

SHAVKAT A. ALIMOV AND ONUR ALP ILHAN

National University of Uzbekistan
Department of Mathematical Physics
Tashkent, Uzbekistan
EMail: shavkat_alimov@hotmail.com

EMail: onuralp@sarkor.uz



volume 5, issue 3, article 67,
2004.

*Received 23 March, 2004;
accepted 25 May, 2004.*

Communicated by: N.K. Govil

[Abstract](#)

[Contents](#)



[Home Page](#)

[Go Back](#)

[Close](#)

[Quit](#)

The object of our paper is to determine the order of growth to infinity of some family of entire functions. For an arbitrary $\alpha > 0$ we introduce the following function

$$(1) \quad \Phi(z, \alpha) = \sum_{k=0}^{\infty} \frac{z^k}{(k!)^\alpha}, \quad \alpha > 0, \quad z \in \mathbb{C}.$$

Note that

$$\Phi(z, 1) = e^z.$$

It is easy to show that if $\alpha > 0$ then the function $\Phi(z, \alpha)$ is defined by series (1) for all z in the complex plane \mathbb{C} .

Proposition 1. *The radius of convergence of the series (1) is equal to infinity.*

Proof. According to the Cauchy formula (see, e.g., [2, 2.6]) the radius of convergence of the series

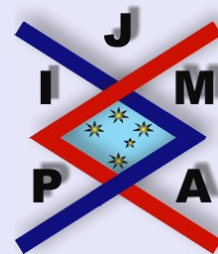
$$\sum_{n=0}^{\infty} c_n z^n$$

is equal to

$$R = \frac{1}{\lim_{n \rightarrow \infty} \sqrt[n]{|c_n|}}.$$

In our case $c_n = (n!)^{-\alpha}$. We may use the Stirling formula (see [2, 12.33]) in the following form

$$(2) \quad n! = \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \left(1 + \frac{\theta_n}{11n}\right), \quad 0 < \theta_n < 1, \quad n = 1, 2, \dots$$



An Inequality Associated with Some Entire Functions

Shavkat A. Alimov and
Onur Alp Ilhan

Title Page

Contents



Go Back

Close

Quit

Page 2 of 15

As a result we get

$$\begin{aligned} \frac{1}{\sqrt[n]{|c_n|}} &= (n!)^{\alpha/n} \\ &= \left[\sqrt{2\pi n} \left(\frac{n}{e}\right)^n \left(1 + \frac{\theta_n}{11n}\right) \right]^{\alpha/n} \\ &= \left(\frac{n}{e}\right)^\alpha (2\pi)^{\alpha/2n} e^{\alpha(\ln n)/2n} \left(1 + \frac{\theta_n}{11n}\right)^{\alpha/n} \\ &= \left(\frac{n}{e}\right)^\alpha (1 + \varepsilon_n) \rightarrow \infty, \quad n \rightarrow \infty, \end{aligned}$$

where $\varepsilon_n = o(1)$, $n \rightarrow \infty$. □

Corollary 2. *The function $\Phi(z, \alpha)$, $\alpha > 0$, is entire function of z .*

The function $\Phi(z, \alpha)$ with $\alpha = \frac{1}{q}$ arises in estimates of the solutions of some Volterra type integral equations with kernel from L_p , where $\frac{1}{p} + \frac{1}{q} = 1$. We mention also the equation with convolution on the circle which these functions satisfy. For two arbitrary 2π -periodical functions $f(\theta)$ and $g(\theta)$ introduce their convolution

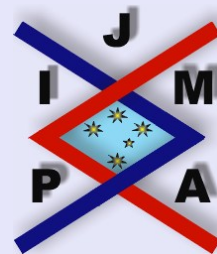
$$(f * g)(\theta) = \frac{1}{2\pi} \int_0^{2\pi} f(\theta - \varphi)g(\varphi)d\varphi.$$

If we denote

$$(3) \quad f_\alpha(\theta) = \Phi(e^{i\theta}, \alpha),$$

then it is easy to check that this function satisfies the following equation

$$(4) \quad (f_\alpha * f_\beta)(\theta) = f_{\alpha+\beta}(\theta), \quad f_1(\theta) = \exp e^{i\theta}.$$



An Inequality Associated with
Some Entire Functions

Shavkat A. Alimov and
Onur Alp Ilhan

Title Page

Contents



Go Back

Close

Quit

Page 3 of 15

It easy to show that every solution of equation (4) has the form (3).

It is well known that for $\Phi(z, \alpha)$ the following formula

$$\ln \Phi(x, \alpha) = \alpha x^{1/\alpha} + o(x^{1/\alpha}), \quad x \rightarrow +\infty$$

is valid (see, e.g. [1, 4.1, Th. 68]). However, in some applications, an explicit estimate for the error of the above asymptotic approximation is desirable.

We are going to prove the following inequality.

Theorem 3. *Let $0 < \alpha \leq 1$. Then for all $x \geq 1$ the inequality*

$$(5) \quad \ln \Phi(x, \alpha) \leq \alpha x^{1/\alpha} + \frac{1 - \alpha}{\alpha} \ln x + \ln(12\alpha^{-2})$$

is valid.

Remark 1. *The order in estimate (5) is precise, at least when $\alpha = 1/q$, where q is natural, because in this case for all $x \geq 1$ the inequality*

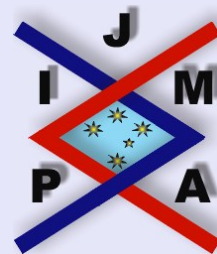
$$(6) \quad \ln \Phi(x, \alpha) \geq \alpha x^{1/\alpha}$$

is true. As it easy to verify, for $\alpha = 1$ the inequality (6) becomes equality.

At first we prove the inequality (5) for $\alpha = \frac{1}{q}$, where q is natural, and after that we use the interpolation technique to prove it for all α , $0 < \alpha \leq 1$.

Lemma 4. *Let q be a natural number and $Q(x)$ be the following polynomial*

$$(7) \quad Q(x) = \sum_{k=0}^{q-1} (k+1) \frac{x^k}{[(k+q)!]^{1/q}}.$$



An Inequality Associated with Some Entire Functions

Shavkat A. Alimov and
Onur Alp Ilhan

Title Page

Contents



Go Back

Close

Quit

Page 4 of 15

Then there exists a constant $c_1 \leq 2$ so that

$$(8) \quad \int_0^\infty e^{-\frac{1}{q}t^q} Q(t) dt \leq c_1 q^2.$$

Proof. It follows from (7) that the inequality

$$(9) \quad Q(t) = \sum_{k=1}^q k \frac{t^{k-1}}{[(k+q-1)!]^{1/q}} \leq \sum_{k=1}^q kt^{k-1}$$

is valid for all $t > 0$. Then

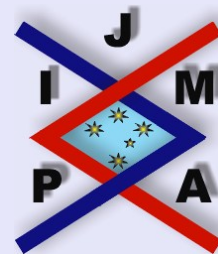
$$(10) \quad \int_0^1 e^{-\frac{1}{q}t^q} Q(t) dt \leq \int_0^1 e^{-\frac{1}{q}t^q} \sum_{k=1}^q kt^{k-1} dt \\ \leq \sum_{k=1}^q k \int_0^1 t^{k-1} dt = \sum_{k=1}^q 1 = q.$$

Further, for $t \geq 1$ it follows from (9) that

$$Q(t) \leq \sum_{k=1}^q kt^{k-1} \leq t^{q-1} \sum_{k=1}^q k = t^{q-1} \frac{q(q+1)}{2}.$$

Using this estimate we get

$$(11) \quad \int_1^\infty e^{-\frac{1}{q}t^q} Q(t) dt \leq \frac{q(q+1)}{2} \int_1^\infty e^{-\frac{1}{q}t^q} t^{q-1} dt \\ = \frac{q(q+1)}{2} e^{-1/q} < \frac{q(q+1)}{2}.$$



An Inequality Associated with Some Entire Functions

Shavkat A. Alimov and
Onur Alp Ilhan

Title Page

Contents



Go Back

Close

Quit

Page 5 of 15

Taking into consideration (10) and (11) we may write

$$\int_0^\infty e^{-\frac{1}{q}t^q} Q(t) dt = \int_0^1 e^{-\frac{1}{q}t^q} Q(t) dt + \int_1^\infty e^{-\frac{1}{q}t^q} Q(t) dt$$

$$\leq q + \frac{q(q+1)}{2} \leq 2q^2,$$

and this inequality proves Lemma 4. □

We consider the auxiliary function

$$(12) \quad F_q(x) = \sum_{k=q}^{\infty} \frac{x^{k-q+1}}{(k!)^{1/q}}, \quad x \geq 0.$$

Lemma 5. *Let $q \in \mathbb{N}$. Then with some constant $c_1 \leq 2$ the following inequality*

$$(13) \quad F_q(x) \leq c_1 q^2 e^{\frac{1}{q}x^q}, \quad x \geq 0,$$

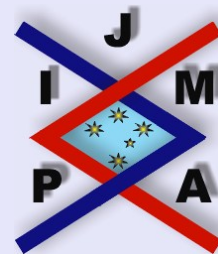
is valid.

Proof. Consider the derivative of the function (12), which equals to

$$(14) \quad F'_q(x) = \sum_{k=q}^{\infty} (k-q+1) \frac{x^{k-q}}{(k!)^{1/q}} = \sum_{k=0}^{\infty} (k+1) \frac{x^k}{[(k+q)!]^{1/q}}.$$

By introducing the following polynomial

$$(15) \quad Q(x) = \sum_{k=0}^{q-1} (k+1) \frac{x^k}{[(k+q)!]^{1/q}},$$



An Inequality Associated with Some Entire Functions

Shavkat A. Alimov and
Onur Alp Ilhan

Title Page

Contents



Go Back

Close

Quit

Page 6 of 15

and comparing (14) and (15) we get

$$F'(x) - Q(x) = \sum_{k=q}^{\infty} (k+1) \frac{x^k}{[(k+q)!]^{1/q}}.$$

Further we use the following equality

$$\begin{aligned} (16) \quad \sum_{k=q}^{\infty} (k+1) \frac{x^k}{[(k+q)!]^{1/q}} &= x^{q-1} \sum_{k=q}^{\infty} (k+1) \frac{x^{k-q+1}}{[(k+q)!]^{1/q}} \\ &= x^{q-1} \sum_{k=q}^{\infty} B_k(q) \frac{x^{k-q+1}}{(k!)^{1/q}}, \end{aligned}$$

where

$$B_k(q) = \frac{k+1}{[(k+1)(k+2)\cdots(k+q)]^{1/q}}.$$

Hence, according to definition (12) and equality (16),

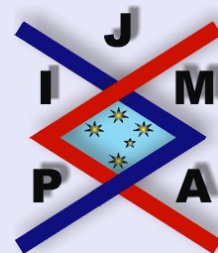
$$(17) \quad F'(x) - Q(x) = x^{q-1} \sum_{k=q}^{\infty} B_k(q) \frac{x^{k-q+1}}{(k!)^{1/q}}.$$

It is clear, that $B_k(q) \leq 1$. Then it follows from equality (17) that

$$(18) \quad F'(x) - Q(x) \leq x^{q-1} F(x), \quad x > 0.$$

In as much as

$$e^{\frac{1}{q}x^q} \left[e^{-\frac{1}{q}x^q} F(x) \right]' = F'(x) - x^{q-1} F(x),$$



**An Inequality Associated with
Some Entire Functions**

Shavkat A. Alimov and
Onur Alp Ilhan

Title Page

Contents



Go Back

Close

Quit

Page 7 of 15

we get from the inequality (18) that

$$\left[e^{-\frac{1}{q}x^q} F(x) \right]' \leq e^{-\frac{1}{q}x^q} Q(x), \quad x > 0.$$

By integrating this inequality and taking into consideration that $F(0) = 0$ we get

$$e^{-\frac{1}{q}x^q} F(x) \leq \int_0^x e^{-\frac{1}{q}t^q} Q(t) dt, \quad x > 0.$$

According to Lemma 4

$$\int_0^x e^{-\frac{1}{q}t^q} Q(t) dt \leq c_1 q^2, \quad x > 0,$$

and consequently

$$F(x) \leq c_1 q^2 e^{\frac{1}{q}x^q}, \quad x > 0.$$

□

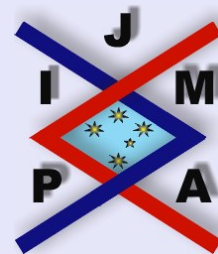
Lemma 6. Let q be a natural number and $P(x)$ be the following polynomial

$$(19) \quad P_q(x) = \sum_{k=0}^{q-1} \frac{x^k}{(k!)^{1/q}}.$$

Then the estimate

$$(20) \quad P_q(x) e^{-\frac{1}{q}x^q} \leq q, \quad x > 0,$$

is valid.



An Inequality Associated with Some Entire Functions

Shavkat A. Alimov and
Onur Alp Ilhan

Title Page

Contents



Go Back

Close

Quit

Page 8 of 15

Proof. It is clear that for any $p > 0$ the maximum of the function

$$f_p(x) = x^p e^{-x}, \quad x \geq 0,$$

equals to

$$\max f_p(x) = f_p(p) = p^p e^{-p}.$$

Then

$$\max_{x \geq 0} x^k e^{-\frac{1}{q}x^q} = q^{k/q} \max_{y \geq 0} y^{k/q} e^{-y} = q^{k/q} \left(\frac{k}{q}\right)^{k/q} e^{-k/q} = k^{k/q} e^{-k/q}.$$

Hence,

$$(21) \quad \frac{x^k}{(k!)^{1/q}} e^{-\frac{1}{q}x^q} \leq \frac{k^{k/q} e^{-k/q}}{(k!)^{1/q}}.$$

Taking into account the Stirling formula (2)

$$(k!)^{1/q} = (2\pi k)^{1/2q} k^{k/q} e^{-k/q} \left[1 + \frac{\theta_k}{11k}\right]^{1/q} \geq (2\pi k)^{1/2q} k^{k/q} e^{-k/q},$$

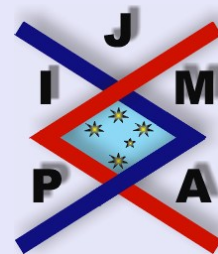
and using estimate (21) we get

$$\frac{x^k}{(k!)^{1/q}} e^{-\frac{1}{q}x^q} \leq \frac{k^{k/q} e^{-k/q}}{(2\pi k)^{1/2q} k^{k/q} e^{-k/q}} = (2\pi k)^{-1/2q} \leq 1.$$

Then according to definition (19)

$$P_q(x) e^{-\frac{1}{q}x^q} = \sum_{k=0}^{q-1} \frac{x^k}{(k!)^{1/q}} e^{-\frac{1}{q}x^q} \leq \sum_{k=0}^{q-1} 1 = q.$$

□



An Inequality Associated with
Some Entire Functions

Shavkat A. Alimov and
Onur Alp Ilhan

Title Page

Contents



Go Back

Close

Quit

Page 9 of 15

Lemma 7. Let $\alpha = \frac{1}{q}$ and $q \in \mathbb{N}$. Then with some constant $c_2 < 3$ the following inequality

$$(22) \quad \Phi\left(x, \frac{1}{q}\right) \leq c_2 q^2 x^{q-1} e^{\frac{1}{q}x^q}, \quad x \geq 1,$$

is valid.

Proof. Obviously,

$$\Phi\left(x, \frac{1}{q}\right) = \sum_{k=0}^{\infty} \frac{x^k}{(k!)^{1/q}} = \sum_{k=0}^{q-1} \frac{x^k}{(k!)^{1/q}} + x^{q-1} \sum_{k=q}^{\infty} \frac{x^{k-q+1}}{(k!)^{1/q}}, \quad x \geq 1.$$

Hence, taking into account definitions (12) and (19), we may write

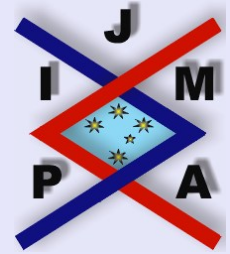
$$(23) \quad \Phi\left(x, \frac{1}{q}\right) = P(x) + x^{q-1} F_q(x).$$

We may estimate the function in the right hand side of (23) by inequalities (20) and (13):

$$\Phi\left(x, \frac{1}{q}\right) \leq q e^{\frac{1}{q}x^q} + x^{q-1} c_1 q^2 e^{\frac{1}{q}x^q} \leq (1 + c_1) q^2 x^{q-1} e^{\frac{1}{q}x^q}, \quad x \geq 1,$$

where $c_1 \leq 2$, according to Lemma 5. □

We proved estimate (22) for integers $q \geq 1$ only. Using this estimate we may prove it for an arbitrary $q \geq 1$ by complex interpolation. For this purpose we



An Inequality Associated with Some Entire Functions

Shavkat A. Alimov and
Onur Alp Ilhan

Title Page

Contents



Go Back

Close

Quit

Page 10 of 15

introduce the following function

$$(24) \quad f(\zeta) = f(\zeta, b) = b^{\zeta-1} e^{-b\zeta} \sum_{k=0}^{\infty} \frac{b^{k\zeta}}{(k!)^{\zeta}},$$

where $\zeta = \xi + i\eta$, $\xi > 0$, $-\infty < \eta < \infty$, $b \geq 1$.

Lemma 8. Let $0 < \xi \leq 1$. Then with some constant $c_0 \leq 12$ the inequality

$$(25) \quad |f(\xi + i\eta)| \leq \frac{c_0}{\xi^2}, \quad 0 < \xi \leq 1, \quad -\infty < \eta < \infty, \quad b > 0,$$

is valid.

Proof. According to definition (24),

$$f(\xi + i\eta) = b^{\xi+i\eta-1} e^{-b(\xi+i\eta)} \sum_{k=0}^{\infty} \frac{b^{k(\xi+i\eta)}}{(k!)^{(\xi+i\eta)}},$$

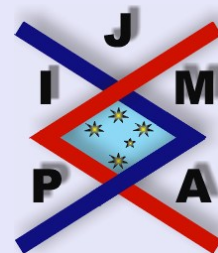
and hence

$$|f(\xi + i\eta)| \leq b^{\xi-1} e^{-b\xi} \sum_{k=0}^{\infty} \frac{b^{k\xi}}{(k!)^{\xi}} = b^{\xi-1} e^{-b\xi} \Phi(b^{\xi}, \xi),$$

where the function Φ is defined by equality (1).

Putting $\xi = 1/q$ we get

$$(26) \quad \left| f\left(\frac{1}{q} + i\eta\right) \right| \leq b^{(1-q)/q} e^{-b/q} \Phi\left(b^{1/q}, \frac{1}{q}\right).$$



An Inequality Associated with
Some Entire Functions

Shavkat A. Alimov and
Onur Alp Ilhan

Title Page

Contents



Go Back

Close

Quit

Page 11 of 15

According to Lemma 7 for all integers $q \geq 1$ the following inequality

$$(27) \quad \Phi \left(b^{1/q}, \frac{1}{q} \right) \leq c_2 q^2 b^{(q-1)/q} e^{b/q}, \quad b \geq 1,$$

is fulfilled. Hence, if $q \in \mathbb{N}$ then it follows from (26) and (27) that

$$(28) \quad \left| f \left(\frac{1}{q} + i\eta \right) \right| \leq c_2 q^2, \quad -\infty < \eta < \infty,$$

where $c_2 \leq 3$.

Let us suppose now that $1/(q+1) < \xi < 1/q$. We may use the Phragmen-Lindelöf theorem (see [3, XII.1.1]) and applying it to (28) we get for some t , $0 < t < 1$, the following estimate

$$(29) \quad |f(\xi + i\eta)| \leq c_2 (1+q)^{2(1-t)} q^{2t}, \quad \xi = \frac{1-t}{q+1} + \frac{t}{q}, \quad -\infty < \eta < \infty.$$

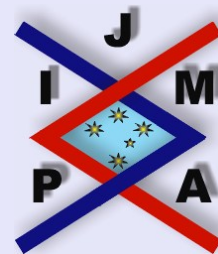
In as much as $1+q \leq 2q$ and $q \leq 1/\xi$ we have

$$(1+q)^{2(1-t)} q^{2t} \leq 2^{2(1-t)} q^2 \leq 4/\xi^2.$$

In that case it follows from the inequality (29) that

$$|f(\xi + i\eta)| \leq \frac{4c_2}{\xi^2}.$$

This inequality coincides with required inequality (25). □



**An Inequality Associated with
Some Entire Functions**

Shavkat A. Alimov and
Onur Alp Ilhan

Title Page

Contents



Go Back

Close

Quit

Page 12 of 15

Proof of Theorem 3. Follows immediately from Lemma 8 and from definitions (1) and (24):

$$\Phi(x, \alpha) = \sum_{k=0}^{\infty} \frac{x^k}{(k!)^\alpha} = x^{(1-\alpha)/\alpha} e^{\alpha x^{1/\alpha}} f(\alpha, x^{1/\alpha}) \leq 4c_0 \alpha^{-2} x^{(1-\alpha)/\alpha} e^{\alpha x^{1/\alpha}},$$

where $c_0 < 3$. Obviously, this inequality is equivalent to (5). □

In closing we prove the inequality (6) (see Remark 1).

Proposition 9. *Let $q \in \mathbb{N}$. Then*

$$\Phi\left(x, \frac{1}{q}\right) \geq e^{\frac{1}{q}x^q}, \quad x \geq 0.$$

Proof. Denote

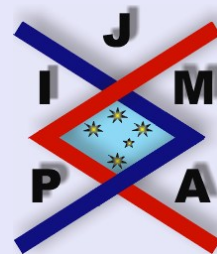
$$g(x) = \Phi\left(x, \frac{1}{q}\right).$$

Obviously,

$$\begin{aligned} g'(x) &= \sum_{k=1}^{\infty} k \frac{x^{k-1}}{(k!)^{1/q}} \geq \sum_{k=q}^{\infty} k \frac{x^{k-1}}{(k!)^{1/q}} \geq \sum_{k=q}^{\infty} \frac{x^{k-1}}{[(k-q)!]^{1/q}} \\ &= x^{q-1} \sum_{k=0}^{\infty} \frac{x^k}{(k!)^{1/q}} = x^{q-1} g(x). \end{aligned}$$

Hence,

$$(30) \quad g'(x) - x^{q-1}g(x) \geq 0, \quad x > 0.$$



An Inequality Associated with
Some Entire Functions

Shavkat A. Alimov and
Onur Alp Ilhan

Title Page

Contents



Go Back

Close

Quit

Page 13 of 15

In as much as

$$e^{\frac{1}{q}x^q} [e^{-\frac{1}{q}x^q} g(x)]' = g'(x) - x^{q-1}g(x),$$

we get from the inequality (30) that

$$\left[e^{-\frac{1}{q}x^q} g(x) \right]' \geq 0, \quad x > 0.$$

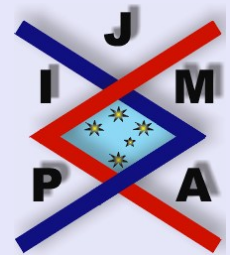
Then since $g(0) = 1$ we have

$$e^{-\frac{1}{q}x^q} g(x) \geq 1.$$

Hence,

$$g(x) \geq e^{\frac{1}{q}x^q}, \quad x > 0.$$

□



**An Inequality Associated with
Some Entire Functions**

Shavkat A. Alimov and
Onur Alp Ilhan

Title Page

Contents



Go Back

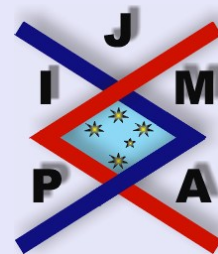
Close

Quit

Page 14 of 15

References

- [1] G. POLYA AND G. SZEGO, *Aufgaben und Lehrsätze aus der Analysis*, 2 Band, Springer-Verlag, 1964.
- [2] E.T. WHITTAKER AND G.N. WATSON, *A Course of Modern Analysis*, Fourth Edition, Cambridge University Press, 1927.
- [3] A. ZYGMUND, *Trigonometric Series*, vol.2, Cambridge University Press, 1959.



An Inequality Associated with Some Entire Functions

Shavkat A. Alimov and
Onur Alp Ilhan

Title Page

Contents



Go Back

Close

Quit

Page 15 of 15