

Journal of Inequalities in Pure and Applied Mathematics

ON CERTAIN CLASSES OF ANALYTIC FUNCTIONS

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©2000 Victoria University
ISSN (electronic): 1443-5756
054-06



volume 7, issue 2, article 49,
2006.

*Received 28 August, 2005;
accepted 21 October, 2005.*

Communicated by: Th.M. Rassias

Abstract

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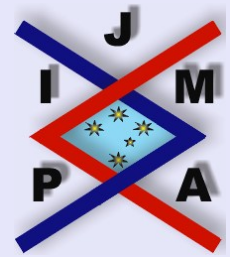


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Abstract

Let \mathcal{A} be the class of functions $f : f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ which are analytic in the unit disk E . We introduce the class $B_k(\lambda, \alpha, \rho) \subset \mathcal{A}$ and study some of their interesting properties such as inclusion results and covering theorem. We also consider an integral operator for these classes.

2000 Mathematics Subject Classification: 30C45, 30C50.

Key words: Analytic functions, Univalent, Functions with positive real part, Convex functions, Convolution, Integral operator.

This research is supported by the Higher Education Commission, Pakistan, through grant No: 1-28/HEC/HRD/2005/90.

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1. Introduction

Let \mathcal{A} denote the class of functions

$$f : f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the unit disk $E = \{z : |z| < 1\}$ and let $S \subset \mathcal{A}$ be the class of functions univalent in E .

Let $P_k(\rho)$ be the class of functions $p(z)$ analytic in E satisfying the properties $p(0) = 1$ and

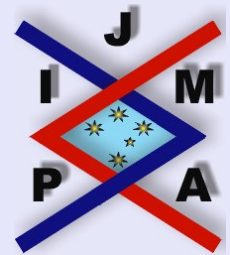
$$(1.1) \quad \int_0^{2\pi} \left| \frac{\operatorname{Re} p(z) - \rho}{1 - \rho} \right| d\theta \leq k\pi,$$

where $z = re^{i\theta}$, $k \geq 2$ and $0 \leq \rho < 1$. This class has been introduced in [7]. We note that, for $\rho = 0$, we obtain the class P_k defined and studied in [8], and for $\rho = 0$, $k = 2$, we have the well known class P of functions with positive real part. The case $k = 2$ gives the class $P(\rho)$ of functions with positive real part greater than ρ .

From (1.1) we can easily deduce that $p \in P_k(\rho)$ if, and only if, there exist $p_1, p_2 \in P(\rho)$ such that, for E ,

$$(1.2) \quad p(z) = \left(\frac{k}{4} + \frac{1}{2}\right) p_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) p_2(z).$$

Let f and g be analytic in E with $f(z) = \sum_{m=0}^{\infty} a_m z^m$ and $g(z) = \sum_{m=0}^{\infty} b_m z^m$



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in E . Then the convolution \star (or Hadamard Product) of f and g is defined by

$$(f \star g)(z) = \sum_{m=0}^{\infty} a_m b_m z^m, \quad m \in \mathbb{N}_0 = \{0, 1, 2, \dots\}.$$

Definition 1.1. Let $f \in \mathcal{A}$. Then $f \in B_k(\lambda, \alpha, \rho)$ if and only if

$$(1.3) \quad \left[(1 - \lambda) \left(\frac{f(z)}{z} \right)^\alpha + \lambda \frac{zf'(z)}{f(z)} \left(\frac{f(z)}{z} \right)^\alpha \right] \in P_k(\rho), \quad z \in E,$$

where $\alpha > 0, \lambda > 0, k \geq 2$ and $0 \leq \rho < 1$. The powers are understood as principal values.

For $k = 2$ and with different choices of λ, α and ρ , these classes have been studied in [2, 3, 4, 10]. In particular $B_2(1, \alpha, \rho)$ is the class of Bazilevic functions studied in [1].

We shall need the following results.

Lemma 1.1 ([9]). If $p(z)$ is analytic in E with $p(0) = 1$ and if λ is a complex number satisfying $\operatorname{Re} \lambda \geq 0, (\lambda \neq 0)$, then

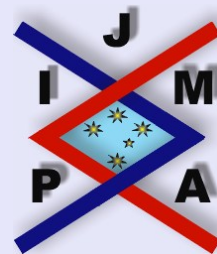
$$\operatorname{Re}[p(z) + \lambda zp'(z)] > \beta \quad (0 \leq \beta < 1)$$

implies

$$\operatorname{Re} p(z) > \beta + (1 - \beta)(2\gamma - 1),$$

where γ is given by

$$\gamma = \gamma_{\operatorname{Re} \lambda} = \int_0^1 (1 + t^{\operatorname{Re} \lambda})^{-1} dt.$$



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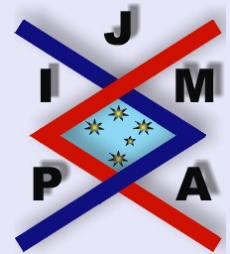
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Lemma 1.2 ([5]). Let $c > 0$, $\lambda > 0$, $\rho < 1$ and $p(z) = 1 + b_1z + b_2z^2 + \dots$ be analytic in E . Let $\operatorname{Re}[p(z) + c\lambda zp'(z)] > \rho$ in E , then

$$\operatorname{Re}[p(z) + czp'(z)] \geq 2\rho - 1 + 2(1 - \rho) \left(1 - \frac{1}{\lambda}\right) \frac{1}{c\lambda} \int_0^1 \frac{u^{\frac{1}{c\lambda}-1}}{1+u} du.$$

This result is sharp.



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2. Main Results

Theorem 2.1. Let $\lambda, \alpha > 0$, $0 \leq \rho < 1$ and let $f \in b_k(\lambda, \alpha, \rho)$. Then $\left(\frac{f(z)}{z}\right)^\alpha \in P_k(\rho_1)$, where ρ_1 is given by

$$(2.1) \quad \rho_1 = \rho + (1 - \rho)(2\gamma - 1),$$

and

$$\gamma = \int_0^1 \left(1 + t^{\frac{\lambda}{\alpha}}\right)^{-1} dt.$$

Proof. Let

$$\left(\frac{f(z)}{z}\right)^\alpha = p(z) = \left(\frac{k}{4} + \frac{1}{2}\right) p_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) p_2(z).$$

Then $p(z) = 1 + \alpha a_2 z + \dots$ is analytic in E , and

$$(2.2) \quad (f(z))^\alpha = z^\alpha p(z).$$

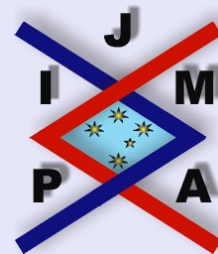
Differentiation of (2.2) and some computation give us

$$(1 - \lambda) \left(\frac{f(z)}{z}\right)^\alpha + \lambda \frac{zf'(z)}{f(z)} \left(\frac{f(z)}{z}\right)^\alpha = p(z) + \frac{\lambda}{\alpha} zp'(z).$$

Since $f \in B_k(\lambda, \alpha, \rho)$, so $\{p(z) + \frac{\lambda}{\alpha} zp'(z)\} \in P_k(\rho)$ for $z \in E$. This implies that

$$\operatorname{Re} \left[p_i(z) + \frac{\lambda}{\alpha} zp'_i(z) \right] > \rho, \quad i = 1, 2.$$

Using Lemma 1.1, we see that $\operatorname{Re}\{p_i(z)\} > \rho_1$, where ρ_1 is given by (2.1). Consequently $p \in P_k(\rho_1)$ for $z \in E$, and the proof is complete. \square



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Corollary 2.2. Let $f = zF'_1$ and $f \in B_2(\lambda, 1, \rho)$. Then F_1 is univalent in E .

Proceeding as in Theorem 2.1 and using Lemma 1.2, we have the following.

Theorem 2.3. Let $\alpha > 0$, $\lambda > 0$, $0 \leq \rho < 1$ and let $f \in B_k(\lambda, \alpha, \rho)$. Then $\frac{zf'(z)}{f(z)} \left(\frac{f(z)}{z}\right)^\alpha \in P_k(\rho_2)$, where

$$\rho_2 = 2\rho - 1 + \frac{1-\rho}{\lambda} + 2(1-\rho) \left(1 - \frac{1}{\lambda}\right) \frac{\alpha}{\lambda} \int_0^1 \frac{u^{\frac{\alpha}{\lambda}-1}}{1+u} du.$$

This result is sharp.

For $k = 2$, we note that f is univalent, see [1].

Theorem 2.4. Let, for $\alpha > 0$, $\lambda > 0$, $0 \leq \rho < 1$, $f \in B_k(\lambda, \alpha, \rho)$ and define $I(f) : \mathcal{A} \rightarrow \mathcal{A}$ as

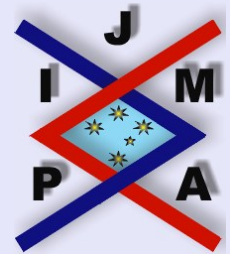
$$(2.3) \quad I(f) = F(z) = \left[\frac{1}{\lambda} z^{\alpha-\frac{1}{\lambda}} \int_0^z t^{\frac{1}{\lambda}-1-\alpha} (f(t))^\alpha dt \right]^{\frac{1}{\alpha}}, \quad z \in E.$$

Then $F \in B_k(\alpha\lambda, \alpha, \rho_1)$ for $z \in E$, where ρ_1 is given by (2.1).

Proof. Differentiating (2.3), we have

$$(1 - \alpha\lambda) \left(\frac{F(z)}{z}\right)^\alpha + \alpha\lambda \frac{zF'(z)}{F(z)} \left(\frac{F(z)}{z}\right)^\alpha = \left(\frac{f(z)}{z}\right)^\alpha.$$

Now, using Theorem 2.1, we obtain the required result. \square



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Theorem 2.5. *Let*

$$f : f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in B_k(\lambda, \alpha, \rho).$$

Then

$$|a_n| \leq \frac{k(1-\rho)}{\lambda + \alpha}.$$

The function $f_{\lambda, \alpha, \rho}(z)$ defined as

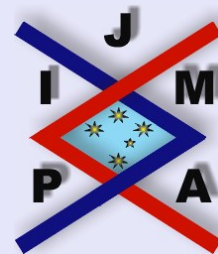
$$\begin{aligned} \left(\frac{f_{\lambda, \alpha, \rho}(z)}{z} \right)^\alpha &= \frac{\alpha}{\lambda} \int_0^1 \left[\left(\frac{k}{4} + \frac{1}{2} \right) u^{\frac{\alpha}{\lambda}-1} \frac{1 + (1-2\rho)uz}{1-uz} \right. \\ &\quad \left. - \left(\frac{k}{4} - \frac{1}{2} \right) u^{\frac{\alpha}{\lambda}-1} \frac{1 - (1-2\rho)uz}{1+uz} \right] du \end{aligned}$$

shows that this inequality is sharp.

Proof. Since $f \in B_k(\lambda, \alpha, \rho)$, so

$$\begin{aligned} (1-\lambda) \left(1 + \sum_{n=2}^{\infty} a_n z^{n-1} \right)^\alpha + \lambda \left(1 + \sum_{n=2}^{\infty} n a_n z^{n-1} \right) \left(1 + \sum_{n=2}^{\infty} a_n z^{n-1} \right)^\alpha \\ = H(z) = \left(1 + \sum_{n=1}^{\infty} c_n z^n \right) \in P_k(\rho). \end{aligned}$$

It is known that $|c_n| \leq k(1-\rho)$ for all n and using this inequality, we prove the required result. \square



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Different choices of k, λ, α and ρ yield several known results.

Theorem 2.6 (Covering Theorem). *Let $\lambda > 0$ and $0 < \rho < 1$. Let $f = zF_1' \in B_2(\lambda, 1, \rho)$. If D is the boundary of the image of E under F_1 , then every point of D has a distance of at least $\frac{\lambda+1}{(3+2\lambda-\rho)}$ from the origin.*

Proof. Let $F_1(z) \neq w_0, w_0 \neq 0$. Then $f_1(z) = \frac{w_0 F_1(z)}{w_0 + F_1(z)}$ is univalent in E since F_1 is univalent. Let

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad F_1(z) = z + \sum_{n=2}^{\infty} b_n z^n.$$

Then $a_2 = 2b_2$. Also

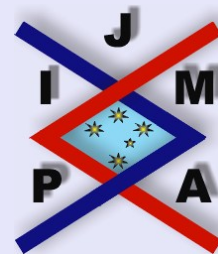
$$f_1(z) = z + \left(b_2 + \frac{1}{w_0} \right) z^2 + \dots,$$

and so $|b_2 + \frac{1}{w_0}| \leq 2$. Since, by Theorem 2.5, $|b_2| \leq \frac{1-\rho}{1+\lambda}$, we obtain $|w_0| \geq \frac{\lambda+1}{3+2\lambda-\rho}$. \square

Theorem 2.7. *For each $\alpha > 0, B_k(\lambda_1, \alpha, \rho) \subset B_k(\lambda_2, \alpha, \rho)$ for $0 \leq \lambda_2 < \lambda_1$.*

Proof. For $\lambda_2 = 0$, the proof is immediate. Let $\lambda_2 > 0$ and let $f \in B_k(\lambda_1, \alpha, \rho)$. Then there exist two functions $h_1, h_2 \in P_k(\rho)$ such that, from Definition 1.1 and Theorem 2.1,

$$(1 - \lambda) \left(\frac{f(z)}{z} \right)^\alpha + \lambda_1 \frac{z f'(z)}{f(z)} \left(\frac{f(z)}{z} \right)^\alpha = h_1(z),$$



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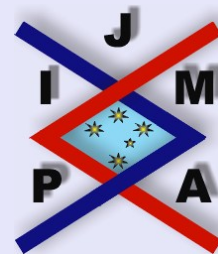
and

$$\left(\frac{f(z)}{z}\right)^\alpha = h_2(z).$$

Hence

$$(2.4) \quad (1-\lambda_2) \left(\frac{f(z)}{z}\right)^\alpha + \lambda_2 \frac{zf'(z)}{f(z)} \left(\frac{f(z)}{z}\right)^\alpha = \frac{\lambda_2}{\lambda_1} h_1(z) + \left(1 - \frac{\lambda_2}{\lambda_1}\right) h_2(z).$$

Since the class $P_k(\rho)$ is a convex set, see [6], it follows that the right hand side of (2.4) belongs to $P_k(\rho)$ and this proves the result. \square



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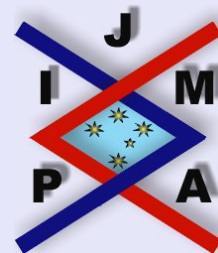
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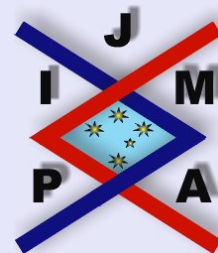
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