Journal of Inequalities in Pure and Applied Mathematics

http://jipam.vu.edu.au/

Volume 4, Issue 1, Article 9, 2003

FUNDAMENTAL INEQUALITIES ON FIRMLY STRATIFIED SETS AND SOME APPLICATIONS

SERGE NICAISE AND OLEG M. PENKIN

UNIVERSITÉ DE VALENCIENNES ET DU HAINAUT CAMBRÉSIS
MACS
INSTITUT DES SCIENCES ET TECHNIQUES DE VALENCIENNES
59313 - VALENCIENNES CEDEX 9, FRANCE.
serge.nicaise@univ-valenciennes.fr

VORONEZH STATE UNIVERSITY
UNIVERSITETSKAJA PL., 1
394000 VORONEZH, RUSSIA.

Received 13 May, 2002; accepted 11 October, 2002
Communicated by M.Z. Nashed

ABSTRACT. We establish different fundamental inequalities on a class of multistructures, more precisely Poincaré's inequality for second and fourth order (scalar) operators as well as Korn's inequality for the elasticity systems. Some consequences to the corresponding variational problems are deduced.

Key words and phrases: Poincaré's inequality, Korn's inequality, Multistructures.

2000 *Mathematics Subject Classification.* 35J50, 35R05, 35Q72.

1. INTRODUCTION

Partial differential equations on multistructures is one of the most popular areas of the general theory of differential equations with a wide range of applications in continuous mechanics, aerodynamics, biology, and others (see for example [3]). In that field the important problems are solvability, regularity of the solution, spectral theory, control problems and numerical approximations of the solutions. For different aspects of that kind of considerations we may refer to [2, 3, 4, 5, 7, 15, 17] and the references cited there.

As usual, the first step is to look at the solvability of the boundary value problems which depends on the smoothness of the coefficients of the differential equations and on the regularity of the boundaries of the domains where the differential equations are considered. For multistructures these aspects have to be combined with the geometry and the algebraic structure of the domain. The main goal of that paper is to answer to this question for different operators on a class of multistructures, called stratified sets. For both examples the main ingredient is the

ISSN (electronic): 1443-5756

This work was supported by Grant of Russian Foundation of Basic Researches: 01-01-00417.
051-02

validity of a fundamental inequality of Poincaré's type that we first establish. Analogous results were presented in [12] in pure geometrical form where we proved that the so-called firmly connectedness of the stratified set guarantees the validity of Poincaré's inequality and then the solvability of the Dirichlet problems in Sobolev's type spaces. For perforated domains a similar answer was found by V.V. Zhikov [23] in a pure analytical form.

This paper may be then considered as a second part of [12] but is devoted to new developments and applications of our previous results. Indeed the results given here are more general on several aspects: first we extend our notion of firmly connectedness, this new notion allows us to combine the algebraic structure and the geometry of the domains with mechanical considerations. We further give applications to second order elliptic (scalar) operators but also to fourth order elliptic (scalar) operators (models of beams and plates) as well as for the elasticity system.

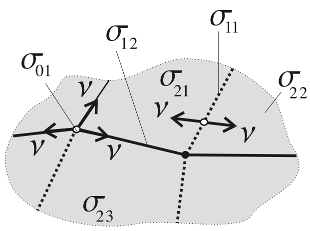

Figure 1.1: An example of stratified set

Before going on let us illustrate our considerations by the following example: consider a mechanical system Ω, lying in the plain Π and consisting of strings and membranes as shown in Figure 1.1. Dotted lines on this figure are the places where the membranes adjoin to each other directly. Full lines represent the strings, in that last case the membranes adjoin to each other indirectly. In both cases we assume that there exist a one-dimensional element (stratum) σ_{1i} between two-dimensional ones. In the case when σ_{1i} is a string we call it elastic, in the opposite case, i.e. when σ_{1i} is a place of direct adjoining of membranes we call it a soft stratum. On the above figure σ_{12} is an elastic stratum and σ_{11} is a soft one. It is convenient to imagine that in both cases we have strings but the soft ones are not stretched. We assume all membranes to be stretched (i.e. all two-dimensional strata are elastic).

Let us denote by $p : \Omega \to \mathbb{R}$ a function which describes the elasticity of the system. The function p then vanishes in the interior of the soft strata and is a positive constant p_{ki} in the elastic stratum σ_{ki}. Let f be a small force which acts orthogonally to the plane Π. Small displacements $u : \Omega \to \mathbb{R}$ caused by this force are solution of the following collection of differential relations (the notation $\sigma_{2j} \succ \sigma_{1i}$ means that σ_{1i} adjoins to σ_{2j}):

$$-p\Delta u(x) = f(x)$$

on two-dimensional strata and

$$-p\frac{\partial^2 u}{\partial \tau^2}(x) - \sum_{\sigma_{2j}\succ\sigma_{1i}} \left(p\frac{\partial u}{\partial \nu}\right)_{|\overline{2j}} (x) = f(x),$$

when x lies in the one-dimensional stratum σ_{1i}. When x lies in σ_{1i} we denote by $\vec{\tau}(x)$ any tangent direction to σ_{1i}. Besides we denote by $\vec{\nu}$ the unit vector directed to the interior of some $\sigma_{k+1j} \succ \sigma_{ki}$ orthogonally to σ_{ki}. The notation

$$w_{|\overline{kj}}(x)$$

means the extension of the restriction $w_{|\sigma_{kj}}$ by continuity to $\overline{\sigma}_{kj}$. When x belongs to some null-dimensional stratum σ_{0i} (like σ_{01} on the above figure), we have

$$-\sum_{\sigma_{1j}\succ\sigma_{0i}}\left(p\frac{\partial u}{\partial\nu}\right)_{|\overline{1j}}(x)=f(x).$$

One can show (see [22]) that the left-hand sides of the last three equations may be rewritten in the divergence form

$$-\nabla(p\nabla u)=f,$$

where the divergence operator ∇ may be defined in a classical manner, as the density of the flow of the vector field with respect to a special "stratified" measure on Ω (more details will be given in the next section).

Adding boundary conditions to the above system, the goal is to find sufficient conditions guaranteeing the solvability of that problem. A positive answer of that problem is given in [12] if all strata are elastic. In the next sections we will extend these results to the case explained here, i.e., when some strata are soft.

The schedule of the paper is the following one: After recalling some basic notions in Section 2, we prove in Section 3 the "standard" Poincaré's inequality on stratified sets under a firmly connectedness property. In Section 4 we give applications to some variational inequalities. Section 5 is devoted to Poincaré's inequality for fourth order operators and an application to the solvability of some boundary value problems with such operators. Finally in Section 6 we prove Korn's inequality on stratified sets and present applications to the elasticity system.

2. SOME PRELIMINARIES

Here we recall some basic definitions on stratified sets. For more details we refer to [12]. Since our considerations are rather sophisticated we also present some examples (see also the simple example of the previous section).

A connected set Ω in $\mathbb{R}^n$ is said to be stratified if there exists a finite sequence of closed subsets of $\mathbb{R}^n$

$$\Omega^{k_0}\subset\Omega^{k_1}\subset\cdots\subset\Omega^{k_m}=\Omega\ \text{ when } k_0<k_1<\cdots<k_m, \tag{2.1}$$

with the following properties:

i) $\Omega^{k_i}\setminus\Omega^{k_{i-1}}$ is a smooth submanifold in $\mathbb{R}^n$ of dimension k_i. Its connected components will be called k_i-dimensional strata and will be denoted by σ_{k_ij}. The second index serves for the numeration of the strata. We shall assume that there is a finite number of strata in Ω and that each of them has a compact closure in $\mathbb{R}^n$. It is important to notice that the boundary of the stratum is piecewise smooth, because it consists of strata. However, it could have some singularities like cracks, cuspidal edges and so on. In order to avoid some serious difficulties we then assume that the boundary of the strata is Lipschitz.

ii) The boundary $\partial\sigma_{ki}=\overline{\sigma}_{ki}\setminus\sigma_{ki}$ of each stratum σ_{ki} with $k\geq 1$ is a union of strata σ_{mj} with $m<k$. We write $\sigma_{mj}\prec\sigma_{ki}$ if $\sigma_{mj}\subset\partial\sigma_{ki}$.

iii) If $\sigma_{k-1,j}\prec\sigma_{ki}$ and $y\in\sigma_{ki}$ tends to $x\in\sigma_{k-1,j}$ along some continuous curve, then the tangent space $T_y\sigma_{ki}$ has a limit position $\lim\limits_{y\to x}T_y\sigma_{ki}$ which contains the tangent space $T_x\sigma_{k-1,j}$.

The sequence (2.1) is called a stratification of Ω. Each set can be stratified in several ways. More exactly a stratified set is a triple (Ω,S,ϕ), where Ω is an initial set, S is a stratification like (2.1) and ϕ describes how to construct Ω using all the pieces σ_{ki}. Nevertheless we shall refer to Ω itself as a stratified set (with fixed S and ϕ).

Before going on, let us present some examples of stratified sets:

- One-dimensional networks (see [2, 4, 15, 16, 17, 21]), where 0-d strata are the vertices and 1-d strata are the edges.
- Two-dimensional polygonal topological networks in the sense of [17], in that case, 1-d strata are the edges and 2-d strata are the faces of the network.
- Take the unit cube of $\mathbb{R}^3$ with the following stratification: the vertices are the 0-d strata, the edges are the 1-d strata, the faces are the 2-d strata and finally the interior of the cube is the 3-d stratum.
- Take for 1-d strata two concentric circles of the plane and as 2-d stratum the area between them.
- In the plane, take as 0-d strata the points $\sigma_{04} = (0,0), \sigma_{02} = (1,0), \sigma_{01} = (2,0)$ and $\sigma_{03} = (0,1)$, as 1-d strata the intervals $(\sigma_{04}, \sigma_{02})$, $(\sigma_{02}, \sigma_{01})$, $(\sigma_{02}, \sigma_{03})$ and $(\sigma_{04}, \sigma_{03})$ and finally as 2-d stratum the triangle of vertices $\sigma_{02}, \sigma_{03}, \sigma_{04}$.
- Take as n-d stratum ($n \geq 1$) a bounded open set O of $\mathbb{R}^n$ with a smooth boundary and as $(n-1)$-d stratum the boundary of O.

The set Ω inherits the topology from $\mathbb{R}^n$. In terms of this topology we fix in Ω some connected and open subset Ω_0 consisting of some strata of Ω and such that $\overline{\Omega}_0 = \Omega$. The complement $\Omega \setminus \Omega_0 = \partial\Omega_0$ is the boundary of Ω_0 in Ω. The set Ω_0 plays the role of a classical domain where a partial differential equation is considered while $\partial\Omega_0$ corresponds to the classical boundary. In this paper we always assume that $\Omega \neq \Omega_0$, i.e. $\partial\Omega_0 \neq \emptyset$.

The set of strata of Ω_0 is divided in two groups. The first one consists of so-called elastic strata. The second one is the set of so-called soft strata. That division is motivated by the example of the previous section as well as problems considered in [1, 6, 7, 9, 17, 18, 20]. We assume null-dimensional strata to be soft (since a point has no mechanical properties like elasticity). So, in contrast to [12] the set Ω_0 has an additional mechanical structure in form of the above mentioned division in elastic and soft strata. In the sequel $E(\Omega_0)$ will denote the set of elastic strata and $S(\Omega_0)$ the set of soft strata.

Now we introduce in Ω a "stratified" measure μ by means of the following expression

$$\mu(\omega) = \sum_{\sigma_{ki} \subset \Omega} \mu_k(\omega \cap \sigma_{ki}),$$

where μ_k is the usual k-dimensional Lebesgue measure on σ_{ki}. A subset ω of Ω for which this formula makes sense will be called μ-measurable. Obviously the μ-measurability of ω is equivalent to the measurability in the Lebesgue sense of all "traces" $\omega \cap \sigma_{ki}$.

We can then define Lebesgue's integral with respect to this measure. One can show that for an integrable function $f : \Omega \to \mathbb{R}$ its integral is equal to the sum of the Lebesgue integrals over the sets σ_{ki}. In other words we have

$$\int_{\Omega} f d\mu = \sum \int_{\sigma_{ki}} f d\mu.$$

In the right-hand side we write $d\mu$ instead of $d\mu_k$ because $d\mu(\omega \cap \sigma_{ki}) = d\mu_k(\omega \cap \sigma_{ki})$ according to our definition.

We now introduce some functional spaces on a stratified set that will be useful later on.

- $C_\sigma(\Omega_0)$ is the set of functions with continuous restrictions u_{ki} (such restrictions might also be denoted by $u_{|\sigma_{ki}}$ or $u_{|ki}$).
- $C(\Omega_0)$ is the set of continuous functions on Ω_0.
- $C^1_\sigma(\Omega_0)$ is the set of functions $u : \Omega \to \mathbb{R}$ such that for each σ_{kj} the restriction u_{kj} has continuous first order partial derivatives with respect to the local coordinates on σ_{kj} and these derivatives may be extended by continuity to those $\sigma_{k-1i} \prec \sigma_{kj}$ which are not

in $\partial\Omega_0$. Note that a function in $C^1_\sigma(\Omega_0)$ may be discontinuous (jumps are possible by passage from one stratum to another one).

- $C^1(\Omega_0) = C^1_\sigma(\Omega_0) \cap C(\Omega)$.
- $C^1_0(\Omega_0)$ is the set of functions from $C^1(\Omega_0)$ vanishing on the boundary $\partial\Omega_0$.
- $L^2_\mu(\Omega_0)$ is the completion of $C(\Omega_0)$ with respect to the norm in $C(\Omega_0)$ generated by the inner product

$$(u,v) = \int_{\Omega_0} uv d\mu.$$

- $\overset{\circ}{H}{}^1_\mu(\Omega_0)$ is the completion of $C^1_0(\Omega_0)$ with respect to the norm $\|\cdot\|_{<>}$ in $C^1_0(\Omega_0)$ induced by the inner product

$$\langle u,v\rangle = \int_{\Omega_0} uv d\mu + \int_{E(\Omega_0)} \nabla u \cdot \nabla v d\mu.$$

Here above and below, for $f \in C^1(\Omega_0)$, the gradient ∇f is the collection of gradients on each stratum, i.e. on the stratum σ_{ki} it is the usual gradient of the restriction $f_{|\sigma_{ki}}$ of f to σ_{ki}.

Let $\vec{F}$ be a tangent vector field on Ω_0 in the sense that for each $x \in \sigma_{k-1i} \subset \Omega_0$, $\vec{F}(x)$ belongs to the tangent space $T_x(\sigma_{k-1i})$. Let $x \in \sigma_{k-1i}$ and ω be a small portion of Ω_0 containing x. We should imagine ω as the intersection of Ω_0 with some smooth domain G of $\mathbb{R}^n$. If we calculate the flow of $\vec{F}$ through the surface of ω and divide it by $\mu(\omega)$ then we shall obtain an approximated value of the divergence $\nabla\vec{F}(x)$. Exact calculations give the following expression for the divergence

$$\nabla\vec{F}(x) = \nabla_{k-1}\vec{F}(x) + \sum_{\sigma_{kj}\succ\sigma_{k-1i}} \vec{\nu}\cdot\vec{F}_{|\overline{kj}}(x),$$

where ∇_{k-1} is the usual $(k-1)$-dimensional divergence operator on σ_{k-1i}. Note that we use the tradition of Physicians to denote the divergence and the gradient by the same symbol ∇.

For some function $p \in C_\sigma(\Omega_0)$ we can now define the elliptic operator

$$\Delta_p u = \nabla(p\nabla u)$$

on Ω_0. In the full paper we will assume that $p \equiv 0$ on the soft strata and that p is positive on the elastic ones.

In [12] we have considered the Dirichlet problem

$$\begin{aligned} \Delta_p u(x) &= f(x) && x \in \Omega_0, \\ u &= 0 && \text{on } \partial\Omega_0, \end{aligned}$$

when the set of soft strata is empty. The solvability of that problem is based on the so-called Poincaré inequality in Ω_0. Therefore our first goal is to extend this inequality to the case when the set of soft strata is not empty.

3. POINCARÉ'S INEQUALITY

We start with the following definition:

Definition 3.1. The triplet $(E(\Omega_0), S(\Omega_0), \partial\Omega_0)$ is said to be *firmly* connected if for any stratum σ_{ki} of Ω_0, there exists a stratum σ_{mj} of $\partial\Omega_0$ and a *firm* chain joining σ_{ki} to σ_{mj} in the following sense: there exists a connected sequence $\sigma_{k_1 i_1}, \sigma_{k_2 i_2} \dots, \sigma_{k_p i_p}$ with the following properties:

- $\sigma_{k_1 i_1} = \sigma_{ki}$, $\sigma_{k_p i_p} = \sigma_{mj}$ and $\sigma_{k_q i_q} \subset \Omega_0$ when $q \neq p$,
- $|k_{q+1} - k_q| = 1$ for each $q < p$ and either $\sigma_{k_q i_q} \prec \sigma_{k_{q+1} i_{q+1}}$ or $\sigma_{k_q i_q} \succ \sigma_{k_{q+1} i_{q+1}}$,

- For $1 \leq q \leq p-1$, if $\sigma_{k_q i_q}$ is a soft stratum then both $\sigma_{k_{q-1} i_{q-1}}$ and $\sigma_{k_{q+1} i_{q+1}}$ are elastic and have a dimension equal to $k_q + 1$ (except if $q = 1$ when only $\sigma_{k_2 i_2}$ is elastic and is of dimension equal to $k_1 + 1$).

We remark that in the above definition $\sigma_{k_{p-1} i_{p-1}}$ is always elastic (since $\sigma_{k_p i_p}$ is not elastic). This implies that each stratum σ_{ki} such that $\partial\sigma_{ki} \cap \partial\Omega_0 \neq \emptyset$ is elastic. We further remark that from this definition strata of higher dimension are elastic as well.

Remark 3.1. In [12] we take a subdomain Ω_1 of Ω with the same properties than Ω_0 and assume that $\partial\Omega_1 = \Gamma_D \cup \Gamma_N$ (Γ_D and Γ_N being also union of strata of Ω), we finally introduce the notion of a firmly connected pair (Ω_1, Γ_D). This definition is a particular case of our definition since we can verify that if (Ω_1, Γ_D) is firmly connected (in the sense of [12]), then the triplet $(E(\Omega_0), S(\Omega_0), \partial\Omega_0)$ is firmly connected with the choice: $E(\Omega_0) = \Omega_1, S(\Omega_0) = \Gamma_N$ and $\partial\Omega_0 = \Gamma_D$. The applications given in [12] are also particular cases of applications given below.

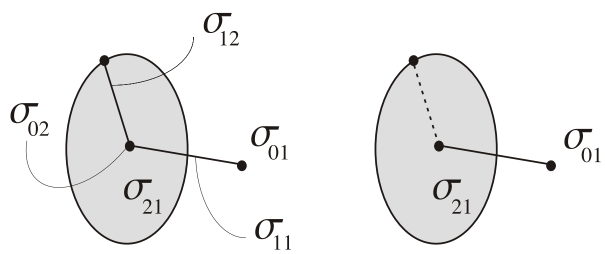

Figure 3.1: Firm and not firm triplets

Figure 3.1 shows examples of firm and not firm triplets. The right example presents a non firm triplet $(E(\Omega_0), S(\Omega_0), \partial\Omega_0)$, when $E(\Omega_0) = \sigma_{21} \cup \sigma_{22} \cup \sigma_{11}$, $S(\Omega_0) = \sigma_{12} \cup \sigma_{02}$ and $\partial\Omega_0 = \sigma_{01} \cup \sigma_{03}$, since there exists no firm chain joining σ_{21} to σ_{01}. On the left we can see a firm triplet $(E(\Omega_0), S(\Omega_0), \partial\Omega_0)$, with $E(\Omega_0) = \sigma_{21} \cup \sigma_{22} \cup \sigma_{11} \cup \sigma_{12}$, $S(\Omega_0) = \sigma_{02}$ and $\partial\Omega_0$ as before (the desired chain joining σ_{21} to σ_{01} is here $\sigma_{21}, \sigma_{12}, \sigma_{02}, \sigma_{11}, \sigma_{01}$).

If Ω_0 is a 1-d network (with the stratification described above) with elastic strata equal to 1-d strata, with a nonempty boundary $\partial\Omega_0$ equal to a subset of 0-d strata, the other 0-d strata being soft, then the triplet $(E(\Omega_0), S(\Omega_0), \partial\Omega_0)$ is firmly connected. Let Ω_0 be a two-dimensional polygonal topological network Ω_0 (with the stratification described above), and take as elastic strata the two-dimensional strata, as well as a part of the one-dimensional strata, the other ones being either soft or on the external boundary, then we get a firmly connected triplet.

Now we can formulate the main result of this section.

Theorem 3.2. *Let $(E(\Omega_0), S(\Omega_0), \partial\Omega_0)$ be a firmly connected stratified triplet. Then there exists a positive constant C such that*

$$\int_{\Omega_0} u^2 d\mu \leq C \int_{E(\Omega_0)} |\nabla u|^2 d\mu \tag{3.1}$$

for all $u \in \overset{\circ}{H}{}^1_\mu(\Omega_0)$.

Our proof is based on the following two lemmas proved in [12].

Lemma 3.3. *Let $\sigma_{k-1i} \prec \sigma_{kj}$. Then there exists a positive constant C such that for all $u \in H^1(\sigma_{kj})$ the following inequality holds*

$$\int\limits_{\sigma_{k-1i}} u^2 d\mu \le C \left(\int\limits_{\sigma_{kj}} u^2 d\mu + \int\limits_{\sigma_{kj}} |\nabla u|^2 d\mu \right). \tag{3.2}$$

Lemma 3.4. *Under the assumption of the previous lemma the following inequality also holds*

$$\int\limits_{\sigma_{kj}} u^2 d\mu \le C \left(\int\limits_{\sigma_{k-1i}} u^2 d\mu + \int\limits_{\sigma_{kj}} |\nabla u|^2 d\mu \right). \tag{3.3}$$

Now we are ready to prove (3.1).

Proof of Theorem 3.2. Let σ_{kj} be an arbitrary stratum of Ω_0. We can connect it with some stratum in $\partial\Omega_0$ by means of a firm chain of strata $\sigma_{k_1 i_1}, \dots \sigma_{k_p i_p}$ like in Definition 3.1. For $1 \le q < p$ we consider the stratum $\sigma_{k_q i_q}$. If $\sigma_{k_q i_q} \subset S(\Omega_0)$ then $\sigma_{k_{q+1} i_{q+1}} \subset E(\Omega_0)$ and $k_{q+1} = k_q + 1$ according to the definition of firmly connectedness and we can apply (3.2) to the pair $\sigma_{k_q i_q}, \sigma_{k_{q+1} i_{q+1}}$. As a result we have

$$\int\limits_{\sigma_{k_q i_q}} u^2 d\mu \le C_q \left(\int\limits_{\sigma_{k_{q+1} i_{q+1}}} u^2 d\mu + \int\limits_{\sigma_{k_{q+1} i_{q+1}}} |\nabla u|^2 d\mu \right), \tag{3.4}$$

for some $C_q > 0$. In the case $\sigma_{k_q i_q} \subset E(\Omega_0)$ and $\sigma_{k_{q+1} i_{q+1}} \subset S(\Omega_0)$ (or $\partial\Omega_0$) we have $k_{q+1} = k_q - 1$ and we can apply (3.3) to obtain

$$\int\limits_{\sigma_{k_q i_q}} u^2 d\mu \le C_q \left(\int\limits_{\sigma_{k_{q+1} i_{q+1}}} u^2 d\mu + \int\limits_{\sigma_{k_q i_q}} |\nabla u|^2 d\mu \right). \tag{3.5}$$

Finally let us consider the case when both $\sigma_{k_q i_q}$ and $\sigma_{k_{q+1} i_{q+1}}$ are included in $E(\Omega_0)$. In this case both possibilities $k_{q+1} = k_q + 1$ and $k_{q+1} = k_q - 1$ are possible. Using (3.2) or (3.3) we obtain (3.4) or (3.5).

It is important to note that in the right-hand sides of (3.4) and (3.5) we have integrals of $|\nabla u|^2$ only over elastic strata, in other words (3.4) or (3.5) implies that

$$\int\limits_{\sigma_{k_q i_q}} u^2 d\mu \le C_q \left(\int\limits_{\sigma_{k_{q+1} i_{q+1}}} u^2 d\mu + \sum_{q'=q,q+1: \sigma_{k_{q'} i_{q'}} \subset E(\Omega_0)} \int\limits_{\sigma_{k_{q'} i_{q'}}} |\nabla u|^2 d\mu \right). \tag{3.6}$$

By induction we get

$$\int\limits_{\sigma_{kj}} u^2 d\mu \le C_{kj} \left(\int\limits_{\sigma_{k_p i_p}} u^2 d\mu + \sum_{1 \le q' \le p-1: \sigma_{k_{q'} i_{q'}} \subset E(\Omega_0)} \int\limits_{\sigma_{k_{q'} i_{q'}}} |\nabla u|^2 d\mu \right), \tag{3.7}$$

with $C_{kj} = 2 \max\limits_{1 \le i \le p-1} \{C_1 \cdots C_i\}$. Taking into account that u vanishes on $\partial\Omega_0$ we obtain

$$\int\limits_{\sigma_{kj}} u^2 d\mu \le C_{kj} \sum_{1 \le q' \le p-1: \sigma_{k_{q'} i_{q'}} \subset E(\Omega_0)} \int\limits_{\sigma_{k_{q'} i_{q'}}} |\nabla u|^2 d\mu \le C_{kj} \int\limits_{E(\Omega_0)} |\nabla u|^2 d\mu.$$

Taking the sum on all strata we obtain (3.1) with $C = \sum C_{kj}$. □

4. APPLICATION TO SOME VARIATIONAL INEQUALITIES

Here we discuss a standard obstacle problem. This is a generalization of the mechanical problem illustrated by Figure 4.1 consisting of a finite number of membranes (two-dimensional strata) and strings (one dimensional strata) initially stretched in the plane. For a general stratified set Ω_0 subdivided into the elastic strata $E(\Omega_0)$ and the soft ones $S(\Omega_0)$, let us fix $p, q \in C_\sigma(\Omega_0)$ such that $q \geq 0$ on Ω_0, $p > 0$ on $E(\Omega_0)$ and $p \equiv 0$ on $S(\Omega_0)$. Consider further $f \in L^2_\mu(\Omega_0)$ as a small force acting on our system and let ϕ be the obstacle which is assumed to be in $\overset{\circ}{H}{}^1_\mu(\Omega_0)$. Then the displacement u of the points of Ω_0 is described by means of the following variational problem

$$\int_{\Omega_0} (p|\nabla u|^2 + qu^2 - 2fu)d\mu = \min_{v \in K} \int_{\Omega_0} (p|\nabla v|^2 + qv^2 - 2fv)d\mu, \tag{4.1}$$

where K is the convex and closed subset of $\overset{\circ}{H}{}^1_\mu(\Omega_0)$ defined by

$$K = \{u \in \overset{\circ}{H}{}^1_\mu(\Omega_0) : u(x) \geq \phi(x)\ (x \in \Omega_0)\}. \tag{4.2}$$

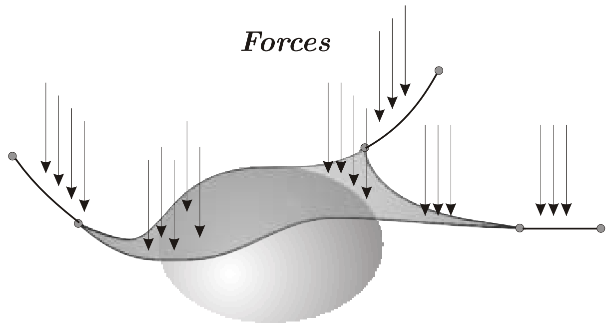

Figure 4.1: A mechanical system with an obstacle.

Remark 4.1. We give the weak formulation (in $\overset{\circ}{H}{}^1_\mu(\Omega_0)$) of the problem, because it has no classical solution even in the case when our mechanical system has no contact with the obstacle. The classical solvability requires stronger conditions than firmly connectedness.

By the standard approach problem (4.1) may be reduced to the variational inequality: Find $u \in K$ solution of

$$a(u, v - u) - (f, v - u)_\mu \geq 0,\ \forall v \in K, \tag{4.3}$$

where the form $a(u, v)$ is defined by

$$a(u, v) = \int_{\Omega_0} (p\nabla u \cdot \nabla v + quv)d\mu. \tag{4.4}$$

Theorem 4.2. *Under the above assumptions if the triplet $(E(\Omega_0), S(\Omega_0), \partial\Omega_0)$ is firmly connected, then the problem (4.3) has a unique solution in K.*

Proof. The bounded and bilinear form $a(u,v)$ is clearly coercive as an easy consequence of Poincaré's inequality. So, the assertion is a consequence of the well-known theorem about variational inequalities in a Hilbert space (see, for example [10, 11]). □

Remark 4.3. The set $\mathcal{N} = \{x \in \Omega_0 : u(x) > \phi(x)\}$ is called the noncoincidence set of the solution u. This set is clearly open and one can show that the above mentioned solution u of the variational inequality (4.3) is a weak solution of

$$\int_{\mathcal{N}} (p\nabla u \nabla\varphi + qu\varphi)d\mu = \int_{\mathcal{N}} f\varphi d\mu, \forall\varphi \in \mathcal{D}(\mathcal{N}).$$

If we take $K = \overset{\circ}{H}{}^1_\mu(\Omega_0)$, then the variational inequality (4.3) becomes the variational identity:

$$a(u,v) = (f,v)_\mu,\ \forall v \in \overset{\circ}{H}{}^1_\mu(\Omega_0).$$

In this case using Green's formula on each stratum, u is a weak solution of

$$-\Delta_p u_{ki} + qu_{ki} - \sum_{\sigma_{ki}\prec\sigma_{k+1,j}} p_{k+1,j}\left(\frac{\partial}{\partial\nu}u_{k+1,j}\right)_{|\sigma_{ki}} = f_{ki} \text{ in } \sigma_{ki},$$
$$u = 0 \text{ on } \partial\Omega_0.$$

In the setting of Remark 3.1 this problem is exactly the one studied in Section 5 of [12]. Let us further remark that this problem extends particular problems considered in [2, 4, 7, 15, 17, 19].

5. POINCARÉ'S INEQUALITY FOR FOURTH ORDER OPERATORS

In this section and the next one, we assume that the strata are flat in the sense that each σ_{ki} is included into a hyperplane of $\mathbb{R}^n$ of dimension k. Consequently we may fix a global system of Cartesian coordinates on each stratum. Under this assumption we shall prove the following Poincaré inequality, useful for boundary value problems involving fourth order operators (see below for some applications).

As before we introduce the space $H^2_\mu(\Omega_0)$ as the closure of $C^2(\Omega_0)$ for the norm $\|\cdot\|_2$ induced by the inner product

$$(u,v)_2 = \int_{\Omega_0} (u^2 + |\nabla u|^2)d\mu + \int_{E(\Omega_0)} \|H(u)\|^2 d\mu,$$

where $H(u)$ is the Hessian matrix of u defined on each stratum σ_{ki} with Cartesian coordinates $(y_1,\dots,y_k)$ by

$$H(u_{ki}) = \left(\frac{\partial^2 u_{ki}}{\partial y_l \partial y_m}\right)_{l,m=1,\dots,k}.$$

The space $C^2(\Omega_0)$ is defined exactly as $C^1(\Omega_0)$ replacing first order derivatives by first and second order derivatives.

Theorem 5.1. *Let the triplet $(E(\Omega_0), S(\Omega_0), \partial\Omega_0)$ be firmly connected such that each stratum is flat in the above sense. Then there exists a positive constant C such that*

$$\int_{\Omega_0} (u^2 + |\nabla u|^2)d\mu \le C \int_{E(\Omega_0)} \|H(u)\|^2 d\mu, \tag{5.1}$$

for all $u \in H^2_\mu(\Omega_0) \cap \overset{\circ}{H}{}^1_\mu(\Omega_0)$.

The proof of this estimate relies on Lemmas 3.3 and 3.4 as well as the so-called interpolation inequalities (see for instance Theorem 1.4.3.3 of [14]):

Lemma 5.2. *There exists a constant C such that for all $\epsilon > 0$ and all $u \in H^2(\sigma_{ki})$ it holds*

$$\int_{\sigma_{ki}} |\nabla u|^2 d\mu \le \epsilon \int_{\sigma_{ki}} u^2 d\mu + \frac{C}{\epsilon^2} \int_{\sigma_{ki}} \|H(u)\|^2 d\mu. \tag{5.2}$$

Lemma 5.3. *Let $\sigma_{k-1,i} \prec \sigma_{kj}$. Then there exists a constant C such that for all $u \in H^2(\sigma_{kj})$ we have*

$$\int_{\sigma_{k-1,i}} (u^2 + |\nabla u|^2) d\mu \le C \left(\int_{\sigma_{kj}} u^2 d\mu + \int_{\sigma_{kj}} \|H(u)\|^2 d\mu \right). \tag{5.3}$$

Proof. Applying Lemma 3.3 to $\frac{\partial u}{\partial y_l}$ with $l = 1, \dots, k$ and summing on l, we may write

$$\int_{\sigma_{k-1,i}} |\nabla u|^2 d\mu \le \sum_{l=1}^{k} \int_{\sigma_{k-1,i}} \left| \frac{\partial u}{\partial y_l} \right|^2 d\mu \le C \left(\int_{\sigma_{kj}} |\nabla u|^2 d\mu + \int_{\sigma_{kj}} \|H(u)\|^2 d\mu \right).$$

Thanks to the estimate (5.2) we obtain

$$\int_{\sigma_{k-1,i}} |\nabla u|^2 d\mu \le C \left(\int_{\sigma_{kj}} u^2 d\mu + \int_{\sigma_{kj}} \|H(u)\|^2 d\mu \right). \tag{5.4}$$

The estimates (3.2) and (5.2) directly yields

$$\int_{\sigma_{k-1,i}} u^2 d\mu \le C \left(\int_{\sigma_{kj}} u^2 d\mu + \int_{\sigma_{kj}} \|H(u)\|^2 d\mu \right). \tag{5.5}$$

We conclude by taking the sum of (5.4) and (5.5). □

Lemma 5.4. *Under the assumptions of the previous lemma the following inequality holds*

$$\int_{\sigma_{kj}} (u^2 + |\nabla u|^2) d\mu \le C \left(\int_{\sigma_{k-1i}} u^2 d\mu + \int_{\sigma_{kj}} \|H(u)\|^2 d\mu \right). \tag{5.6}$$

Proof. By the estimate (5.2), for any $\epsilon > 0$ we have

$$\int_{\sigma_{kj}} |\nabla u|^2 d\mu \le \epsilon \int_{\sigma_{kj}} u^2 d\mu + \frac{C}{\epsilon^2} \int_{\sigma_{kj}} \|H(u)\|^2 d\mu.$$

Lemma 3.4 then yields

$$\int_{\sigma_{kj}} |\nabla u|^2 d\mu \le C\epsilon \int_{\sigma_{k-1i}} u^2 d\mu + C\epsilon \int_{\sigma_{kj}} |\nabla u|^2 d\mu + \frac{C}{\epsilon^2} \int_{\sigma_{kj}} \|H(u)\|^2 d\mu.$$

Choosing $\epsilon > 0$ such that $C\epsilon < 1/2$ we obtain

$$\int_{\sigma_{kj}} |\nabla u|^2 d\mu \le C \left(\int_{\sigma_{k-1i}} u^2 d\mu + \int_{\sigma_{kj}} \|H(u)\|^2 d\mu \right).$$

This estimate and (3.3) directly yield (5.6). □

Proof of Theorem 5.1. The arguments of Theorem 3.2 replacing Lemma 3.3 (resp. Lemma 3.4) by Lemma 5.3 (resp. Lemma 5.4) directly lead to the conclusion. □

Let us shortly give an application of the above Poincaré inequality to some boundary value problems with fourth order operators. The problem considered below is actually an extension of particular problems studied in [17, 18, 7, 9]. For each elastic stratum σ_{ki} we introduce the Young modulus $E_{ki} > 0$ and the Poisson coefficient $\nu_{ki} \in (0, 1)$ of the constitutive material of the stratum σ_{ki}. We then set $p_{ki} = \frac{E_{ki}}{1-\nu_{ki}^2}$ for each elastic stratum σ_{ki} and $p_{ki} = 0$ for each soft stratum σ_{ki} . With these notation we define the bilinear form a on any closed subspace V of $H^2_\mu(\Omega_0) \cap \overset{\circ}{H}{}^1_\mu(\Omega_0)$ by

$$a(u, v) = \sum_{\sigma_{ki} \subset E(\Omega_0)} p_{ki} a_{ki}(u_{ki}, v_{ki}),$$

where we set

$$a_{ki}(u, v) = \int_{\sigma_{ki}} \left\{ \Delta u \Delta v - (1 - \nu_{ki}) \sum_{l \neq m} \left[\frac{\partial^2 u}{\partial y_l^2} \frac{\partial^2 v}{\partial y_m^2} - \frac{\partial^2 u}{\partial y_l \partial y_m} \frac{\partial^2 v}{\partial y_l \partial y_m} \right] \right\} dy.$$

Owing to Theorem 5.1 we shall show that this bilinear form is coercive on V, namely we have (compare with Lemma 2.5 of [17] or Lemma 2.1 of [18]):

Lemma 5.5. *There exists a positive constant α such that for all $u \in V$ we have*

(5.7) $$a(u, u) \geq \alpha \|u\|_2^2.$$

Proof. By direct calculations we see that

$$a_{ki}(u, u) = \int_{\sigma_{ki}} \left\{ \sum_{l=1}^{k} \left| \frac{\partial^2 u}{\partial y_l^2} \right|^2 + \nu_{ki} \sum_{l \neq m} \frac{\partial^2 u}{\partial y_l^2} \frac{\partial^2 u}{\partial y_m^2} + (1 - \nu_{ki}) \sum_{l \neq m} \left(\frac{\partial^2 u}{\partial y_l \partial y_m} \right)^2 \right\} dy.$$

By Young's type inequality

$$\sum_{l \neq m} a_l a_m \geq - \sum_{l=1}^{k} |a_l|^2,$$

valid for all real numbers a_l, we arrive at

$$\begin{aligned} a_{ki}(u, u) &\geq (1 - \nu_{ki}) \int_{\sigma_{ki}} \left\{ \sum_{l=1}^{k} \left| \frac{\partial^2 u}{\partial y_l^2} \right|^2 + \sum_{l \neq m} \left(\frac{\partial^2 u}{\partial y_l \partial y_m} \right)^2 \right\} dy \\ &\geq (1 - \nu_{ki}) \int_{\sigma_{ki}} \|H(u)\|^2 dy. \end{aligned}$$

The conclusion follows from Poincaré's inequality (5.1). □

The so-called Lax-Milgram lemma allows to conclude the existence and uniqueness of the solution $u \in V$ of

(5.8) $$a(u, v) = \int_{\Omega_0} f v \, d\mu, \forall v \in V,$$

for any $f \in L^2_\mu(\Omega_0)$.

Let us give the interpretation of problem (5.8) in terms of partial differential equations in the special case $V = H^2_\mu(\Omega_0) \cap \overset{\circ}{H}{}^1_\mu(\Omega_0)$. In that case for each stratum σ_{ki} we introduce the boundary operators

$$M_{ki}u := p_{ki}\left(\nu_{ki}\Delta u + (1-\nu_{ki})\frac{\partial^2 u}{\partial \nu^2}\right), \tag{5.9}$$

$$N_{ki}u := p_{ki}\left(\frac{\partial \Delta u}{\partial \nu} + (1-\nu_{ki})\Delta_T\frac{\partial u}{\partial \nu}\right) \tag{5.10}$$

on its boundary. Then by applications of Green's formula we see that for u and v sufficiently regular we have

$$p_{ki}a_{ki}(u,v) = p_{ki}\int_{\sigma_{ki}} \Delta^2 uv dy - \int_{\partial\sigma_{ki}} \left\{M_{ki}u\frac{\partial v}{\partial \nu} - N_{ki}uv\right\} d\sigma \\ + p_{ki}(1-\nu_{ki})\int_{\partial(\partial\sigma_{ki})} \frac{\partial}{\partial \nu}\left(\frac{\partial u}{\partial \nu}_{|\partial\sigma_{ki}}\right) vd\sigma.$$

Using this identity in (5.8) we see that the solution $u \in H^2_\mu(\Omega_0) \cap \overset{\circ}{H}{}^1_\mu(\Omega_0)$ of (5.8) is a weak solution of

$$p_{ki}\Delta^2 u_{ki} + \sum_{\sigma_{ki}\prec\sigma_{k+1,j}} N_{k+1,j}u_{k+1,j} \\ + \sum_{\sigma_{ki}\prec\sigma_{k+1,j}\prec\sigma_{k+2,l}} p_{k+2,l}(1-\nu_{k+2,l})\frac{\partial}{\partial \nu}\left(\frac{\partial u}{\partial \nu}_{|\sigma_{k+1,j}}\right)_{|\sigma_{ki}} = f_{ki} \text{ in } \sigma_{ki},$$

$$M_{ki}u_{ki} = 0 \text{ on } \partial\sigma_{ki},$$
$$u = 0 \text{ on } \partial\Omega_0.$$

Note that this problem extends boundary value problems studied in [7, 9] on one-dimensional networks and in [17, 18] on two-dimensional ones.

6. Korn's Inequality

The so-called Korn's inequality is the basic ingredient for coerciveness property of problems involving the elasticity system [8, 11]. We now show that this inequality is also valid on stratified sets. An application to the elasticity system on such sets will be presented at the end of the section.

Let us first recall Korn's inequality on one stratum σ_{ki} (see for instance [13] for a proof of the estimate below in the case of domains with a Lipschitz boundary), which says that there exists a positive constant C such that

$$\int_{\sigma_{ki}} |\nabla u|^2\, dy \le C\left(\int_{\sigma_{ki}} \|\epsilon(u)\|^2\, dy + \int_{\sigma_{ki}} |u|^2\, dy\right), \forall u \in H^1(\sigma_{ki})^k, \tag{6.1}$$

where, as usual, $\epsilon(u) = (\epsilon_{lm}(u))_{l,m=1}^k$ is the strain tensor: $\epsilon_{lm}(u) = \frac{1}{2}\left(\frac{\partial u_l}{\partial y_m} + \frac{\partial u_m}{\partial y_l}\right)$ and for shorthness we write $|\nabla u|^2 = \sum_{l,m=1}^k \left|\frac{\partial u_l}{\partial y_m}\right|^2$.

This estimate and Lemma 3.3 directly lead to

Lemma 6.1. *Let $\sigma_{k-1i} \prec \sigma_{kj}$. Then there exists a constant C such that for all $u \in H^1(\sigma_{kj})^k$*

$$\int_{\sigma_{k-1i}} |u|^2 d\mu \le C\left(\int_{\sigma_{kj}} |u|^2 d\mu + \int_{\sigma_{kj}} \|\epsilon(u)\|^2 d\mu\right). \tag{6.2}$$

The equivalent of Lemma 3.4 requires a more careful analysis and, to our knowledge, seems to be new:

Lemma 6.2. *Under the assumptions of the previous lemma we have*

$$\int_{\sigma_{kj}} |u|^2 d\mu \leq C \left(\int_{\sigma_{k-1i}} |u_t|^2 d\mu + \int_{\sigma_{kj}} \|\epsilon(u)\|^2 d\mu \right), \tag{6.3}$$

where u_t is the tangent component of u on σ_{k-1i}, i.e.,

$$u_t = u - (u \cdot \nu_{kj})\nu_{kj} \text{ on } \sigma_{k-1i}.$$

Proof. Assume that the estimate (6.3) does not hold then there exists a sequence (u_n) such that

$$\int_{\sigma_{kj}} |u_n|^2 d\mu = 1, \tag{6.4}$$

$$\int_{\sigma_{k-1i}} |u_{t,n}|^2 d\mu + \int_{\sigma_{kj}} \|\epsilon(u_n)\|^2 d\mu = \frac{1}{n}. \tag{6.5}$$

By Korn's inequality (6.1) the sequence (u_n) is bounded in $H^1(\sigma_{kj})^k$ and by the compact embedding of $H^1(\sigma_{kj})$ into $L^2(\sigma_{kj})$ (Rellich-Kondrasov's theorem), there exists a subsequence, still denoted by (u_n), which is convergent in $L^2(\sigma_{kj})^k$. By (6.5) and (6.1) the sequence is convergent in $H^1(\sigma_{kj})^k$. Denote its limit by u. By (6.4) and (6.5) it fulfils

$$\int_{\sigma_{kj}} |u|^2 d\mu = 1, \tag{6.6}$$

$$u_t = 0 \text{ on } \sigma_{k-1i}, \tag{6.7}$$

$$\epsilon(u) = 0 \text{ in } \sigma_{kj}. \tag{6.8}$$

This last property implies that u is a rigid body motion, i.e.,

$$u(y) = Ay + b, \forall y \in \sigma_{kj},$$

for some $b \in \mathbb{R}^k$ and an antisymmetric matrix A. Owing to the boundary condition (6.7), we conclude that $b = 0$ and $A = 0$ and so $u = 0$, which is in contradiction with (6.6). □

Now we define the space of vector valued functions on Ω_0: We first define $\mathbf{C}(\Omega_0)$ as the set of functions such that its restriction u_{ki} to σ_{ki} is in $C(\overline{\sigma_{ki}}, \mathbb{R}^k)$ and such that for any strata σ_{ki} and $\sigma_{ki'}$ with a common boundary $\sigma_{k-1,j}$, we have the "continuity" conditions:

$$u_{ki} = u_{ki'} \text{ on } \sigma_{k-1,j},$$
$$u_{k-1,j} = u_{ki} - (u_{ki} \cdot \nu_{ki})\nu_{ki} \text{ on } \sigma_{k-1,j}.$$

Note that the first condition implies that

$$u_{ki} - (u_{ki} \cdot \nu_{ki})\nu_{ki} = u_{ki'} - (u_{ki'} \cdot \nu_{ki'})\nu_{ki'} \text{ on } \sigma_{k-1,j}.$$

We may now define $\mathbf{C}^1(\Omega_0)$ as the subspace of $\mathbf{C}(\Omega_0)$ such that its restriction u_{ki} to σ_{ki} is in $C^1(\overline{\sigma_{ki}}, \mathbb{R}^k)$, while $\mathbf{C}^1_0(\Omega_0)$ is the subspace of $\mathbf{C}^1(\Omega_0)$ of functions which are zero on $\partial\Omega_0$. Finally we take $\overset{\circ}{\mathbf{H}}{}^1_\mu(\Omega_0)$ as the closure of $\mathbf{C}^1_0(\Omega_0)$ for the norm induced by the inner product

$$(u, v) = \int_{\Omega_0} uv d\mu + \sum_{\sigma_{ki} \subset E(\Omega_0)} \int_{\sigma_{ki}} \sum_{l,m=1}^{k} \frac{\partial u_l}{\partial y_m} \frac{\partial v_l}{\partial y_m} dy.$$

The arguments of the proof of Theorem 3.2 replacing Lemma 3.3 (resp. Lemma 3.4) by Lemma 6.1 (resp. Lemma 6.2) lead to the following result that we may call Korn's inequality on stratified sets.

Theorem 6.3. *Let the triplet $(E(\Omega_0), S(\Omega_0), \partial\Omega_0)$ be firmly connected such that each stratum is flat in the above sense. Then there exists a constant C such that*

$$\int\limits_{\Omega_0} |u|^2 d\mu \leq C \int\limits_{E(\Omega_0)} \|\epsilon(u)\|^2 d\mu \tag{6.9}$$

for all $u \in \overset{\circ}{\mathbf{H}}{}^1_\mu(\Omega_0)$.

We finish this paper with an application to the elasticity system, extension of the notion of transmission problems for the elasticity operators considered in [19, 20]: For each elastic stratum σ_{ki}, we suppose that Hooke's law holds, namely the stress tensor $\sigma^{(ki)}(u_{ki}) = (\sigma_{lm}(u_{ki}))_{l,m=1,\dots,k}$ is related to the strain tensor by the relation

$$\sigma^{(ki)}_{lm}(u_{ki}) = \sum_{l',m'=1}^{k} c^{(ki)}_{lml'm'} \epsilon_{l'm'}(u_{ki}),$$

where the elastic moduli $c^{(ki)}_{lml'm'}$ are real constants, satisfy the standard symmetry relations $c^{(ki)}_{lml'm'} = c^{(ki)}_{l'm'lm} = c^{(ki)}_{mll'm'} = c^{(ki)}_{lmm'l'}$, and the positiveness condition: there exists $\alpha > 0$ such that

$$\sum_{l,m,l',m'=1,\dots,k} c^{(ki)}_{lml'm'} \xi_{lm}\xi_{l'm'} \geq \alpha \sum_{l,m=1,\dots,k} |\xi_{lm}|^2, \forall \xi_{lm} \in \mathbb{R}.$$

We now introduce the bilinear form a_k on $H^1(\sigma_{ki})^k$ by

$$a_{ki}(u,v) = \int_{\sigma_{ki}} \sum_{l,m=1,\dots,k} \sigma^{(ki)}_{lm}(u)\epsilon_{lm}(v)\, dy.$$

Note that the Lamé system is a particular case of the above one for a particular choice of $c^{(ki)}_{lml'm'}$, in that case one has

$$a_k(u,v) = \int_{\sigma_{ki}} \left\{ \lambda_{ki} \operatorname{div} u \operatorname{div} v + 2\mu_{ki} \sum_{l,m=1}^{k} \epsilon_{lm}(u)\epsilon_{lm}(v) \right\} dy,$$

where $\lambda_{ki} > 0$ and $\mu_{ki} > 0$ are the Lamé constants of σ_{ki}.

Finally we define the bilinear form a on the space $\overset{\circ}{\mathbf{H}}{}^1_\mu(\Omega_0)$ by

$$a(u,v) = \sum_{\sigma_{ki} \subset E(\Omega_0)} a_{ki}(u_{ki}, v_{ki}).$$

The positiveness assumption on $c^{(ki)}_{lml'm'}$ and Korn's inequality (6.9) lead to the coerciveness of the form a. Consequently by the Lax-Milgram lemma, there exists a unique solution $u \in \overset{\circ}{\mathbf{H}}{}^1_\mu(\Omega_0)$ of

$$a(u,v) = \sum_{ki} \int_{\sigma_{ki}} f_{ki} \cdot v_{ki}\, dy, \forall v \in \overset{\circ}{\mathbf{H}}{}^1_\mu(\Omega_0), \tag{6.10}$$

for any $f_{ki} \in L^2(\sigma_{ki})^k$.

Applying Green's formula we see that

$$a_k(u,v) = -\int_{\sigma_{ki}} \sum_{l=1}^{k} \frac{\partial}{\partial y_l}\sigma_{lm}^{(ki)}(u)v_m dy - \int_{\partial\sigma_{ki}} (\sigma^{(ki)}(u)\cdot\nu)\cdot v d\sigma.$$

Therefore the solution $u \in \overset{\circ}{\mathbf{H}}{}^1_\mu(\Omega_0)$ of (6.10) is a weak solution of

$$-\sum_{l=1}^{k} \frac{\partial}{\partial y_l}\sigma_{lm}^{(ki)}(u_{ki}) - \sum_{\sigma_{ki}\prec\sigma_{k+1,j}} (\sigma^{(k+1,j)}(u_{k+1,j})\cdot\nu)_t = f_{ki} \text{ in } \sigma_{ki},$$

$$(\sigma^{(ki)}(u_{k+1,j})\cdot\nu)\cdot\nu = 0 \text{ on } \partial\sigma_{ki} \subset \Omega_0,$$

$$u = 0 \text{ on } \partial\Omega_0,$$

with the convention that $\sigma^{(ki)}(u_{ki}) = 0$ on a soft stratum σ_{ki}.

REFERENCES

[1] F. ALI MEHMETI, Regular solutions of transmission and interaction problems for wave equations, *Math. Meth. Appl. Sc.*, **11** (1989), 665–685.

[2] F. ALI MEHMETI, *Nonlinear Waves in Networks*, Akademie-Verlag, 1994.

[3] F. ALI MEHMETI, J. VON BELOW AND S. NICAISE, (Eds), Partial differential equations on multistructures, *Lect. Notes in Pure and Appl. Math.*, **219**, Marcel Dekker, 2001.

[4] J. VON BELOW, A characteristic equation associated to an eigenvalue problem on c^2-networks, *Linear Algebra and Appl.*, **71** (1985), 309–325.

[5] J. VON BELOW, *Parabolic Network Equations*, Habilitation Thesis, Eberhard-Karls-Universität Tübingen, 1993.

[6] J. VON BELOW AND S. NICAISE, Dynamical interface transition with diffusion in ramified media, *Communications in Partial Differential Equations*, **21** (1996), 255–279.

[7] A. BOROVSKIKH, R. MUSTAFOKULOV, K. LAZAREV AND YU. POKORNYI, A class of fourth-order differential equations on a spatial net, *Doklady Math.*, **52** (1995), 433–435.

[8] P.G. CIARLET, The finite element method for elliptic problems, *St. in Math. and its Appl.*, **4** (1978), North-Holland.

[9] B. DEKONINCK AND S. NICAISE, The eigenvalue problem for networks of beams, *Linear Algebra and its Applications*, **314** (2000), 165–189.

[10] A. FRIEDMAN, *Variation Principles and Free-boundary Problems*, John Wiley and Sons, 1982.

[11] G. FICHERA, *Existence Theorems in Elasticity*, Handbuch der physik, Bd. 6a/2, Springer-Verlag, 1972.

[12] A. GAVRILOV, S. NICAISE AND O. PENKIN, Poincaré's inequality on stratified sets and applications, *Proceedings of the 7th International Conference EVEQ2000*, to appear.

[13] J. GOBERT, Une inégalité fondamentale de la théorie de l'élasticité, *Bull. Soc. Royale des Sciences de Liège*, **3-4** (1962), 182–191.

[14] P. GRISVARD, *Elliptic Problems in Nonsmooth Domains*, Pitman, Boston-London-Melbourne, 1985.

[15] J.E. LAGNESE, G. LEUGERING AND E.J.P.G. SCHMIDT, *Modeling, Analysis and Control of Dynamic Elastic Multi-link Structures*, Birkhäuser, Boston, 1994.

[16] G. LUMER, Espaces ramifiés et diffusions sur les réseaux topologiques, *C.R. Acad. Sc. Paris, Série A*, **291** (1980), 627–630.

[17] S. NICAISE, Polygonal interface problems, *Methoden und Verfahren Math. Physik*, **39** (1993), Peter Lang Verlag.

[18] S. NICAISE, Polygonal interface problems for the biharmonic operator, *Mathematical Methods in the Applied Sciences*, **17** (1994), 21–39.

[19] S. NICAISE AND A.M. SÄNDIG, General interface problems I-II, *Mathematical Methods in the Applied Sciences*, **17** (1994), 395–429, 431–450.

[20] S. NICAISE AND A.M. SÄNDIG, Transmission problems for the Laplace and elasticity operators: Regularity and boundary integral formulation, *Mathematical Models and Methods in Applied Sciences*, **9** (1999), 855–898.

[21] O.M. PENKIN, YU.V. POKORNYI AND E.N. PROVOTOROVA, On a vectorial boundary value problem, *Boundary value problems, Interuniv. Collect. Sci. Works*, Perm' 1983, 64–70.

[22] O. PENKIN, About a geometrical approach to multistructures and some qualitative properties of solutions, in: F. Ali Mehmeti, J. von Belov and S. Nicaise (Eds.), *Partial Differential Equations on Multistructures*, Marcel Dekker, 2001, 183–191.

[23] V.V. ZHIKOV, Connectedness and homogenization. Examples of fractal conductivity, *Sb. Math.*, **187**(8) (1996), 1109–1147.