we have

$$\begin{aligned}\|\phi_t *_\alpha f\|_{p,\alpha} &\leq c\left(\int_0^t \left(\frac{x}{t}\right)^{2(\alpha+1)} \omega_p^\alpha(f)(x)\frac{dx}{x} + \int_t^{+\infty} \left(\frac{t}{x}\right)^r \omega_p^\alpha(f)(x)\frac{dx}{x}\right) \\ &\leq c \sup_{x\in(0,+\infty)} \frac{\omega_p^\alpha(f)(x)}{x^\beta}\left(t^{-2(\alpha+1)}\int_0^t x^{2\alpha+1+\beta}dx + t^r\int_t^{+\infty} x^{-\beta-r-1}dx\right) \\ &\leq ct^\beta \sup_{x\in(0,+\infty)} \frac{\omega_p^\alpha(f)(x)}{x^\beta},\end{aligned}$$

then we deduce that $f \in \mathcal{C}^{p,+\infty}_{\phi,\beta,\alpha}$.

Case when $1 < q < +\infty$. By (3.1) again, we have for $t > 0$

$$\frac{\|\phi_t *_\alpha f\|_{p,\alpha}}{t^\beta} \leq c\int_0^{+\infty} \left(\frac{x}{t}\right)^\beta \min\left\{\left(\frac{x}{t}\right)^{2(\alpha+1)}, \left(\frac{t}{x}\right)^r\right\}\frac{\omega_p^\alpha(f)(x)}{x^\beta}\frac{dx}{x}.$$

Put

$$K(x,t) = \left(\frac{x}{t}\right)^\beta \min\left\{\left(\frac{x}{t}\right)^{2(\alpha+1)}, \left(\frac{t}{x}\right)^r\right\}.$$

Using Hölder's inequality and (3.10), we can write

$$\begin{aligned}\frac{\|\phi_t *_\alpha f\|_{p,\alpha}}{t^\beta} &\leq c\int_0^{+\infty} (K(x,t))^{\frac{1}{q'}}\left((K(x,t))^{\frac{1}{q}}\frac{\omega_p^\alpha(f)(x)}{x^\beta}\right)\frac{dx}{x} \\ &\leq c\left(\int_0^{+\infty} K(x,t)\left(\frac{\omega_p^\alpha(f)(x)}{x^\beta}\right)^q\frac{dx}{x}\right)^{\frac{1}{q}}.\end{aligned}$$

Then by Fubini's theorem and (3.11), we have

$$\begin{aligned}\int_0^{+\infty}\left(\frac{\|\phi_t *_\alpha f\|_{p,\alpha}}{t^\beta}\right)^q\frac{dt}{t} &\leq c\int_0^{+\infty}\left(\frac{\omega_p^\alpha(f)(x)}{x^\beta}\right)^q\left(\int_0^{+\infty} K(x,t)\frac{dt}{t}\right)\frac{dx}{x} \\ &\leq c\int_0^{+\infty}\left(\frac{\omega_p^\alpha(f)(x)}{x^\beta}\right)^q\frac{dx}{x} < +\infty,\end{aligned}$$

which proves the result.

- Let us now prove the equality (3.13). Assume $f \in \mathcal{C}^{p,q}_{\phi,\beta,\alpha}$, $\phi \in \mathcal{A}$ and $0 < \beta < 1$. For $1 < p < +\infty$ and $1 \leq q \leq +\infty$, we have to show only that $f \in \mathcal{BD}^{p,q}_{\beta,\alpha}$.

Case when $q = 1$. By (3.5) and Fubini's theorem, we have

$$\begin{aligned}\int_0^{+\infty}\frac{\omega_p^\alpha(f)(x)}{x^\beta}\frac{dx}{x} &\leq c\int_0^{+\infty}\int_0^{+\infty}\min\left\{1,\frac{x}{t}\right\}\|\phi_t *_\alpha f\|_{p,\alpha}x^{-\beta-1}\frac{dt}{t}dx \\ &\leq c\int_0^{+\infty}\|\phi_t *_\alpha f\|_{p,\alpha}\left(\int_0^{+\infty}\min\left\{1,\frac{x}{t}\right\}x^{-\beta-1}dx\right)\frac{dt}{t} \\ &\leq c\int_0^{+\infty}\|\phi_t *_\alpha f\|_{p,\alpha}\left(\frac{1}{t}\int_0^t x^{-\beta}dx + \int_t^{+\infty} x^{-\beta-1}dx\right)\frac{dt}{t} \\ &\leq c\int_0^{+\infty}\frac{\|\phi_t *_\alpha f\|_{p,\alpha}}{t^\beta}\frac{dt}{t} < +\infty,\end{aligned}$$

then we obtain the result.

Case when $q = +\infty$. By (3.5), we get

$$\begin{aligned}\omega_p^\alpha(f)(x) &\leq c\left(\int_0^x \|\phi_t *_\alpha f\|_{p,\alpha}\frac{dt}{t} + \int_x^{+\infty} \frac{x}{t}\|\phi_t *_\alpha f\|_{p,\alpha}\frac{dt}{t}\right)\\ &\leq c \sup_{t\in(0,+\infty)} \frac{\|\phi_t *_\alpha f\|_{p,\alpha}}{t^\beta}\left(\int_0^x t^{\beta-1}dt + x\int_x^{+\infty} t^{\beta-2}dt\right)\\ &\leq cx^\beta \sup_{t\in(0,+\infty)} \frac{\|\phi_t *_\alpha f\|_{p,\alpha}}{t^\beta},\end{aligned}$$

so, we deduce that $f \in \mathcal{BD}^{p,+\infty}_{\beta,\alpha}$.

Case when $1 < q < +\infty$. By (3.5) again, we have for $x > 0$

$$\frac{\omega_p^\alpha(f)(x)}{x^\beta} \leq c\int_0^{+\infty}\left(\frac{t}{x}\right)^\beta \min\left\{1, \frac{x}{t}\right\}\frac{\|\phi_t *_\alpha f\|_{p,\alpha}}{t^\beta}\frac{dt}{t}.$$

Put

$$R(x,t) = \left(\frac{t}{x}\right)^\beta \min\left\{1, \frac{x}{t}\right\}.$$

Using Hölder's inequality and (3.10), we can write

$$\begin{aligned}\frac{\omega_p^\alpha(f)(x)}{x^\beta} &\leq c\int_0^{+\infty}(R(x,t))^{\frac{1}{q'}}\left((R(x,t))^{\frac{1}{q}}\frac{\|\phi_t *_\alpha f\|_{p,\alpha}}{t^\beta}\right)\frac{dt}{t}\\ &\leq c\left(\int_0^{+\infty} R(x,t)\left(\frac{\|\phi_t *_\alpha f\|_{p,\alpha}}{t^\beta}\right)^q\frac{dt}{t}\right)^{\frac{1}{q}},\end{aligned}$$

then by Fubini's theorem and (3.11), we have

$$\begin{aligned}\int_0^{+\infty}\left(\frac{\omega_p^\alpha(f)(x)}{x^\beta}\right)^q\frac{dx}{x} &\leq c\int_0^{+\infty}\left(\frac{\|\phi_t *_\alpha f\|_{p,\alpha}}{t^\beta}\right)^q\left(\int_0^{+\infty}R(x,t)\frac{dx}{x}\right)\frac{dt}{t}\\ &\leq c\int_0^{+\infty}\left(\frac{\|\phi_t *_\alpha f\|_{p,\alpha}}{t^\beta}\right)^q\frac{dt}{t} < +\infty,\end{aligned}$$

thus the result is established. □

Remark 3.7. By proceeding in the same manner as in Lemma 3.4 and (2) of Theorem 3.6, we can assert that for $1 < p < +\infty$ and $0 < \beta < 1$, we have $\mathcal{C}^{p,q}_{\phi,\beta,\alpha} \subset \widetilde{\mathcal{BD}}^{p,q}_{\beta,\alpha}$, hence from (3.13) we conclude that $\mathcal{BD}^{p,q}_{\beta,\alpha} \subset \widetilde{\mathcal{BD}}^{p,q}_{\beta,\alpha}$.

REFERENCES

[1] C. ABDELKEFI AND M. SIFI, On the uniform convergence of partial Dunkl integrals in Besov-Dunkl spaces, *Fractional Calculus and Applied Analysis*, **9**(1) (2006), 43–56.

[2] C. ABDELKEFI AND M. SIFI, Further results of integrability for the Dunkl transform, *Communications in Mathematical Analysis*, **2**(1) (2007), 29–36.

[3] J.L. ANSORENA AND O. BLASCO, Characterization of weighted Besov spaces, *Math. Nachr.*, **171** (1995), 5–17.

[4] O.V. BESOV, On a family of function spaces in connection with embeddings and extentions, *Trudy Mat. Inst. Steklov*, **60** (1961), 42–81.

[5] C.F. DUNKL, Differential-difference operators associated to reflection groups, *Trans.Amer. Math. Soc.*, **311**(1) (1989), 167–183.

[6] C.F. DUNKL, Hankel transforms associated to finite reflection groups in *Proc. of Special Session on Hypergeometric Functions on Domains of Positivity, Jack Polynomials and Applications*. Proceedings, Tampa 1991, *Contemp. Math.*, **138** (1992), 123–138.

[7] J. GOSSELIN AND K. STEMPAK, A weak type estimate for Fourier-Bessel multipliers, *Proc. Amer. Math. Soc.*, **106** (1989), 655–662.

[8] L. KAMOUN, Besov-type spaces for the Dunkl operator on the real line, *J. Comp. and Appl. Math.*, **199**(1) (2007), 56–67.

[9] M.A. MOUROU, Transmutation operators associated with a Dunkl-type differential-difference operator on the real line and certain of their applications, *Integral transforms Spec. Funct.*, **12**(1) (2001), 77–88.

[10] M.A. MOUROU AND K. TRIMÈCHE, Calderon's reproducing formula related to the Dunkl operator on the real line, *Monatshefte für Mathematik*, **136** (2002), 47–65.

[11] M.A. MOUROU, Taylor series associated with a differential-difference operator on the real line, *J. Comp. and Appl. Math.*, **153** (2003), 343–354.

[12] J. PEETRE, *New Thoughts on Besov Spaces*, Duke Univ. Math. Series, Durham, NC, 1976.

[13] M. RÖSLER, Bessel-Type signed hypergroup on $\mathbb{R}$, in Probability measure on groups and related structures, *Proc. Conf Oberwolfac*h, (1994), H. Heyer and A. Mukherjea (Eds) World scientific Publ, 1995, pp 292–304.

[14] F. SOLTANI, L^p-Fourier multipliers for the Dunkl operator on the real line, *J. Funct. Analysis*, **209** (2004), 16–35.

[15] S. THANGAVELYU AND Y. XU, Convolution operator and maximal function for Dunkl transform, *J. Anal. Math.*, **97** (2005), 25–56.

[16] A. TORCHINSKY, *Real-variable Methods in Harmonic Analysis*, Academic Press, 1986.

[17] H. TRIEBEL, *Theory of Function Spaces*, Monographs in Math., Vol. 78, Birkhäuser Verlag, Basel, 1983.

[18] G.N. WATSON, *A Treatise on the Theory of Bessel Functions*, Camb. Univ. Press, Cambridge 1966.