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GENERALIZED INTEGRAL OPERATOR AND MULTIVALENT FUNCTIONS

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Abstract

Let $\mathcal{A}(p)$ be the class of functions $f : f(z) = z^p + \sum_{j=1}^{\infty} a_j z^{p+j}$ analytic in the open unit disc E. Let, for any integer $n > -p$, $\quad f_{n+p-1}(z) = \frac{z^p}{(1-z)^{n+p}}$. We define $f_{n+p-1}^{(-1)}(z)$ by using convolution $\star$ as $f_{n+p-1}(z) \star f_{n+p-1}^{(-1)}(z) = \frac{z^P}{(1-z)^{n+p}}$. A function p, analytic in E with $p(0) = 1$, is in the class $P_k(\rho)$ if $\int_0^{2\pi} \left| \frac{Rep(z)-\rho}{p-\rho} \right| d\theta \leq k\pi$, where $z = re^{i\theta}, k \geq 2$ and $0 \leq \rho < p$. We use the class $P_k(\rho)$ to introduce a new class of multivalent analytic functions and define an integral operator $I_{n+p-1}(f) = f_{n+p-1}^{(-1)} \star f(z)$ for $f(z)$ belonging to this class. We derive some interesting properties of this generalized integral operator which include inclusion results and radius problems.

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Contents

1. Introduction

Let $\mathcal{A}(p)$ denote the class of functions f given by

$$f(z) = z^p + \sum_{j=1}^{\infty} a_j z^{p+j}, \quad p \in N = \{1, 2, \dots\}$$

which are analytic in the unit disk $E = \{z : |z| < 1\}$. The Hadamard product or convolution $(f \star g)$ of two functions with

$$f(z) = z^p + \sum_{j=1}^{\infty} a_{j,1} z^{p+j} \quad \text{and} \quad g(z) = z^p + \sum_{j=1}^{\infty} z^{p+j}$$

is given by

$$(f \star g)(z) = z^p + \sum_{j=1}^{\infty} a_{j,1} a_{j,2} z^{p+j}.$$

The integral operator $I_{n+p-1} : \mathcal{A}(p) \longrightarrow \mathcal{A}(p)$ is defined as follows, see [2].

For any integer n greater than $-p$, let $f_{n+p-1}(z) = \frac{z^p}{(1-z)^{n+p}}$ and let $f_{n+p-1}^{(-1)}(z)$ be defined such that

(1.1) $$f_{n+p-1}(z) \star f_{n+p-1}^{(-1)}(z) = \frac{z^p}{(1-z)^{p+1}}.$$

Then

(1.2) $$I_{n+p-1} f(z) = f_{n+p-1}^{(-1)}(z) \star f(z) = \left[\frac{z^p}{(1-z)^{n+p}}\right]^{(-1)} \star f(z).$$

From (1.1) and (1.2) and a well known identity for the Ruscheweyh derivative [1, 8], it follows that

$$z\left(I_{n+p}f(z)\right)' = (n+p)I_{n+p-1}f(z) - nI_{n+p}f(z). \tag{1.3}$$

For $p=1$, the identity (1.3) is given by Noor and Noor [3].

Let $P_k(\rho)$ be the class of functions $p(z)$ analytic in E satisfying the properties $p(0)=1$ and

$$\int_0^{2\pi}\left|\frac{\operatorname{Re}p(z)-\rho}{p-\rho}\right|d\theta \le k\pi, \tag{1.4}$$

where $z=re^{i\theta}$, $k\ge 2$ and $0\le\rho<p$. For $p=1$, this class was introduced in [5] and for $\rho=0$, see [6]. For $\rho=0$, $k=2$, we have the well known class P of functions with positive real part and the class $k=2$ gives us the class $P(\rho)$ of functions with positive real part greater than ρ. Also from (1.4), we note that $p\in P_k(\rho)$ if and only if there exist $p_1,p_2\in P_k(\rho)$ such that

$$p(z) = \left(\frac{k}{4}+\frac{1}{2}\right)p_1(z) - \left(\frac{k}{4}-\frac{1}{2}\right)p_2(z). \tag{1.5}$$

It is known [4] that the class $P_k(\rho)$ is a convex set.

Definition 1.1. *Let $f\in\mathcal{A}(p)$. Then $f\in T_k(\alpha,p,n,\rho)$ if and only if*

$$\left[(1-\alpha)\frac{I_{n+p-1}f(z)}{z^p}+\alpha\frac{I_{n+p}f(z)}{z^p}\right]\in P_k(\rho),$$

for $\alpha\ge 0, n>-p, 0\le\rho<p, k\ge 2$ and $z\in E$.

2. Preliminary Results

Lemma 2.1. *Let* $p(z) = 1 + b_1 z + b_2 z^2 + \cdots \in P(\rho)$. *Then*

$$\operatorname{Re} p(z) \geq 2\rho - 1 + \frac{2(1-\rho)}{1+|z|}.$$

This result is well known.

Lemma 2.2 ([7]). *If* $p(z)$ *is analytic in* E *with* $p(0) = 1$ *and if* λ_1 *is a complex number satisfying* $\operatorname{Re} \lambda_1 \geq 0, \quad (\lambda_1 \neq 0)$, *then* $\operatorname{Re}\{p(z) + \lambda_1 z p'(z)\} > \beta \quad (0 \leq \beta < p)$ *implies*

$$\operatorname{Re} p(z) > \beta + (1-\beta)(2\gamma_1 - 1),$$

where γ_1 *is given by*

$$\gamma_1 = \int_0^1 \left(1 + t^{\operatorname{Re} \lambda_1}\right)^{-1} dt.$$

Lemma 2.3 ([9]). *If* $p(z)$ *is analytic in* E, $\quad p(0) = 1$ *and* $\operatorname{Re} p(z) > \frac{1}{2}, \quad z \in E$, *then for any function* F *analytic in* E, *the function* $p \star F$ *takes values in the convex hull of the image* E *under* F.

3. Main Results

Theorem 3.1. *Let $f \in T_k(\alpha, p, n, \rho_1)$ and $g \in T_k(\alpha, p, n, \rho_2)$, and let $F = f \star g$. Then $F \in T_k(\alpha, p, n, \rho_3)$ where*

$$\rho_3 = 1 - 4(1-\rho_1)(1-\rho_2)\left[1 - \frac{n+p}{1-\alpha}\int_0^1 \frac{u^{\left(\frac{n+p}{1-\alpha}\right)-1}}{1+u}du\right]. \tag{3.1}$$

This results is sharp.

Proof. Since $f \in T_k(\alpha, p, n, \rho_1)$, it follows that

$$H(z) = \left[(1-\alpha)\frac{I_{n+p-1}f(z)}{z^p} + \alpha\frac{I_{n+p}f(z)}{z^p}\right] \in P_k(\rho_1),$$

and so using (1.3), we have

$$I_{n+p}f(z) = \frac{n+p}{1-\alpha}z^{-\left(\frac{n+p}{1-\alpha}\right)}\int_0^z t^{\frac{n+p}{1-\alpha}-1}H(t)dt. \tag{3.2}$$

Similarly

$$I_{n+p}g(z) = \frac{n+p}{1-\alpha}z^{-\left(\frac{n+p}{1-\alpha}\right)}\int_0^z t^{\frac{n+p}{1-\alpha}-1}H^\star(t)dt, \tag{3.3}$$

where $H^\star \in P_k(\rho_2)$.

Using (3.1) and (3.2), we have

$$I_{n+p}F(z) = \frac{n+p}{1-\alpha}z^{-\left(\frac{n+p}{1-\alpha}\right)}\int_0^z t^{\frac{n+p}{1-\alpha}-1}Q(t)dt, \tag{3.4}$$

where

$$Q(z) = \left(\frac{k}{4}+\frac{1}{2}\right) q_1(z) - \left(\frac{k}{4}-\frac{1}{2}\right) q_2(z)$$

(3.5) $$= \frac{n+p}{1-\alpha} z^{-\left(\frac{n+p}{1-\alpha}\right)} \int_0^z t^{\frac{n+p}{1-\alpha}-1} (H \star H^\star)(t)dt.$$

Now

$$H(z) = \left(\frac{k}{4}+\frac{1}{2}\right) h_1(z) - \left(\frac{k}{4}-\frac{1}{2}\right) h_2(z)$$

(3.6) $$H(z)^\star = \left(\frac{k}{4}+\frac{1}{2}\right) h_1^\star(z) - \left(\frac{k}{4}-\frac{1}{2}\right) h_2^\star(z),$$

where $h_i \in P(\rho_1)$ and $h_i^\star \in P_k(\rho_2), \quad i = 1, 2.$

Since

$$p_i^\star(z) = \frac{h_i^\star(z) - \rho_2}{2(1-\rho_2)} + \frac{1}{2} \in P\left(\frac{1}{2}\right), \quad i = 1, 2,$$

we obtain that $(h_i \star p_i^\star)(z) \in P(\rho_1)$, by using the Herglotz formula.

Thus

$$(h_i \star h_i^\star)(z) \in P(\rho_3)$$

with

(3.7) $$\rho_3 = 1 - 2(1-\rho_1)(1-\rho_2).$$

Using (3.4), (3.5), (3.6), (3.7) and Lemma 2.1, we have

$$\begin{aligned}\operatorname{Re} q_i(z) &= \frac{n+p}{1-\alpha}\int_0^1 u^{\frac{n+p}{1-\alpha}-1}\operatorname{Re}\{(h_i \star h_i^\star)(uz)\}du \\ &\geq \frac{n+p}{1-\alpha}\int_0^1 u^{\frac{n+p}{1-\alpha}-1}\left(2\rho_3 - 1 + \frac{2(1-\rho_3)}{1+u|z|}\right)du \\ &> \frac{n+p}{1-\alpha}\int_0^1 u^{\frac{n+p}{1-\alpha}-1}\left(2\rho_3 - 1 + \frac{2(1-\rho_3)}{1+u}\right)du \\ &= 1-4(1-\rho_1)(1-\rho_2)\left[1 - \frac{n+p}{1-\alpha}\int_0^1 \frac{u^{\frac{n+p}{1-\alpha}-1}}{1+u}du\right].\end{aligned}$$

From this we conclude that $F \in T_k(\alpha, p, n, \rho_3)$, where ρ_3 is given by (3.1).

We discuss the sharpness as follows:

We take

$$\begin{aligned}H(z) &= \left(\frac{k}{4}+\frac{1}{2}\right)\frac{1+(1-2\rho_1)z}{1-z} - \left(\frac{k}{4}-\frac{1}{2}\right)\frac{1-(1-2\rho_1)z}{1+z}, \\ H^\star(z) &= \left(\frac{k}{4}+\frac{1}{2}\right)\frac{1+(1-2\rho_2)z}{1-z} - \left(\frac{k}{4}-\frac{1}{2}\right)\frac{1-(1-2\rho_2)z}{1+z}.\end{aligned}$$

Since

$$\left(\frac{1+(1-2\rho_1)z}{1-z}\right)\star\left(\frac{1+(1-2\rho_2)z}{1-z}\right) = 1-4(1-\rho_1)(1-\rho_2)+\frac{4(1-\rho_1)(1-\rho_2)}{1-z},$$

it follows from (3.5) that

$$q_i(z) = \frac{n+p}{1-\alpha}\int_0^1 u^{\frac{n+p}{1-\alpha}-1}\left\{1-4(1-\rho_1)(1-\rho_2)+\frac{4(1-\rho_1)(1-\rho_2)}{1-uz}\right\}du$$
$$\longrightarrow 1-4(1-\rho_1)(1-\rho_2)\left\{1-\frac{n+p}{1-\alpha}\int_0^1 \frac{u^{\frac{n+p}{1-\alpha}-1}}{1+u}du\right\} \text{ as } z\longrightarrow 1.$$

This completes the proof. □

We define $J_c : \mathcal{A}(p) \longrightarrow \mathcal{A}(p)$ as follows:

(3.8) $$J_c(f) = \frac{c+p}{z^c}\int_0^z t^{c-1} f(t)dt,$$

where c is real and $c > -p$.

Theorem 3.2. *Let* $f \in T_k(\alpha, p, n, \rho)$ *and* $J_c(f)$ *be given by (3.8). If*

(3.9) $$\left[(1-\alpha)\frac{I_{n+p}f(z)}{z^p}+\alpha\frac{I_{n+p}J_c(f)}{z^p}\right] \in P_k(\rho),$$

then

$$\left\{\frac{I_{n+p}J_c(f)}{z^p}\right\} \in P_k(\gamma), \quad z \in E$$

and

$$\gamma = \rho(1-\rho)(2\sigma-1)$$

(3.10) $$\sigma = \int_0^1 \left[1+t^{\operatorname{Re}\frac{1-\alpha}{\lambda+p}}\right]^{-1} dt.$$

Proof. From (3.8), we have

$$(c+p)I_{n+p}f(z) = cI_{n+p}J_c(f) + z(I_{n+p}J_c(f))'.$$

Let

$$H_c(z) = \left(\frac{k}{4}+\frac{1}{2}\right)s_1(z) - \left(\frac{k}{4}-\frac{1}{2}\right)s_2(z) = \frac{I_{n+p}J_c(f)}{z^p}. \tag{3.11}$$

From (3.9), (3.10) and (3.11), we have

$$\left[(1-\alpha)\frac{I_{n+p}f(z)}{z^p} + \alpha\frac{I_{n+p}J_c(f)}{z^p}\right] = \left[H_c(z) + \frac{1-\alpha}{\lambda+p}zH_c'(z)\right]$$

and consequently

$$\left[s_i(z) + \frac{1-\alpha}{\lambda+p}zs_i'(z)\right] \in P(\rho), \quad i=1,2.$$

Using Lemma 2.2, we have $\operatorname{Re}\{s_i(z)\} > \gamma$ where γ is given by (3.10). Thus

$$H_c(z) = \frac{I_{n+p}J_c(f)}{z^p} \in P_k(\gamma)$$

and this completes the proof. □

Let

$$J_n(f(z)) := J_n(f) = \frac{n+p}{z^p}\int_0^z t^{n-1}f(t)dt. \tag{3.12}$$

Then

$$I_{n+p-1}J_n(f) = I_{n+p}(f),$$

and we have the following.

Theorem 3.3. *Let* $f \in T_k(\alpha, p, n+1, \rho)$*. Then* $J_n(f) \in T_k(\alpha, p, n, \rho)$ *for* $z \in E$*.*

Theorem 3.4. *Let* $\phi \in C_p$*, where* C_p *is the class of* p*-valent convex functions, and let* $f \in T_k(\alpha, p, n, \rho)$*. Then* $\phi \star f \in T_k(\alpha, p, n, \rho)$ *for* $z \in E$*.*

Proof. Let $G = \phi \star f$. Then

$$\begin{aligned}
(1-\alpha)&\frac{I_{n+p-1}G(z)}{z^p} + \alpha\frac{I_{n+p}G(z)}{z^p} \\
&= (1-\alpha)\frac{I_{n+p-1}(\phi\star f)(z)}{z^p} + \alpha\frac{I_{n+p}(\phi\star f)(z)}{z^p} \\
&= \frac{\phi(z)}{z^p} \star \left[(1-\alpha)\frac{I_{n+p-1}f(z)}{z^p} + \alpha\frac{I_{n+p}f(z)}{z^p}\right] \\
&= \frac{\phi(z)}{z^p} \star H(z), \quad H \in P_k(\rho) \\
&= \left(\frac{k}{4}+\frac{1}{2}\right)\left\{(p-\rho)\left(\frac{\phi(z)}{z^p}\star h_1(z)\right)+\rho\right\} \\
&\qquad - \left(\frac{k}{4}-\frac{1}{2}\right)\left\{(p-\rho)\left(\frac{\phi(z)}{z^p}\star h_2(z)\right)+\rho\right\}, \quad h_1, h_2 \in P.
\end{aligned}$$

Since $\phi \in C_p$, $\quad \mathrm{Re}\left\{\frac{\phi(z)}{z^p}\right\} > \frac{1}{2}, z \in E$ and so using Lemma 2.3, we conclude that $G \in T_k(\alpha, p, n, \rho)$. □

3.1. Applications

(1) We can write $J_c(f)$ defined by (3.8) as

$$J_c(f) = \phi_c \star f,$$

where ϕ_c is given by

$$\phi_c(z) = \sum_{m=p}^{\infty} \frac{p+c}{m+c} z^m, \quad (c > -p)$$

and $\phi_c \in C_p$. Therefore, from Theorem 3.4, it follows that $J_c(f) \in T_k(\alpha, p, n, \rho)$.

(2) Let $J_n(f)$, defined by (3.12), belong to $T_k(\alpha, p, n, \rho)$. Then $f \in T_k(\alpha, p, n, \rho)$ for $|z| < r_n = \frac{(1+n)}{2+\sqrt{3+n^2}}$. In fact, $J_n(f) = \Psi_n \star f$, where

$$\Psi_n(z) = z^p + \sum_{j=2}^{\infty} \frac{n+j-1}{n+1} z^{j+p-1}$$
$$= \frac{n}{n+1} \cdot \frac{z^p}{1-z} + \frac{1}{n+1} \cdot \frac{z^p}{(1-z)^2}$$

and $\Psi_n \in C_p$ for

$$|z| < r_n = \frac{1+n}{2+\sqrt{3+n^2}}.$$

Now $I_{n+p-1} J_n(f) = \Psi_n \star I_{n+p-1} f$, and using Theorem 3.4, we obtain the result.

Theorem 3.5. *For* $0 \le \alpha_2 < \alpha_1, \quad T_k(\alpha_1, p, n, \rho) \subset T_k(\alpha_2, p, n, \rho), \quad z \in E.$

Proof. For $\alpha_2 = 0$, the proof is immediate. Let $\alpha_2 > 0$ and let $f \in T_k(\alpha_1, p, n, \rho)$.

Then

$$(1-\alpha_2)\frac{I_{n+p-1}f(z)}{z^p}+\alpha_2\frac{I_{n+p}f(z)}{z^p}$$
$$+\frac{\alpha_2}{\alpha_1}\left[\left(\frac{\alpha_1}{\alpha_2}-1\right)\frac{I_{n+p-1}f(z)}{z^p}+(1-\alpha_1)\frac{I_{n+p-1}f(z)}{z^p}+\alpha_1\frac{I_{n+p-1}f(z)}{z^p}\right]$$
$$=\left(1-\frac{\alpha_2}{\alpha_1}\right)H_1(z)+\frac{\alpha_2}{\alpha_1}H_2(z),\quad H_1,H_2\in P_k(\rho).$$

Since $P_k(\rho)$ is a convex set, we conclude that $f\in T_k(\alpha_2,p,n,\rho)$ for $z\in E$. □

Theorem 3.6. *Let $f\in T_k(0,p,n,\rho)$. Then $f\in T_k(\alpha,p,n,\rho)$ for*

$$|z|<r_\alpha=\frac{1}{2\alpha+\sqrt{4\alpha^2-2\alpha+1}},\quad \alpha\neq\frac{1}{2},\quad 0<\alpha<1.$$

Proof. Let

$$\Psi_\alpha(z)=(1-\alpha)\frac{z^p}{1-z}+\alpha\frac{z^p}{(1-z)^2}$$
$$=z^p+\sum_{m=2}^{\infty}(1+(m-1)\alpha)z^{m+p-1}.$$

$\Psi_\alpha\in C_p$ for

$$|z|<r_\alpha=\frac{1}{2\alpha+\sqrt{4\alpha^2-2\alpha+1}}\quad\left(\alpha\neq\frac{1}{2},\quad 0<\alpha<1\right)$$

We can write

$$\left[(1-\alpha)\frac{I_{n+p-1}f(z)}{z^p}+\alpha\frac{I_{n+p}f(z)}{z^p}\right]=\frac{\Psi_\alpha(z)}{z^p}\star\frac{I_{n+p-1}f(z)}{z^p}.$$

Applying Theorem 3.4, we see that $f\in T_k(\alpha,p,n,\rho)$ for $|z|<r_\alpha$. □

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