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SOME RESULTS CONCERNING BEST UNIFORM COAPPROXIMATION

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ABSTRACT. This paper provides some conditions to obtain best uniform coapproximation. Some error estimates are determined. A relation between interpolation and best uniform coapproximation is exhibited. Continuity properties of selections for the metric projection and the cometric projection are studied.

Key words and phrases: Best approximation, Best coapproximation, Chebyshev space, Cometric projection, Interpolation, Metric projection, Selection and Weak Chebyshev space.

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1. INTRODUCTION

A new kind of approximation was first introduced in 1972 by Franchetti and Furi [3] to characterize real Hilbert spaces among real reflexive Banach spaces. This was christened 'best coapproximation' by Papini and Singer [16]. Subsequently, Geetha S. Rao and coworkers have developed this theory to a considerable extent [4] – [13]. This theory is largely concerned with the questions of existence, uniqueness and characterizations of best coapproximation. It also deals with the continuity properties of the cometric projection and selections for the cometric projection, apart from related maps and strongly unique best coapproximation. This paper mainly deals with the role of Chebyshev subspaces in the best uniform coapproximation problems and a selection for the cometric projection. Section 2 gives the fundamental concepts of best approximation and best coapproximation that are used in the sequel. Section 3 provides some conditions to obtain a best uniform coapproximation. Section 4 deals with the error estimates and a relation between interpolation and best uniform coapproximation. Selections for the metric projection and the cometric projection and their continuity properties are studied in Section 5.

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2. PRELIMINARIES

Definition 2.1. Let G be a nonempty subset of a real normed linear space X. An element $g_f \in G$ is called a *best coapproximation* to $f \in X$ from G if for every $g \in G$,

$$\|g - g_f\| \leq \|f - g\| .$$

The set of all best coapproximations to $f \in X$ from G is denoted by $R_G(f)$. The subset G is called an *existence set* if $R_G(f)$ contains at least one element, for every $f \in X$. The subset G is called a *uniqueness* set if $R_G(f)$ contains at most one element, for every $f \in X$. The subset G is called an *existence and uniqueness set* if $R_G(f)$ contains exactly one element, for every $f \in X$. The set

$$D(R_G) := \{f \in X : R_G(f) \neq \emptyset\}$$

is called the domain of R_G.

Proposition 2.1. [16]*Let G be a linear subspace of a real normed linear space X. If $f \in D(R_G)$ and $\alpha \in \mathbb{R}$, then $\alpha f \in D(R_G)$ and $R_G(\alpha f) = \alpha R_G(f)$, where $\mathbb{R}$ denotes the set of real numbers. That is, R_G is homogeneous.*

Remark 2.2. If G is a subset of a real normed linear subspace of X such that $\alpha g \in G$ for every $g \in G$, $\alpha \geq 0$, then Proposition 2.1 holds for $\alpha \geq 0$.

Definition 2.2. Let G be a nonempty subset of a real normed linear space X. The set-valued mapping $R_G : X \to POW(G)$ which associates for every $f \in X$, the set $R_G(f)$ of the best coapproximations to f from G is called the *cometric projection* onto G, where $POW(G)$ denotes the set of all subsets of G.

Definition 2.3. Let G be a nonempty subset of a real normed linear space X. An element $g_f \in G$ is called a *best approximation* to $f \in X$ from G if for every $g \in G$,

$$\|f - g_f\| \leq \|f - g\|$$

i.e., if

$$\|f - g_f\| = \inf_{g \in G} \|f - g\| = d(f, G),$$

where $d(f, G) :=$ distance between the element f and the set G.

The set of all best approximations to $f \in X$ from G is denoted by $P_G(f)$.

The subset G is called a *proximinal* or *existence set* if $P_G(f)$ contains at least one element for every $f \in X$. G is called a *semi Chebyshev* or *uniqueness set* if $P_G(f)$ contains exactly one element for every $f \in X$.

Definition 2.4. Let G be a nonempty subset of a real normed linear space X. The set-valued mapping $P_G : X \to POW(G)$ which associates for every $f \in X$, the set $P_G(f)$ of the best approximations to f from G is called the *metric projection* onto G.

Let $[a, b]$ be a closed and bounded interval of the real line. A space of continuous real valued functions on $[a, b]$ is defined by

$$C[a, b] = \{f : [a, b] \to \mathbb{R} : f \text{ is continuous}\} .$$

If r is a positive integer, then the space of $r-$times continuously differentiable functions on $[a, b]$ is defined by

$$C^r[a, b] = \left\{f : [a, b] \to \mathbb{R} : f^{(r)} \in C[a, b]\right\} .$$

Definition 2.5. The *sign* of a function $g \in C[a,b]$ is defined by

$$\operatorname{sgn} g(t) = \begin{cases} -1 & \text{if } g(t) < 0 \\ 0 & \text{if } g(t) = 0 \\ 1 & \text{if } g(t) > 0. \end{cases}$$

Definition 2.6. Let G be a subset of a real normed linear space $C[a,b]$, $f \in C[a,b] \setminus G$. Let $\{t_1, \dots, t_n\} \in [a,b]$. A function $g \in G$ is said to *interpolate* f at the points $\{t_1, \dots, t_n\}$ if

$$g(t_i) = f(t_i), \quad i = 1, \dots, n.$$

Definition 2.7. An $n-$dimensional subspace G of $C[a,b]$ is called a *Chebyshev subspace (Tchebycheff subspace,* in brief, *$T-$subspace) or Haar subspace,* if there exists a basis $\{g_1, \dots, g_n\}$ of G such that

$$D\begin{pmatrix} g_1, \dots, g_n \\ \\ t_1, \dots, t_n \end{pmatrix} = \begin{vmatrix} g_1(t_1) & \cdots & g_n(t_1) \\ \vdots & & \vdots \\ g_1(t_1) & \cdots & g_n(t_n) \end{vmatrix} > 0,$$

for all $t_1 < \dots < t_n$ in $[a,b]$.

Definition 2.8. Let $\{g_1, \dots, g_n\}$ be a set of bounded real valued functions defined on a subset I of $\mathbb{R}$. The system $\{g_i\}_1^n$ is said to be a *weak Chebyshev system* (or *Weak Tchebycheff system;* in brief, WT-system) if they are linearly independent, and

$$D\begin{pmatrix} g_1, \dots, g_n \\ \\ t_1, \dots, t_n \end{pmatrix} = \begin{vmatrix} g_1(t_1) & \cdots & g_n(t_1) \\ \vdots & & \vdots \\ g_1(t_1) & \cdots & g_n(t_n) \end{vmatrix} \geq 0,$$

for all $t_1 < \dots < t_n \in I$. The space spanned by a weak Chebyshev system is called a *weak Chebychev space.*

In contrast to the definitions of Chebychev space, there the functions are defined on arbitrary subsets I of $\mathbb{R}$ and they are not required to be continuous on T. It is clear that every Chebyshev space is a weak Chebyshev space.

Best coapproximation problems can be considered with respect to various norms, e.g., L_1-norm, L_2-norm, and $L_\infty-$norm. The choice of the norms depends on the given minimization problem. Since the L_2-norm induces an inner product and best coapproximation coincides with best approximation in inner product spaces, all the results of best approximation with respect to the L_2-norm can be carried over to best coapproximation with respect to L_2-norms. Hence, the best coapproximation problems will be considered with respect to the L_1 and L_∞ norms.

Definition 2.9. For all functions $f \in C[a,b]$, the *uniform norm* or *$L_\infty-$norm* or *supremum norm* is defined by

$$\|f\|_\infty = \sup_{t \in [a,b]} |f(t)|.$$

Best coapproximation (respectively, best approximation) with respect to this norm is called *best uniform coapproximation* (respectively, *best uniform approximation*).

Definition 2.10. The set $E(f)$ of *extreme points* of a function $f \in C[a,b]$ is defined by

$$E(f) = \{t \in [a,b] : |f(t)| = \|f\|_\infty\}.$$

For the sake of brevity, the terminology subspace is used instead of a linear subspace. Unless otherwise stated, all normed linear spaces considered in this paper are existence subsets and existence subspaces with respect to best coapproximation. It is easy to deal with $C[a,b]$ instead of an arbitrary normed linear space. Since best coapproximation (respectively, best approximation) of an element in a subset from the same subset is the element of itself, i.e., if $G \subset X$, $f \in G \Longrightarrow R_G(f) = f$ and $P_G(f) = f$, it is sufficient to deal with the element to which a best coapproximation (respectively, best approximation) to be found, which lies outside the subset, i.e., $f \in X \backslash G$.

3. CHARACTERIZATION OF BEST UNIFORM COAPPROXIMATION

The following theorem is a characterization best uniform coapproximation due to Geetha S. Rao and R. Saravanan [14].

Theorem 3.1. *Let G be a subspace of $C[a,b]$, $f \in C[a,b] \backslash G$ and $g_f \in G$. Then the following statements are equivalent:*

(i) *The function g_f is a best uniform coapproximation to f from G.*
(ii) *For every function $g \in G$,*

$$\min_{t \in E(g)} (f(t) - g_f(t))\, g(t) \leq 0.$$

The next result generalizes one part of Theorem 3.1.

Theorem 3.2. *Let G be a subset of $C[a,b]$ such that $\alpha g \in G$ for all $g \in G$ and $\alpha \in [0, \infty)$. Let $f \in C[a,b] \backslash G$ and $g_f \in G$. If g_f is a unique best uniform coapproximation to f from G, then for every function $g \in G \backslash \{g_f\}$ and every set U containing $E(g - g_f)$,*

$$\inf_{t \in U} (f(t) - g_f(t))(g(t) - g_f(t)) < 0.$$

Proof. Assume to the contrary that there exists a function $g_1 \in G \backslash \{g_f\}$ and a set U containing $E(g_1 - g_f)$ such that

$$\inf_{t \in U} (f(t) - g_f(t))(g_1(t) - g_f(t)) \geq 0.$$

Then for all $t \in U$, it follows that

$$(f(t) - g_f(t))(g_1(t) - g_f(t)) \geq 0. \tag{3.1}$$

Let

$$V = \left\{ t \in [a,b] : |g_1(t) - g_f(t)| \geq \frac{1}{2} \|g_1 - g_f\|_\infty \right\}. \tag{3.2}$$

Assume without loss of generality that $E(g_1 - g_f) \subset U \subset V$. Let

$$c = \|g_1 - g_f\|_\infty - \max\{|g_1(t) - g_f(t)| : t \in V \backslash U\}. \tag{3.3}$$

It is clear that $c > 0$. By multiplying $f - g_f$ with an appropriate positive factor and using Remark 2.2, assume without loss of generality that

$$\|f - g_f\|_\infty \leq \min\left\{ c, \frac{1}{2} \|g_1 - g_f\|_\infty \right\}. \tag{3.4}$$

Case 1. Let $t \in [a,b] \backslash V$. Then it follows that

$$\begin{aligned}
|f(t) - g_1(t)| &= |(f(t) - g_f(t)) - (g_1(t) - g_f(t))| \\
&\leq |f(t) - g_f(t)| + |g_1(t) - g_f(t)| \\
&\leq \|f - g_f\|_\infty + \frac{1}{2} \|g_1 - g_f\|_\infty \quad \text{by (3.2)} \\
&\leq \frac{1}{2} \|g_1 - g_f\|_\infty + \frac{1}{2} \|g_1 - g_f\|_\infty \quad \text{by (3.4)} \\
&= \|g_1 - g_f\|_\infty .
\end{aligned}$$

Case 2. Let $t \in V \backslash U$. Then it follows that

$$\begin{aligned} |f(t) - g_1(t)| &= |(f(t) - g_f(t)) - (g_1(t) - g_f(t))| \\ &\leq |f(t) - g_f(t)| + |g_1(t) - g_f(t)| \\ &\leq |f(t) - g_f(t)| + \|g_1 - g_f\|_\infty - c \quad \text{by (3.3)} \\ &\leq \|g_1 - g_f\|_\infty \quad \text{by (3.4).} \end{aligned}$$

Case 3. Let $t \in U$. Then it follows that

$$\begin{aligned} |f(t) - g_1(t)| &= |(f(t) - g_f(t)) - (g_1(t) - g_f(t))| \\ &= ||f(t) - g_f(t)| - |g_1(t) - g_f(t)|| \quad \text{by (3.1)} \\ &= |g_1(t) - g_f(t)| - |f(t) - g_f(t)| \quad \text{by (3.2) and (3.4)} \\ &\leq \|g_1 - g_f\|_\infty . \end{aligned}$$

Thus for all $t \in [a, b]$,

$$|f(t) - g_1(t)| \leq \|g_1 - g_f\|_\infty .$$

This implies that

$$\|g_1 - g_f\|_\infty \geq \|f - g_1\|_\infty ,$$

which shows that g_f is not a unique best uniform coapproximation to f from G, a contradiction. □

If G is considered as a subspace of $C[a, b]$, then Theorem 3.2 can be written as:

Theorem 3.3. *Let G be a subspace of $C[a, b]$, $f \in C[a, b] \backslash G$ and $g_f \in G$. If g_f is a unique best uniform coapproximation to f from G, then for every nontrivial function $g \in G$ and every set U containing $E(g)$,*

$$\inf_{t \in U} (f(t) - g_f(t))(g(t)) < 0.$$

Proof. Assume to the contrary that there exist a nontrivial function $g_1 \in G$ and a set U containing $E(g_1)$ such that

$$\inf_{t \in U} (f(t) - g_f(t))(g_1(t)) \geq 0.$$

Let $g_2 = g_1 + g_f$. Then for all $t \in U$, it follows that

$$(f(t) - g_f(t))(g_2(t) - g_f(t)) \geq 0.$$

The remaining part of the proof is the same as that of Theorem 3.2. □

Remark 3.4. Theorems 3.2 and 3.3 remain true if the interval $[a, b]$ is replaced by a compact Hausdorff space.

Let X be a normed linear space and G be a subset of X. Let $g_f \in G$ be fixed. For each $g \in G$, define a set $\mathcal{L}(g, g_f)$ of continuous linear functionals depending upon g and g_f by

$$\mathcal{L}(g, g_f) = \{L \in G^* : L(g - g_f) = \|g - g_f\| \text{ and } \|L\| = 1\},$$

where G^* denotes the set of continuous linear functionals defined on G.

Some conditions to obtain best coapproximation are established.

Proposition 3.5. *Let G be a subset of a normed linear space X, $f \in X \backslash G$ and $g_f \in G$. If for each $g \in G$,*

$$\min_{L \in \mathcal{L}(g, g_f)} L(f - g_f) \leq 0,$$

or if for each $g \in G$, there exists $L \in \mathcal{L}(g, g_f)$ such that

$$L(g_f) \geq L(f),$$

then g_f is a best approximation to f from G.

Proof. Let $\min_{L\in\mathcal{L}(g,g_f)} L(g_f - f) \leq 0$. Then there exists a continuous linear functional $L \in \mathcal{L}(g, g_f)$ such that $L(f - g_f) \leq 0$. It follows that

$$\|g - g_f\| = L(g - g_f) = L(g) - L(g_f) = L(g) - L(f) = L(g - f) \leq \|g - f\|.$$

The other case can be proved similarly. □

Let G be a subspace of a normed linear space X. For $x \in X$, let $d(x, G)$ denote the distance between x and G, i.e.,

$$d(x, G) = \inf_{g\in G} \|x - g\|.$$

Then the quotient space X/G is equipped with the norm,

$$\|x + g\| = d(x, G).$$

Theorem 3.6. *Let G and H be subspaces of a normed linear space X such that $G \subset H$ and let $f \in X\backslash H$ and $h \in H$. If h is a best coapproximation to f from H, then $h + G$ is a best coapproximation to $f + G$ from the quotient space $H\backslash G$.*

Proof. Assume that $h + G$ is not a best coapproximation to $f + G$ from H/G. Then there exists $h' + G \in H/G$ such that

$$|||h' + G - (h + G)||| > |||f + G - (h' + G)|||.$$

That is,

$$|||h' - h + G||| > |||f - h' + G|||.$$

That is,

$$d(f - h', G) < d(h' - h, G).$$

This implies that there exists $g \in G$ such that

$$\begin{aligned} \|f - h' - g\| &< d(h' - h, G) \\ &< \|h' - h + g\|. \end{aligned}$$

That is,

$$\|(g + h') - h\| > \|f - (g + h')\|.$$

Thus h is not a best coapproximation to f from H, a contradiction. □

4. BEST UNIFORM COAPPROXIMATION AND CHEBYSHEV SUBSPACES

Let G be a subset of $C[a, b]$, $f \in C[a, b]\backslash G$ and $g_f \in G$ be a best uniform coapproximation to f from G. It is known that for every $g \in G$,

$$\|f - g_f\| \leq 2\|f - g\|.$$

If the subset G is considered as a Chebyshev subspace, then a lower bound for $\|f - g_f\|_\infty$ is obtained, for which the following definition and results are required.

Definition 4.1. The points $t_1 < \cdots < t_p$ in $[a, b]$ are called *alternating extreme points* of a function $f \in C[a, b]$, if there exists a sign $\sigma \in \{-1, 1\}$ such that

$$\sigma(-1)^i f(t_i) = \|f\|_\infty, \qquad i = 1, \ldots, p.$$

Theorem 4.1. [1]*Let G be an $n-$dimensional weak Chebyshev subspace of $C[a, b]$, $f \in C[a, b]\backslash G$ and $g_f \in G$. If the error $f - g_f$ has at least $n + 1$ alternating extreme points in $[a, b]$, g_f is a best uniform approximations to f from G.*

Theorem 4.2. [15]*Let G be an $n-$dimensional weak Chebyshev subspace of $C\left[a,b\right]$. Then for all integers $m\in\{1,\ldots,n\}$ and all points $a=t_0<t_1<\cdots<t_{m-1}<t_m=b$, there exists a nontrivial function $g\in G$ such that*

$$\left(-1\right)^i g\left(t\right)\geq 0,\ t\in\left[t_{i-1},t_i\right],\qquad i=1,\ldots,m.$$

Now a lower bound for $\left\|f-g_f\right\|_\infty$ can be established as follows:

Theorem 4.3. *Let G be an $n-$dimensional weak Chebyshev subspace of $C\left[a,b\right]$, $f\in C\left[a,b\right]\backslash G$ and $g_f\in G$. If g_f is a best uniform coapproximation but not a best uniform approximation to f from G, then there exists a nontrivial function $g\in G$ such that*

$$\left\|g\right\|_\infty\leq\left\|f-g_f\right\|_\infty.$$

Proof. Since g_f is not a best uniform approximation to f from G, by Theorem 4.1, $f-g_f$ cannot have more than n alternating extreme points in $\left[a,b\right]$. Let $t_1<\cdots<t_p$, $p\leq n$ be the alternating extreme points of $f-g_f$ in $\left[a,b\right]$. Assume first that $f\left(t_1\right)-g_f\left(t_1\right)=\left\|f-g_f\right\|_\infty$. Then there exist points $x_0,x_1,\ldots,x_p$ in $\left[a,b\right]$ and a real number $c>0$ such that

$$\begin{aligned} a &= x_0<x_1<\cdots<x_{pi1}<x_p=b,\\ x_i &\in \left(t_i,t_{i-1}\right),\ \ i=0,\ldots,p-1. \end{aligned}$$

and

$$\left(-1\right)^{i+1}\left(f\left(t\right)-g_f\left(t\right)\right)\leq\left\|f-g_f\right\|_\infty-c,\ t\in\left[x_i,x_{i+1}\right],\ i=0,\ldots,p-1.$$

Since $p\leq n$, by Theorem 4.2 there exists a nontrivial function $g\in G$ such that

$$\left(-1\right)^i g\left(t\right)\geq 0,\ t\in\left[x_i,x_{i+1}\right],\ i=0,\ldots,p-1.$$

By multiplying g with an appropriate positive factor, assume without loss of generality that $\left\|g\right\|_\infty\leq c$. Then for all $t\in\left[x_i,x_{i+1}\right]$, it follows that

$$\begin{aligned} -\left\|f-g_f\right\|_\infty &\leq \left(-1\right)^{i+1}\left(f\left(t\right)-g_f\left(t\right)\right)\\ &\leq \left(-1\right)^{i+1}\left(f\left(t\right)-g_f\left(t\right)\right)+\left(-1\right)^i g\left(t\right)\\ &= \left(-1\right)^{i+1}\left(f\left(t\right)-g_f\left(t\right)\right)-\left(-1\right)^{i+1} g\left(t\right)\\ &\leq \left\|f-g_f\right\|_\infty-c+\left\|g\right\|_\infty\\ &\leq \left\|f-g_f\right\|_\infty. \end{aligned}$$

That is,

$$-\left\|f-g_f\right\|_\infty\leq\left(-1\right)^{i+1}\left(f\left(t\right)-g_f\left(t\right)\right)-\left(-1\right)^{i+1}g\left(t\right)\leq\left\|f-g_f\right\|_\infty.$$

This implies that for all $i\in\{0,1,\ldots,p-1\}$ and for all $t\in\left[x_i,x_{i+1}\right]$,

$$\left|\left(-1\right)^{i+1}\left(\left(f\left(t\right)-g_f\left(t\right)\right)-g\left(t\right)\right)\right|\leq\left\|f-g_f\right\|_\infty.$$

Hence

$$\left\|f-g_f-g\right\|_\infty\leq\left\|f-g_f\right\|_\infty.$$

For the second case, $f\left(t_1\right)-g_f\left(t_1\right)=-\left\|f-g_f\right\|_\infty$, the inequality

$$\left\|f-g_f-g\right\|_\infty\leq\left\|f-g_f\right\|_\infty$$

can be proved similarly.

Since $g_f-\left(g_f+g\right)$ is a best uniform approximation to $f-\left(g_f+g\right)$ from G it follows that

$$\left\|g_f-\left(g_f+g\right)\right\|_\infty\leq\left\|f-\left(g_f+g\right)\right\|_\infty.$$

Hence

$$\left\|g\right\|_\infty\leq\left\|f-g_f\right\|_\infty.$$

□

In order to approximate a given function $f \in C[a,b]$ by functions from a finite dimensional subspace, it is required that the approximating function coincides with f at certain points of the interval $[a,b]$. In order to establish a similar fact for coapproximation, the following theorems are required.

Theorem 4.4. [1]*Let G be a Chebyshev subspace of $C[a,b]$. Then for every function $f \in C[a,b] \setminus G$, there exists a unique best uniform approximation from G.*

Theorem 4.5. *Let G be an $n-$dimensional Chebyshev subspace of $C[a,b]$, $f \in C[a,b] \setminus G$ and $g_f \in G$. Then the following statements are equivalent:*

(i) *The function g_f is a best uniform approximation to f from G.*
(ii) *The error $f - g_f$ has at least $n+1$ alternating extreme points in $[a,b]$.*

Now a relation between interpolation and best uniform coapproximation is obtained as follows:

Theorem 4.6. *Let G be an $n-$dimensional Chebyshev subspace of $C[a,b]$, $f \in C[a,b] \setminus G$ and $g_f \in G$. If g_f is a best uniform coapproximation to f from G, then g_f interpolates f at at least n points of $[a,b]$.*

Proof. Since G is an $n-$dimensional Chebyshev space of $C[a,b]$, by Theorem 4.4 and Theorem 4.5 there exists a unique function, say $g_1 \in G$, such that $f - g_1$ has at least $n+1$ alternating extreme points in $[a,b]$. Therefore, there exist points $t_1 < \cdots < t_p$, $p \geq n+1$, in $[a,b]$ and a sign $\sigma \in \{-1, 1\}$ such that

$$\sigma(-1)^i (f(t_i) - g_1(t_i)) = \|f - g_1\|_\infty, \quad i = 1, \dots, p.$$

Since g_f is a best uniform coapproximation to f from G, it follows that for $i = 1, \dots, p$,

$$\sigma(-1)^i (g_f(t_i) - g_1(t_i)) \leq \|g_f - g_1\|_\infty \leq \|f - g_1\|_\infty = \sigma(-1)^i (f(t_i) - g_1(t_i)).$$

This implies that

$$\sigma(-1)^i (g_f(t_i) - f(t_i)) \leq 0, \quad i = 1, \dots, p.$$

Hence the function $f - g_f$ has at least $p-1$ zeros, say $x_1, \dots, x_{p-1}$. Thus g_f interpolates f at at least n points $x_1, \dots, x_{p-1}$. □

Remark 4.7. Theorem 4.6 can be proved in the context of weak Chebyshev subspaces.

The following theorem is required to establish an upper bound for the error $\|f - g_f\|_\infty$ under some conditions.

Theorem 4.8. [1]*If $f \in C^n[a,b]$, if g is a polynomial of degree n which interpolates f at n points $x_1, \dots, x_n$ in $[a,b]$ and if $w(x) = (x - x_1) \cdots (x - x_n)$, then*

$$\|f - g\|_\infty \leq \frac{1}{n!} \left\|f^{(n)}\right\|_\infty \|w\|_\infty.$$

Now, an upper bound can be determined as follows:

Corollary 4.9. *Let G be a space of polynomials of degree n defined on $[a,b]$ and $f \in C^n[a,b] \setminus G$. If $g_f \in G$ is a best uniform coapproximation to f from G, then*

$$\|f - g_f\|_\infty \leq \frac{1}{n!} \left\|f^{(n)}\right\|_\infty \|w\|_\infty,$$

where $w(x) = (x - x_1) \cdots (x - x_n)$ and $x_1, \dots, x_n$ are the points in $[a,b]$ at which g_f interpolates f.

Proof. Since a space of polynomials is a Chebyshev space, by Theorem 4.6, there exist n points $x_1, \ldots, x_n$ in $[a,b]$ at which g_f interpolates f. Hence by Theorem 4.8,

$$\|f - g_f\|_\infty \leq \frac{1}{n!} \left\|f^{(n)}\right\|_\infty \|w\|_\infty .$$

□

Remark 4.10. It is clear that the error $\|f - g_f\|_\infty$ is minimum when the x_i's are taken as the zeros of Chebyshev polynomials.

Proposition 4.11. *Let G be a subspace of $C[a,b]$, $f \in C[a,b] \backslash G$ and $g_f \in G$ be a best uniform coapproximation to f from G. Then there does not exist a function in G, which interpolates $f - g_f$ at its extreme points.*

Proof. Suppose to the contrary that there exists a function $g_0 \in G$ such that g_0 interpolates $f - g_f$ at its extreme points. Let $E(g_0) = \{t_1, \ldots, t_n\}$. So

$$g_0(t_i) = f(t_i) - g_f(t_i), \qquad i = 1, \ldots, n.$$

This implies that

$$g_0(t_i)(f(t_i) - g_f(t_i)) > 0, \qquad i = 1, \ldots, n.$$

Hence

$$\min_{t \in E(g_0)} g_0(t)(f(t) - g_f(t)) > 0.$$

Thus by Theorem 3.1, g_f is not a best uniform coapproximation to f from G, a contradiction.

□

The following result answers the question:
When does a best uniform approximation imply a best uniform coapproximation?

Theorem 4.12. *Let G be a subset of $C[a,b]$, $f \in C[a,b] \backslash G$ and $g_f \in G$ be a best uniform approximation to f from G. If for every function $g \in G$,*

$$\min_{t \in E(g - g_f)} (f(t) - g(t))(g_f(t) - g(t)) \leq 0, \tag{4.1}$$

then the function g_f is a best uniform coapproximation to f from G.

Proof. For every function $g \in G$, there exists a point $t \in E(g - g_f)$ such that

$$(f(t) - g(t))(g_f(t) - g(t)) \leq 0.$$

Therefore, it follows that

$$\begin{aligned} \|f - g\|_\infty &\geq \|f - g_f\|_\infty \\ &\geq |f(t) - g_f(t)| \\ &= |(f(t) - g(t)) - (g_f(t) - g(t))| \\ &= |f(t) - g(t)| + |g_f(t) - g(t)| \\ &= \|g_f - g\|_\infty . \end{aligned}$$

□

Remark 4.13. In Theorem 4.12, the result holds even if the condition (4.1) is replaced by the condition:

$$sgn(f(t) - g(t)) = sgn(g(t) - g_f(t)),$$

for some $t \in E(g - g_f)$.

If $g_f \in R_G(f)$ and $g_0 \in P_G(f)$, then it is clear that $\frac{1}{2}\|f - g_f\| \leq \|f - g_0\|$.
The following result improves this lower bound. The proof is obvious.

Proposition 4.14. *Let G be a subset of a normed linear space X. Let $f_1, f_2 \in X \backslash G$, $g_{f_1} \in R_G(f_1)$, $g_{f_2} \in R_G(f_2)$, $g_1 \in P_G(f_1)$ and $g_2 \in P_G(f_2)$. Then*

$$\max\left\{\frac{\|f_1 - g_{f_1}\|}{2}, \frac{\|g_{f_1} - g_{f_2}\| - \|f_1 - f_2\|}{2}\right\} \leq \|f_1 - g_1\|$$

and

$$\max\left\{\frac{\|f_2 - g_{f_2}\|}{2}, \frac{\|g_{f_1} - g_{f_2}\| - \|f_1 - f_2\|}{2}\right\} \leq \|f_2 - g_2\|.$$

5. Selection for the Metric Projection and the Cometric Projection

Definition 5.1. Let G be a subset of a normed linear space X and let $P_G : X \to POW(G)$ (respectively, $R_G : X \to POW(G)$) be the metric projection (respectively, cometric projection) onto G. A *selection* for the metric projection P_G (respectively, cometric projection R_G) is an onto map $S : X \to G$ such that $S(f) \in P_G(f)$ (respectively, $S(f) \in R_G(f)$) for all $f \in X$. If S is continuous, then it is called a continuous selection for the metric projection (respectively, cometric projection).

Definition 5.2. A selection S for the metric projection P_G (respectively, cometric projection R_G) is said to be *sunny* if $S(f_\alpha) = S(f)$ for all $f \in X$ and $\alpha \geq 0$, where $f_\alpha := \alpha f + (1-\alpha) S(f)$.

The following result shows that every selection for a cometric projection onto a subspace is a sunny selection.

Theorem 5.1. *Let G be a subspace of a normed linear space X. Then every selection for a cometric projection $R_G : X \to POW(G)$ is a sunny selection.*

Proof. Let S be a selection. It is enough to prove that $S(f_\alpha) = S(f)$, for all $f \in X$ and $\alpha \geq 0$, where $f_\alpha := \alpha f + (1-\alpha) S(f)$. It follows from Proposition 2.1 that

$$\begin{aligned}
S(f_\alpha) &= S(\alpha f + (1-\alpha) S(f)) \\
&= S(\alpha(f - S(f)) + S(f)) \\
&= S(\alpha(f - S(f))) + S(f) \\
&= \alpha S(f - S(f)) + S(f) \\
&= \alpha(S(f) - S(f)) + S(f) \\
&= S(f).
\end{aligned}$$

Thus every selection is sunny. ☐

Let B_∞ denote the closed unit sphere in $C[a,b]$ with center at origin with respect to $L_\infty-$norm. That is,

$$B_\infty := \{f \in C[a,b] : \|f\|_\infty \leq 1\}.$$

Definition 5.3. A map $T : C[a,b] \to B_\infty$ defined by

$$(T(f))(x) := \max\{-1, \min\{1, f(x)\}\}, \quad f \in C[a,b], \; x \in [a,b],$$

is called an *orthogonal projection.*

Remark 5.2. By the definition of orthogonal projection, it can be written as

$$(T(f))(x) = \begin{cases} sgn\, f(x), & x \in M(f), \\ f(x), & \text{otherwise,} \end{cases}$$

where

$$M(f) := \{x \in [a,b] : |f(x)| > 1\}.$$

The next result shows that the orthogonal projection is a continuous selection for the cometric projection.

Theorem 5.3. *The orthogonal projection* $T : C[a,b] \to B_\infty$ *is a continuous selection for the cometric projection* $R_{B_\infty} : C[a,b] \to POW(B_\infty)$ *under the* L_p-*norm,* $1 \le p \le \infty$.

Proof. Since the inequality $|b - sgna| \le |a-b|$ holds for all real a and such that $|a| \ge 1$ and $|b| \le 1$, it can be shown that T is a selection for the cometric projection R_{B_∞} by taking $a = f(x)$ and $b = g(x)$. For if $a = f(x)$, then $|f(x)| \ge 1$. Therefore, $\|f\|_\infty \ge 1$, hence either f belongs to the boundary of B_∞ or f belongs to $C[a,b] \setminus B_\infty$. If $b = g(x)$, then $|g(x)| \le 1$. Therefore, $\|g\|_\infty \le 1$, hence $g \in B_\infty$. Then for any $f \in C[a,b]$ and $g \in B_\infty$, it can be shown that

$$|g(x) - (T(f))(x)| \le |f(x) - g(x)|,$$

for all $x \in [a,b]$.

Case 1. For all $x \in [a,b]$ such that $|f(x)| > 1$, it follows that

$$|g(x) - sgnf(x)| \le |f(x) - g(x)|.$$

Hence by Remark 5.2 it follows that

$$|g(x) - (T(f))(x)| \le |f(x) - g(x)|.$$

Case 2. For all $x \in [a,b]$ such that $|f(x)| \le 1$, it follows that

$$|g(x) - (T(f))(x)| = |g(x) - f(x)|.$$

By monotonicity of the norm, it follows that $\|g - T(f)\|_p \le \|f - g\|_p$. Hence $T(f) \in R_{B_\infty}(f)$. Thus T is a selection for the cometric projection R_{B_∞}.

To prove T is continuous, it is enough to prove that

$$\|T(f_1) - T(f_2)\|_p \le \|f_1 - f_2\|_p, \tag{5.1}$$

for $f_1, f_2 \in C[a,b]$.

Case 1. Let $x \in [a,b]$ such that $|f_1(x)| > 1$ and $|f_2(x)| > 1$. Since the inequality $|\text{sgn}\, a - \text{sgn}\, b| \le |a-b|$ holds, whenever $|a| \ge 1$, $|b| \ge 1$, inequality (5.1) follows by taking $a = f_1(x)$ and $b = f_2(x)$ and by using remark 5.2 and monotonicity of the norm.

Case 2. Let $x \in [a,b]$ such that $|f_1(x)| \le 1$ and $|f_2(x)| \le 1$. By Remark 5.2 and monotonicity of the norm, inequality (5.1) is obvious.

Case 3. Let $x \in [a,b]$ such that $|f_1(x)| \le 1$ and $|f_2(x)| \ge 1$. Since the inequality $|a - sgnb| \le |a-b|$ holds, whenever $|a| \le 1$, $|b| \ge 1$, inequality (5.1) follows by taking $a = f_1(x)$ and $b = f_2(x)$ and by using Remark 5.2 and monotonicity of the norm. Thus $\|T(f_1) - T(f_2)\|_p \le \|f_1 - f_2\|_p$.

□

Exponential sums are functions of the form

$$h(x) = \sum_{i=1}^{n} p_i(x)\, et_i^x,$$

where t_i are real and distinct and p_i are polynomials. The expression

$$d(h) := \sum_{i=1}^{m} (\partial p_i + 1),$$

is called as the degree of exponential sum h. here ∂p denotes the degree of p. Let V_n denote the set of all exponential sums of degree less than or equal to n. E. Schmidt [17] studied about the continuity properties of the metric projection

$$P_{V_n} : C\left[a,b\right] \to POW\left(V_n\right).$$

The following definition and results are required to prove the next result, which answers the question:
When does the metric projection P_{V_n} have a continuous selection?

In a normed linear space X, the $\varepsilon-$neighbourhood of a nonempty set A in X is given by

$$B_\varepsilon\left(A\right) := \left\{x \in X : d\left(x,A\right) < \varepsilon\right\},$$

where

$$d\left(x,A\right) := \inf_{a\in A} \left\|x-a\right\|.$$

Definition 5.4. [2]Let G be a subset of a normed linear space X. Then a set-valued map $F : X \to POW\left(G\right)$ is said to be $2-$*lower semicontinuous* at $f \in X$, if for each $\varepsilon > 0$, there exists a neighbourhood U of f such that

$$B_\varepsilon\left(F\left(f_1\right)\right) \cap B_\varepsilon\left(F\left(f_2\right)\right) \neq \emptyset$$

for each choice of points $f_1, f_2 \in U$. F is said to be 2-lower semicontinuous if F is 2-lower semicontinuous at each point of X.

Theorem 5.4. [2]*Let G be he complete subspace of a normed linear space X and let $F : X \to POW\left(G\right)$ be a set-valued map. Let $H\left(F\right) = \{x \in X : F\left(x\right)$ is a singleton set$\}$. Suppose that F has closed images and $H\left(F\right)$ is dense in X. Then F has a continuous selection if and only if F is 2-lower semicontinuous. Moreover, if F has a continuous selection, then it is unique.*

Theorem 5.5. [17]*The set of functions of $C\left[a,b\right]$ which have a unique best approximation from V_n is dense in $C\left[a,b\right]$.*

Now a result which provides a necessary and sufficient condition for the metric projection P_{V_n} to have a continuous selection can be stated. The proof follows from Theorem 5.4 and Theorem 5.5.

Theorem 5.6. *The metric projection*

$$P_{V_n} : C\left[a,b\right] \to POW\left(V_n\right)$$

has a continuous selection if and only if P_{V_n} is 2-lower semicontinuous. Moreover, if P_{V_n} has a continuous selection, then it is unique.

Theorem 5.7. [2]*Let G be a subset of normed linear space X and let $F : X \to POW\left(G\right)$. If F is a singleton-valued map, then F is 2-lower semicontinuous if and only if f is continuous.*

Theorem 5.8. [7]*Let G be an existence and uniqueness subspace with respect to best coapproximation of a normed linear space X. Then each of the following statements implies that the cometric projection R_G is continuous.*

(i) *G is a finite dimensional space.*
(ii) *G is a hyperplane.*
(iii) *G is closed and $R_G^{-1}\left(0\right)$ is boundedly compact.*
(iv) *R_G is continuous at the points of $R_G^{-1}\left(0\right)$.*
(v) *$R_G^{-1}\left(0\right) + R_G^{-1}\left(0\right) \subset R_G^{-1}\left(0\right)$.*

As a consequence of Theorems 5.4, 5.7 and 5.8, the next result follows.

Theorem 5.9. *Let G be an existence and uniqueness subspace with respect to best coapproximation of a normed linear space X. Then each of the statements (i), (ii), (iii), (iv) and (v) of Theorem 5.8 implies that the cometric projection R_G has a unique continuous selection.*

Remark 5.10. Theorem 5.4 can be stated in the context of best coapproximation as follows.:

Let G be a complete subspace of a normed linear space X and let $R_G :\rightarrow POW(G)$ be the cometric projection. Then R_G has a selection which is continuous on the closure of the set $\{f \in X : f$ has a unique best coapproximation from $G\}$ if and only if R_G is 2-lower semicontinuous.

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