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AN INEQUALITY ASSOCIATED WITH SOME ENTIRE FUNCTIONS

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ABSTRACT. For some family of entire functions the estimates of growth on infinity are established. In case when a function from this family coincides with exponent the inequality obtained is precise.

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The object of our paper is to determine the order of growth to infinity of some family of entire functions. For an arbitrary $\alpha > 0$ we introduce the following function

$$\Phi(z,\alpha) = \sum_{k=0}^{\infty} \frac{z^k}{(k!)^\alpha}, \quad \alpha > 0, \quad z \in \mathbb{C}. \tag{1}$$

Note that

$$\Phi(z,1) = e^z.$$

It is easy to show that if $\alpha > 0$ then the function $\Phi(z,\alpha)$ is defined by series (1) for all z in the complex plane $\mathbb{C}$.

Proposition 1. *The radius of convergence of the series (1) is equal to infinity.*

Proof. According to the Cauchy formula (see, e.g., [2, 2.6]) the radius of convergence of the series

$$\sum_{n=0}^{\infty} c_n z^n$$

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is equal to

$$R = \frac{1}{\overline{\lim\limits_{n\to\infty}} \sqrt[n]{|c_n|}}.$$

In our case $c_n = (n!)^{-\alpha}$. We may use the Stirling formula (see [2, 12.33]) in the following form

(2) $$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(1 + \frac{\theta_n}{11n}\right), \quad 0 < \theta_n < 1,\ n = 1, 2, \dots$$

As a result we get

$$\begin{aligned}
\frac{1}{\sqrt[n]{|c_n|}} &= (n!)^{\alpha/n} \\
&= \left[\sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(1 + \frac{\theta_n}{11n}\right)\right]^{\alpha/n} \\
&= \left(\frac{n}{e}\right)^\alpha (2\pi)^{\alpha/2n} e^{\alpha(\ln n)/2n} \left(1 + \frac{\theta_n}{11n}\right)^{\alpha/n} \\
&= \left(\frac{n}{e}\right)^\alpha (1 + \varepsilon_n) \to \infty, \quad n \to \infty,
\end{aligned}$$

where $\varepsilon_n = o(1)$, $n \to \infty$. □

Corollary 2. *The function $\Phi(z, \alpha)$, $\alpha > 0$, is entire function of z.*

The function $\Phi(z, \alpha)$ with $\alpha = \frac{1}{q}$ arises in estimates of the solutions of some Volterra type integral equations with kernel from L_p, where $\frac{1}{p} + \frac{1}{q} = 1$. We mention also the equation with convolution on the circle which these functions satisfy. For two arbitrary 2π-periodical functions $f(\theta)$ and $g(\theta)$ introduce their convolution

$$(f * g)(\theta) = \frac{1}{2\pi}\int_0^{2\pi} f(\theta - \varphi) g(\varphi) d\varphi.$$

If we denote

(3) $$f_\alpha(\theta) = \Phi(e^{i\theta}, \alpha),$$

then it is easy to check that this function satisfies the following equation

(4) $$(f_\alpha * f_\beta)(\theta) = f_{\alpha+\beta}(\theta), \quad f_1(\theta) = \exp e^{i\theta}.$$

It easy to show that every solution of equation (4) has the form (3).

It is well known that for $\Phi(z, \alpha)$ the following formula

$$\ln \Phi(x, \alpha) = \alpha x^{1/\alpha} + o\left(x^{1/\alpha}\right), \quad x \to +\infty$$

is valid (see, e.g. [1, 4.1, Th. 68]). However, in some applications, an explicit estimate for the error of the above asymptotic approximation is desirable.

We are going to prove the following inequality.

Theorem 3. *Let $0 < \alpha \le 1$. Then for all $x \ge 1$ the inequality*

(5) $$\ln \Phi(x, \alpha) \le \alpha x^{1/\alpha} + \frac{1-\alpha}{\alpha}\ln x + \ln(12\alpha^{-2})$$

is valid.

Remark 4. The order in estimate (5) is precise, at least when $\alpha = 1/q$, where q is natural, because in this case for all $x \geq 1$ the inequality

$$\ln \Phi(x, \alpha) \geq \alpha x^{1/\alpha} \tag{6}$$

is true. As it easy to verify, for $\alpha = 1$ the inequality (6) becomes equality.

At first we prove the inequality (5) for $\alpha = \frac{1}{q}$, where q is natural, and after that we use the interpolation technique to prove it for all α, $0 < \alpha \leq 1$.

Lemma 5. *Let q be a natural number and $Q(x)$ be the following polynomial*

$$Q(x) = \sum_{k=0}^{q-1} (k+1) \frac{x^k}{[(k+q)!]^{1/q}}. \tag{7}$$

Then there exists a constant $c_1 \leq 2$ so that

$$\int_0^\infty e^{-\frac{1}{q}t^q} Q(t)dt \leq c_1 q^2. \tag{8}$$

Proof. It follows from (7) that the inequality

$$Q(t) = \sum_{k=1}^{q} k \frac{t^{k-1}}{[(k+q-1)!]^{1/q}} \leq \sum_{k=1}^{q} k t^{k-1} \tag{9}$$

is valid for all $t > 0$. Then

$$\int_0^1 e^{-\frac{1}{q}t^q} Q(t)dt \leq \int_0^1 e^{-\frac{1}{q}t^q} \sum_{k=1}^{q} k t^{k-1} dt \leq \sum_{k=1}^{q} k \int_0^1 t^{k-1} dt = \sum_{k=1}^{q} 1 = q. \tag{10}$$

Further, for $t \geq 1$ it follows from (9) that

$$Q(t) \leq \sum_{k=1}^{q} k t^{k-1} \leq t^{q-1} \sum_{k=1}^{q} k = t^{q-1} \frac{q(q+1)}{2}.$$

Using this estimate we get

$$\int_1^\infty e^{-\frac{1}{q}t^q} Q(t)dt \leq \frac{q(q+1)}{2} \int_1^\infty e^{-\frac{1}{q}t^q} t^{q-1} dt = \frac{q(q+1)}{2} e^{-1/q} < \frac{q(q+1)}{2}. \tag{11}$$

Taking into consideration (10) and (11) we may write

$$\int_0^\infty e^{-\frac{1}{q}t^q} Q(t)dt = \int_0^1 e^{-\frac{1}{q}t^q} Q(t)dt + \int_1^\infty e^{-\frac{1}{q}t^q} Q(t)dt \leq q + \frac{q(q+1)}{2} \leq 2q^2,$$

and this inequality proves Lemma 5. □

We consider the auxiliary function

$$F_q(x) = \sum_{k=q}^{\infty} \frac{x^{k-q+1}}{(k!)^{1/q}}, \quad x \geq 0. \tag{12}$$

Lemma 6. *Let $q \in \mathbb{N}$. Then with some constant $c_1 \leq 2$ the following inequality*

$$F_q(x) \leq c_1 q^2 e^{\frac{1}{q}x^q}, \quad x \geq 0, \tag{13}$$

is valid.

Proof. Consider the derivative of the function (12), which equals to

$$F'(x) = \sum_{k=q}^{\infty}(k-q+1)\frac{x^{k-q}}{(k!)^{1/q}} = \sum_{k=0}^{\infty}(k+1)\frac{x^k}{[(k+q)!]^{1/q}}. \tag{14}$$

By introducing the following polynomial

$$Q(x) = \sum_{k=0}^{q-1}(k+1)\frac{x^k}{[(k+q)!]^{1/q}}, \tag{15}$$

and comparing (14) and (15) we get

$$F'(x) - Q(x) = \sum_{k=q}^{\infty}(k+1)\frac{x^k}{[(k+q)!]^{1/q}}.$$

Further we use the following equality

$$\begin{aligned} \sum_{k=q}^{\infty}(k+1)\frac{x^k}{[(k+q)!]^{1/q}} &= x^{q-1}\sum_{k=q}^{\infty}(k+1)\frac{x^{k-q+1}}{[(k+q)!]^{1/q}} \\ &= x^{q-1}\sum_{k=q}^{\infty}B_k(q)\frac{x^{k-q+1}}{(k!)^{1/q}}, \end{aligned} \tag{16}$$

where

$$B_k(q) = \frac{k+1}{[(k+1)(k+2)\cdots(k+q)]^{1/q}}.$$

Hence, according to definition (12) and equality (16),

$$F'(x) - Q(x) = x^{q-1}\sum_{k=q}^{\infty}B_k(q)\frac{x^{k-q+1}}{(k!)^{1/q}}. \tag{17}$$

It is clear, that $B_k(q) \le 1$. Then it follows from equality (17) that

$$F'(x) - Q(x) \le x^{q-1}F(x), \quad x > 0. \tag{18}$$

In as much as

$$e^{\frac{1}{q}x^q}\left[e^{-\frac{1}{q}x^q}F(x)\right]' = F'(x) - x^{q-1}F(x),$$

we get from the inequality (18) that

$$\left[e^{-\frac{1}{q}x^q}F(x)\right]' \le e^{-\frac{1}{q}x^q}Q(x), \quad x > 0.$$

By integrating this inequality and taking into consideration that $F(0) = 0$ we get

$$e^{-\frac{1}{q}x^q}F(x) \le \int_0^x e^{-\frac{1}{q}t^q}Q(t)dt, \quad x > 0.$$

According to Lemma 5

$$\int_0^x e^{-\frac{1}{q}t^q}Q(t)dt \le c_1q^2, \quad x > 0,$$

and consequently

$$F(x) \le c_1q^2e^{\frac{1}{q}x^q}, \quad x > 0.$$

□

Lemma 7. *Let q be a natural number and $P(x)$ be the following polynomial*

$$P_q(x) = \sum_{k=0}^{q-1} \frac{x^k}{(k!)^{1/q}}. \tag{19}$$

Then the estimate

$$P_q(x)e^{-\frac{1}{q}x^q} \leq q, \quad x > 0, \tag{20}$$

is valid.

Proof. It is clear that for any $p > 0$ the maximum of the function

$$f_p(x) = x^p e^{-x}, \ x \geq 0,$$

equals to

$$\max f_p(x) = f_p(p) = p^p e^{-p}.$$

Then

$$\max_{x\geq 0} x^k e^{-\frac{1}{q}x^q} = q^{k/q} \max_{y \geq 0} y^{k/q} e^{-y} = q^{k/q} \left(\frac{k}{q}\right)^{k/q} e^{-k/q} = k^{k/q} e^{-k/q}.$$

Hence,

$$\frac{x^k}{(k!)^{1/q}} e^{-\frac{1}{q}x^q} \leq \frac{k^{k/q}e^{-k/q}}{(k!)^{1/q}}. \tag{21}$$

Taking into account the Stirling formula (2)

$$(k!)^{1/q} = (2\pi k)^{1/2q} k^{k/q} e^{-k/q} \left[1 + \frac{\theta_k}{11k}\right]^{1/q} \geq (2\pi k)^{1/2q} k^{k/q} e^{-k/q},$$

and using estimate (21) we get

$$\frac{x^k}{(k!)^{1/q}} e^{-\frac{1}{q}x^q} \leq \frac{k^{k/q}e^{-k/q}}{(2\pi k)^{1/2q} k^{k/q} e^{-k/q}} = (2\pi k)^{-1/2q} \leq 1.$$

Then according to definition (19)

$$P_q(x)e^{-\frac{1}{q}x^q} = \sum_{k=0}^{q-1} \frac{x^k}{(k!)^{1/q}} e^{-\frac{1}{q}x^q} \leq \sum_{k=0}^{q-1} 1 = q.$$

□

Lemma 8. *Let $\alpha = \frac{1}{q}$ and $q \in \mathbb{N}$. Then with some constant $c_2 < 3$ the following inequality*

$$\Phi\left(x, \frac{1}{q}\right) \leq c_2 q^2 x^{q-1} e^{\frac{1}{q}x^q}, \quad x \geq 1, \tag{22}$$

is valid.

Proof. Obviously,

$$\Phi\left(x, \frac{1}{q}\right) = \sum_{k=0}^{\infty} \frac{x^k}{(k!)^{1/q}} = \sum_{k=0}^{q-1} \frac{x^k}{(k!)^{1/q}} + x^{q-1} \sum_{k=q}^{\infty} \frac{x^{k-q+1}}{(k!)^{1/q}}, \quad x \geq 1.$$

Hence, taking into account definitions (12) and (19), we may write

$$\Phi\left(x, \frac{1}{q}\right) = P(x) + x^{q-1} F_q(x). \tag{23}$$

We may estimate the function in the right hand side of (23) by inequalities (20) and (13):

$$\Phi\left(x,\frac{1}{q}\right) \le qe^{\frac{1}{q}x^q} + x^{q-1}c_1q^2e^{\frac{1}{q}x^q} \le (1+c_1)q^2x^{q-1}e^{\frac{1}{q}x^q}, \quad x \ge 1,$$

where $c_1 \le 2$, according to Lemma 6. □

We proved estimate (22) for integers $q \ge 1$ only. Using this estimate we may prove it for an arbitrary $q \ge 1$ by complex interpolation. For this purpose we introduce the following function

$$f(\zeta) = f(\zeta, b) = b^{\zeta-1}e^{-b\zeta}\sum_{k=0}^{\infty}\frac{b^{k\zeta}}{(k!)^{\zeta}}, \tag{24}$$

where $\zeta = \xi + i\eta$, $\xi > 0$, $-\infty < \eta < \infty$, $b \ge 1$.

Lemma 9. *Let* $0 < \xi \le 1$*. Then with some constant* $c_0 \le 12$ *the inequality*

$$|f(\xi + i\eta)| \le \frac{c_0}{\xi^2}, \qquad 0 < \xi \le 1,\ -\infty < \eta < \infty,\ b > 0, \tag{25}$$

is valid.

Proof. According to definition (24),

$$f(\xi + i\eta) = b^{\xi+i\eta-1}e^{-b(\xi+i\eta)}\sum_{k=0}^{\infty}\frac{b^{k(\xi+i\eta)}}{(k!)^{(\xi+i\eta)}},$$

and hence

$$|f(\xi+i\eta)| \le b^{\xi-1}e^{-b\xi}\sum_{k=0}^{\infty}\frac{b^{k\xi}}{(k!)^{\xi}} = b^{\xi-1}e^{-b\xi}\Phi(b^{\xi},\xi),$$

where the function Φ is defined by equality (1).

Putting $\xi = 1/q$ we get

$$\left|f\left(\frac{1}{q}+i\eta\right)\right| \le b^{(1-q)/q}e^{-b/q}\Phi\left(b^{1/q},\frac{1}{q}\right). \tag{26}$$

According to Lemma 8 for all integers $q \ge 1$ the following inequality

$$\Phi\left(b^{1/q},\frac{1}{q}\right) \le c_2q^2b^{(q-1)/q}e^{b/q}, \quad b \ge 1, \tag{27}$$

is fulfilled. Hence, if $q \in \mathbb{N}$ then it follows from (26) and (27) that

$$\left|f\left(\frac{1}{q}+i\eta\right)\right| \le c_2q^2, \quad -\infty < \eta < \infty, \tag{28}$$

where $c_2 \le 3$.

Let us suppose now that $1/(q+1) < \xi < 1/q$. We may use the Phragmen-Lindelöf theorem (see [3, XII.1.1]) and applying it to (28) we get for some t, $0 < t < 1$, the following estimate

$$|f(\xi+i\eta)| \le c_2(1+q)^{2(1-t)}q^{2t}, \quad \xi = \frac{1-t}{q+1}+\frac{t}{q}, \quad -\infty < \eta < \infty. \tag{29}$$

In as much as $1+q \le 2q$ and $q \le 1/\xi$ we have

$$(1+q)^{2(1-t)}q^{2t} \le 2^{2(1-t)}q^2 \le 4/\xi^2.$$

In that case it follows from the inequality (29) that

$$|f(\xi+i\eta)| \le \frac{4c_2}{\xi^2}.$$

This inequality coincides with required inequality (25). □

Proof of Theorem 3. Follows immediately from Lemma 9 and from definitions (1) and (24):

$$\Phi(x,\alpha) = \sum_{k=0}^{\infty} \frac{x^k}{(k!)^\alpha} = x^{(1-\alpha)/\alpha} e^{\alpha x^{1/\alpha}} f(\alpha, x^{1/\alpha}) \le 4c_0 \alpha^{-2} x^{(1-\alpha)/\alpha} e^{\alpha x^{1/\alpha}},$$

where $c_0 < 3$. Obviously, this inequality is equivalent to (5). □

In closing we prove the inequality (6) (see Remark 4).

Proposition 10. *Let* $q \in \mathbb{N}$. *Then*

$$\Phi\left(x, \frac{1}{q}\right) \ge e^{\frac{1}{q}x^q}, \quad x \ge 0.$$

Proof. Denote

$$g(x) = \Phi\left(x, \frac{1}{q}\right).$$

Obviously,

$$\begin{aligned} g'(x) &= \sum_{k=1}^{\infty} k \frac{x^{k-1}}{(k!)^{1/q}} \ge \sum_{k=q}^{\infty} k \frac{x^{k-1}}{(k!)^{1/q}} \ge \sum_{k=q}^{\infty} \frac{x^{k-1}}{[(k-q)!]^{1/q}} \\ &= x^{q-1} \sum_{k=0}^{\infty} \frac{x^k}{(k!)^{1/q}} = x^{q-1} g(x). \end{aligned}$$

Hence,

$$g'(x) - x^{q-1} g(x) \ge 0, \quad x > 0. \tag{30}$$

In as much as

$$e^{\frac{1}{q}x^q} [e^{-\frac{1}{q}x^q} g(x)]' = g'(x) - x^{q-1} g(x),$$

we get from the inequality (30) that

$$\left[e^{-\frac{1}{q}x^q} g(x)\right]' \ge 0, \quad x > 0.$$

Then since $g(0) = 1$ we have

$$e^{-\frac{1}{q}x^q} g(x) \ge 1.$$

Hence,

$$g(x) \ge e^{\frac{1}{q}x^q}, \quad x > 0.$$

□

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