

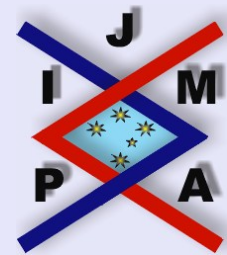
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A CHARACTERIZATION OF λ -CONVEX FUNCTIONS

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Abstract

The main result of this paper shows that λ -convex functions can be characterized in terms of a lower second-order generalized derivative.

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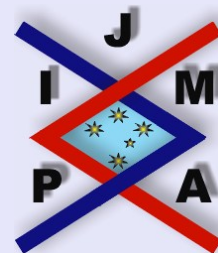
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1. Introduction

Let $I \subseteq \mathbb{R}$ be an open interval and $\lambda : I^2 \rightarrow (0, 1)$ be a fixed function. A real-valued function $f : I \rightarrow \mathbb{R}$ defined on an interval $I \subseteq \mathbb{R}$ is called λ -convex if

$$(1.1) \quad f(\lambda(x, y)x + (1 - \lambda(x, y))y) \leq \lambda(x, y)f(x) + (1 - \lambda(x, y))f(y) \quad \text{for } x, y \in I.$$

Such functions were introduced and discussed by Zs. Páles in [6], who obtained a Bernstein-Doetch type theorem for them. A Sierpiński-type result, stating that measurable λ -convex functions are convex, can be found in [2]. Recently K. Nikodem and Zs. Páles [5] proved that functions satisfying (1.1) with a constant λ can be characterized by use of a second-order generalized derivative. The main results of this paper show that λ -convexity, for λ not necessarily constant, can also be characterized in terms of a properly chosen lower second-order generalized derivative.



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2. Divided Differences and Convexity Triplets

If $f : I \rightarrow \mathbb{R}$ is an arbitrary function then define the second-order divided difference of f for three pairwise distinct points x, y, z of I by

$$(2.1) \quad f[x, y, z] := \frac{f(x)}{(y-x)(z-x)} + \frac{f(y)}{(x-y)(z-y)} + \frac{f(z)}{(x-z)(y-z)}.$$

It is known (cf. e.g.[4], [7]) and easy to check that a function $f : I \rightarrow \mathbb{R}$ is convex if and only if

$$f[x, y, z] \geq 0$$

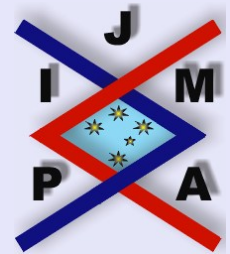
for every pairwise distinct points x, y, z of I . Motivated by this characterization of convexity, a triplet (x, y, z) in I^3 with pairwise distinct points x, y, z is called a *convexity triplet for a function* $f : I \rightarrow \mathbb{R}$ if $f[x, y, z] \geq 0$ and the set of all convexity triplets of f is denoted by $\mathcal{C}(f)$. Using this terminology, f is λ -convex if and only if

$$(2.2) \quad (x, \lambda(x, y)x + (1 - \lambda(x, y))y, y) \in \mathcal{C}(f) \quad \text{for } x, y \in I \text{ with } x \neq y.$$

The following result obtained in [5] will be used in the proof of the main theorem.

Lemma 2.1. (Chain Inequality) Let $f : I \rightarrow \mathbb{R}$ and $x_0 < x_1 < \dots < x_n$ ($n \geq 2$) be arbitrary points in I . Then, for all fixed $0 < j < n$,

$$(2.3) \quad \min_{1 \leq i \leq n-1} f[x_{i-1}, x_i, x_{i+1}] \leq f[x_0, x_j, x_n] \leq \max_{1 \leq i \leq n-1} f[x_{i-1}, x_i, x_{i+1}].$$



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3. Main Results

Assume that $\lambda : I \rightarrow (0, 1)$ is a fixed function and consider the *lower 2nd-order generalized λ -derivative* of a function $f : I \rightarrow \mathbb{R}$ at a point $\xi \in I$ defined by

$$(3.1) \quad \underline{\delta}_\lambda^2 f(\xi) := \liminf_{\substack{(x,y) \rightarrow (\xi, \xi) \\ \xi \in \text{co}\{x,y\}}} 2f[x, \lambda(x,y)x + (1 - \lambda(x,y))y, y].$$

One can easily show that if f is twice continuously differentiable at ξ then

$$\underline{\delta}_\lambda^2 f(\xi) = f''(\xi).$$

Moreover, from (2.2) and (3.1), if a function $f : I \rightarrow \mathbb{R}$ is λ -convex, then $\underline{\delta}_\lambda^2 f(\xi) \geq 0$ for every $\xi \in I$. The following example shows that the reverse implication is not true in general.

Example 3.1. Define $\lambda : \mathbb{R}^2 \rightarrow (0, 1)$ by the formula

$$\lambda(x, y) = \begin{cases} \frac{1}{3} & \text{if } x = y, \\ \frac{1}{2} & \text{if } x \neq y, \end{cases}$$

and take the function $f : \mathbb{R} \rightarrow \mathbb{R}$;

$$f(x) = \begin{cases} 0 & \text{if } x = 0, \\ 1 & \text{if } x \neq 0. \end{cases}$$



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It is easy to check that this function is not λ -convex, but $\underline{\delta}_\lambda^2 f(\xi) \geq 0$ for every $\xi \in \mathbb{R}$.

Now, let $\lambda : I^2 \rightarrow (0, 1)$ be a fixed function. Define

$$M(x, y) := \lambda(x, y)x + (1 - \lambda(x, y))y$$

and write conditions

$$(3.2) \quad \inf_{x, y \in [x_0, y_0]} \lambda(x, y) > 0 \quad \text{and} \quad \sup_{x, y \in [x_0, y_0]} \lambda(x, y) < 1,$$

for all $x_0, y_0 \in I$ with $x_0 \leq y_0$,

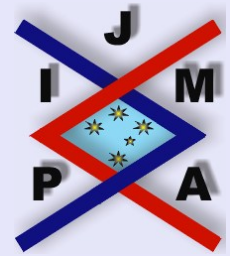
$$(3.3) \quad M(M(x, M(x, y)), M(y, M(x, y))) = M(x, y), \text{ for all } x, y \in I.$$

Of course, the above assumptions are satisfied for arbitrary constant λ . Moreover, observe that if M fulfils the bisymmetry equation (cf. [1], [3]) then it fulfils equation (3.3), too. Thus for each quasi-arithmetic mean M these conditions are also fulfilled.

Using a similar method as in [5] we can prove the following result.

Theorem 3.1. (Mean Value Inequality for λ -convexity) *Let $I \subseteq \mathbb{R}$ be an interval, $\lambda : I^2 \rightarrow (0, 1)$ satisfies assumptions (3.2) – (3.3), $f : I \rightarrow \mathbb{R}$ and $x, y \in I$ with $x \neq y$. Then there exists a point $\xi \in \text{co}\{x, y\}$ such that*

$$(3.4) \quad 2f[x, \lambda(x, y)x + (1 - \lambda(x, y))y, y] \geq \underline{\delta}_\lambda^2 f(\xi).$$



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Proof. In the sequel, a triplet $(x, u, y) \in I^3$ will be called a λ -triplet if

$$u = \lambda(x, y)x + (1 - \lambda(x, y))y$$

or

$$u = \lambda(y, x)y + (1 - \lambda(y, x))x.$$

Let x and y be distinct elements of I . Assume that $x < y$ (the proof in the case $x > y$ is similar). In what follows, we intend to construct a sequence of λ -triplets (x_n, u_n, y_n) such that

$$(3.5) \quad x_0 \leq x_1 \leq x_2 \leq \dots, \quad y_0 \geq y_1 \geq y_2 \geq \dots, \quad x_n < u_n < y_n \quad (n \in \mathbb{N}),$$

$$(3.6) \quad y_n - x_n \leq \left(\max \left\{ 1 - \inf_{x, y \in [x_0, y_0]} \lambda(x, y), \sup_{x, y \in [x_0, y_0]} \lambda(x, y) \right\} \right)^n (y_0 - x_0) \quad (n \in \mathbb{N}),$$

and

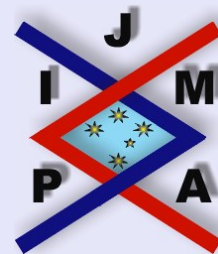
$$(3.7) \quad f[x_0, u_0, y_0] \geq f[x_1, u_1, y_1] \geq f[x_2, u_2, y_2] \geq \dots$$

Define

$$(x_0, u_0, y_0) := (x, \lambda(x, y)x + (1 - \lambda(x, y))y, y)$$

and assume that we have constructed (x_n, u_n, y_n) . Now set

$$z_{n,0} := x_n, \quad z_{n,1} := \lambda(x_n, u_n)x_n + (1 - \lambda(x_n, u_n))u_n, \quad z_{n,2} := u_n, \\ z_{n,3} := \lambda(y_n, u_n)y_n + (1 - \lambda(y_n, u_n))u_n, \quad z_{n,4} := y_n.$$



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Then $(z_{n,i-1}, z_{n,i}, z_{n,i+1})$ are λ -triplets for $i \in \{1, 2, 3\}$ (for $i \in \{1, 3\}$ immediately from the definition of λ -triplets and for $i = 2$ from condition (3.3)).

Using the Chain Inequality, we find that there exists an index $i \in \{1, 2, 3\}$ such that

$$f[x_n, u_n, y_n] \geq f[z_{n,i-1}, z_{n,i}, z_{n,i+1}].$$

Finally, define

$$(x_{n+1}, u_{n+1}, y_{n+1}) := (z_{n,i-1}, z_{n,i}, z_{n,i+1}).$$

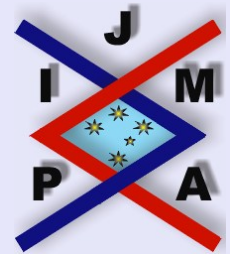
The sequence so constructed clearly satisfies (3.5) and (3.7). We prove (3.6) by induction. It is obvious for $n = 0$. Assume that it holds for n and $u_n = \lambda(x_n, y_n)x_n + (1 - \lambda(x_n, y_n))y_n$ (if $u_n = \lambda(y_n, x_n)y_n + (1 - \lambda(y_n, x_n))x_n$ then the motivation is the same). Consider three cases.

(i)

$$(x_{n+1}, u_{n+1}, y_{n+1}) = (x_n, \lambda(x_n, u_n)x_n + (1 - \lambda(x_n, u_n))u_n, u_n)$$

then

$$\begin{aligned} & y_{n+1} - x_{n+1} \\ &= u_n - x_n \\ &= \lambda(x_n, y_n)x_n + (1 - \lambda(x_n, y_n))y_n - x_n \\ &= (1 - \lambda(x_n, y_n))(y_n - x_n) \\ &\leq \max \left\{ 1 - \inf_{x,y \in [x_0, y_0]} \lambda(x, y), \sup_{x,y \in [x_0, y_0]} \lambda(x, y) \right\} (y_n - x_n) \end{aligned}$$



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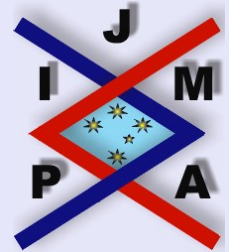
$$\leq \left(\max \left\{ 1 - \inf_{x,y \in [x_0, y_0]} \lambda(x, y), \sup_{x,y \in [x_0, y_0]} \lambda(x, y) \right\} \right)^{n+1} (y_0 - x_0).$$

(ii)

$$\begin{aligned} & (x_{n+1}, u_{n+1}, y_{n+1}) \\ &= (\lambda(x_n, u_n)x_n + (1 - \lambda(x_n, u_n))u_n, u_n, \lambda(y_n, u_n)y_n + (1 - \lambda(y_n, u_n))u_n) \end{aligned}$$

then

$$\begin{aligned} & y_{n+1} - x_{n+1} \\ &= \lambda(x_n, u_n)(u_n - x_n) + \lambda(y_n, u_n)(y_n - u_n) \\ &= \lambda(x_n, u_n)(1 - \lambda(x_n, y_n))(y_n - x_n) + \lambda(y_n, u_n)\lambda(x_n, y_n)(y_n - x_n) \\ &\leq \max \left\{ 1 - \inf_{x,y \in [x_0, y_0]} \lambda(x, y), \sup_{x,y \in [x_0, y_0]} \lambda(x, y) \right\} (1 - \lambda(x_n, y_n))(y_n - x_n) \\ &\quad + \max \left\{ 1 - \inf_{x,y \in [x_0, y_0]} \lambda(x, y), \sup_{x,y \in [x_0, y_0]} \lambda(x, y) \right\} \lambda(x_n, y_n)(y_n - x_n) \\ &= \max \left\{ 1 - \inf_{x,y \in [x_0, y_0]} \lambda(x, y), \sup_{x,y \in [x_0, y_0]} \lambda(x, y) \right\} (y_n - x_n) \\ &\leq \left(\max \left\{ 1 - \inf_{x,y \in [x_0, y_0]} \lambda(x, y), \sup_{x,y \in [x_0, y_0]} \lambda(x, y) \right\} \right)^{n+1} (y_0 - x_0). \end{aligned}$$



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(iii)

$$(x_{n+1}, u_{n+1}, y_{n+1}) = (u_n, \lambda(y_n, u_n)y_n + (1 - \lambda(y_n, u_n))u_n, y_n)$$

then

$$\begin{aligned} & y_{n+1} - x_{n+1} \\ &= y_n - u_n \\ &= y_n - (\lambda(x_n, y_n)x_n + (1 - \lambda(x_n, y_n))y_n) \\ &= \lambda(x_n, y_n)(y_n - x_n) \\ &\leq \max \left\{ 1 - \inf_{x,y \in [x_0, y_0]} \lambda(x, y), \sup_{x,y \in [x_0, y_0]} \lambda(x, y) \right\} (y_n - x_n) \\ &\leq \left(\max \left\{ 1 - \inf_{x,y \in [x_0, y_0]} \lambda(x, y), \sup_{x,y \in [x_0, y_0]} \lambda(x, y) \right\} \right)^{n+1} (y_0 - x_0). \end{aligned}$$

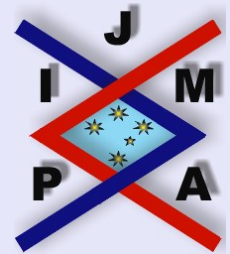
Thus (3.6) is also verified.

Due to the monotonicity properties of the sequences (x_n) , (y_n) and also (3.2), (3.6), there exists a unique element $\xi \in [x, y]$ such that

$$\bigcap_{i=0}^{\infty} [x_n, y_n] = \{\xi\}.$$

Then, by (3.7) and symmetry of the second-order divided difference, we get that

$$\begin{aligned} f[x, \lambda(x, y)x + (1 - \lambda(x, y))y, y] &= f[x_0, u_0, y_0] \\ &\geq \liminf_{n \rightarrow \infty} f[x_n, u_n, y_n] \end{aligned}$$



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$$\begin{aligned} &\geq \liminf_{\substack{(v,w) \rightarrow (\xi,\xi) \\ \xi \in \text{co}\{v,w\}}} f[v, \lambda(v, w)v + (1 - \lambda(v, w))w, w] \\ &= \frac{1}{2} \delta_{\lambda}^2 f(\xi), \end{aligned}$$

which completes the proof. \square

As an immediate consequence of the above theorem, we get the following characterization of λ -convexity.

Theorem 3.2. *Let $\lambda : I^2 \rightarrow (0, 1)$ be a fixed function satisfying assumptions (3.2) – (3.3). A function $f : I \rightarrow \mathbb{R}$ is λ -convex on I if and only if*

$$(3.8) \quad \delta_{\lambda}^2 f(\xi) \geq 0, \text{ for all } \xi \in I.$$

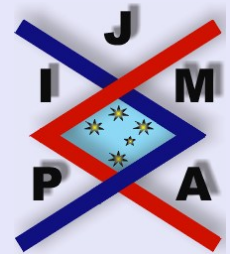
Proof. If f is λ -convex, then, clearly $\delta_{\lambda}^2 f \geq 0$. Conversely, if $\delta_{\lambda}^2 f$ is nonnegative on I , then, by the previous theorem

$$f[x, \lambda(x, y)x + (1 - \lambda(x, y))y, y] \geq 0$$

for all $x, y \in I$, i.e., f is λ -convex. \square

An obvious but interesting consequence of Theorem 3.2 is that the λ -convexity property is *localizable* in the following sense:

Corollary 3.3. *Let $\lambda : I^2 \rightarrow (0, 1)$ be a fixed function satisfying assumptions (3.2) – (3.3). A function $f : I \rightarrow \mathbb{R}$ is λ -convex on I if and only if, for each point $\xi \in I$, there exists a neighborhood U of ξ such that f is λ -convex on $I \cap U$.*



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