

S-GEOMETRIC CONVEXITY OF A FUNCTION INVOLVING MACLAURIN'S ELEMENTARY SYMMETRIC MEAN

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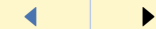
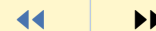
Abstract: Let $x_i > 0, i = 1, 2, \dots, n, x = (x_1, x_2, \dots, x_n)$, the k th elementary symmetric function of x is defined as $E_n(x, k) = \sum_{1 \leq i_1 < \dots < i_k \leq n} \prod_{j=1}^k x_{i_j}$ with $1 \leq k \leq n$, the k th elementary symmetric mean is defined as $P_n(x, k) = \left(\binom{n-1}{k} E_n(x, k) \right)^{\frac{1}{k}}$, and the function f is defined as $f(x) = P_n(x, k-1) - P_n(x, k)$. The paper proves that f is a S-geometrically convex function. The result generalizes the well-known Maclaurin-Inequality.



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1. Introduction

Throughout the paper we assume \mathbb{R}^n be the n -dimensional Euclidean Space,

$$\mathbb{R}_+^n = \{(x_1, x_2, \dots, x_n), x_i > 0, i = 1, 2, \dots, n\},$$

and

$$e^x = (e^{x_1}, e^{x_2}, \dots, e^{x_n}), \quad x^c = (x_1^c, x_2^c, \dots, x_n^c),$$

$$\ln x = (\ln x_1, \ln x_2, \dots, \ln x_n), \quad x \cdot y = (x_1 y_1, x_2 y_2, \dots, x_n y_n),$$

where $c \in \mathbb{R}$, and $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, $y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$. And if $n \geq 2$, $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}_+^n$. They are defined respectively by

$$A_n(x) = \frac{x_1 + x_2 + \dots + x_n}{n}, \quad G_n(a) = \sqrt[n]{x_1 x_2 \dots x_n}.$$

The k th elementary symmetric function of x , k th elementary symmetric mean, and function f are defined respectively as

$$E_n(x, k) = \sum_{1 \leq i_1 < \dots < i_k \leq n} \prod_{j=1}^k x_{i_j}, \quad (1 \leq k \leq n)$$

$$P_n(x, k) = \left(\binom{n}{k}^{-1} E_n(x, k) \right)^{\frac{1}{k}}, \quad (1 \leq k \leq n)$$

$$f(x) = P_n(x, k-1) - P_n(x, k), \quad (2 \leq k \leq n)$$

with $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

The following theorem is true by [1].



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Theorem 1.1 (Maclaurin-Inequality).

$$(1.1) \quad \begin{aligned} A_n(x) &= P_n(x, 1) \\ &\geq P_n(x, 2) \geq \cdots \geq P_n(x, n-1) \geq P_n(x, n) = G_n(x). \end{aligned}$$

$$(1.2) \quad \begin{aligned} \frac{E_n(x, n)}{E_n(x, n-1)} &< \frac{E_n(x, n-1)}{E_n(x, n-2)} \\ &< \cdots < \frac{E_n(x, 3)}{E_n(x, 2)} < \frac{E_n(x, 2)}{E_n(x, 1)} < E_n(x, 1). \end{aligned}$$

References [5], [4], [2], [7], [8], [9], [10] and [6] give the definitions of n dimensional geometrically convex functions, S-geometrically convex functions and logarithm majorization, and a large number of results have been obtained. Since many functions have geometric convexity or geometric concavity, research into geometrically convex functions make sense. For a comprehensive list of recent results on geometrically convex functions, see the book [10] and the papers [5], [4], [2] [7] [8], [9] and [6] where further results are given.

The main aim of this paper is to prove the following theorem.

Theorem 1.2. *Let $n = 2$, or $n \geq 3$, $2 \leq k-1 \leq n-1$, and $f(x) = P_n(x, k-1) - P_n(x, k)$, then f is a S-geometrically convex function.*

The result generalizes the Maclaurin-Inequality.

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2. Relative Definition and a Lemma

Lemma 2.1 ([3]). Let $H \subseteq \mathbb{R}^n$ be a symmetric convex set with a nonempty interior, $\phi : H \rightarrow \mathbb{R}$ be continuously differentiable on the interior of H and continuous on H . Necessary and sufficient conditions for ϕ to be S -convex(concave) on H are that ϕ is symmetric on H , and

$$(x_1 - x_2) \left(\frac{\partial \phi}{\partial x_1} - \frac{\partial \phi}{\partial x_2} \right) \geq (\leq) 0,$$

for all x in the interior of H .

Definition 2.1 ([9], [10, p. 89], [6]). Let $x \in \mathbb{R}_+^n$, $y \in \mathbb{R}_+^n$, $(x_{[1]}, x_{[2]}, \dots, x_{[n]})$ and $(y_{[1]}, y_{[2]}, \dots, y_{[n]})$ be the decreasing queues of (x_1, x_2, \dots, x_n) and (y_1, y_2, \dots, y_n) respectively. We say (x_1, x_2, \dots, x_n) logarithm majorizes (y_1, y_2, \dots, y_n) , denote $\ln x \succ \ln y$ if

$$\begin{cases} \prod_{i=1}^k x_i \geq \prod_{i=1}^k y_i, & k = 1, 2, \dots, n-1, \\ \prod_{i=1}^n x_i = \prod_{i=1}^n y_i. \end{cases}$$

Remark 1. If x logarithm majorizes y , then

$$\begin{cases} \ln \left(\prod_{i=1}^k x_i \right) \geq \ln \left(\prod_{i=1}^k y_i \right), & k = 1, 2, \dots, n-1, \\ \ln \left(\prod_{i=1}^n x_i \right) = \ln \left(\prod_{i=1}^n y_i \right). \end{cases}$$

$$\begin{cases} \sum_{i=1}^k \ln x_i \geq \sum_{i=1}^k \ln y_i, & k = 1, 2, \dots, n-1, \\ \sum_{i=1}^n \ln x_i = \sum_{i=1}^n \ln y_i. \end{cases}$$

So we can denote $\ln x \succ \ln y$ in the Definition 2.1.

Lemma 2.2 ([10, p. 97]). $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}_+^n$ logarithm majorizes

$$\bar{G}(x) = (G_n(x), G_n(x), \dots, G_n(x)).$$

Definition 2.2 ([7]). Let $E \subseteq \mathbb{R}_+^n$, then E is said to be a logarithm convex set, if for any $x, y \in E$, $\alpha, \beta > 0$, $\alpha + \beta = 1$, it have $x^\alpha y^\beta \in E$.

Remark 2. Let $E \subseteq \mathbb{R}_+^n$, $\ln E = \{\ln x | x \in E\}$. Then $x, y \in E$ if only if $\ln x, \ln y \in \ln E$, and $x^\alpha y^\beta \in E$ if only if $\alpha \ln x + \beta \ln y \in \ln E$, so E is a logarithm convex set if and only if $\ln E$ is a convex set.

Definition 2.3 ([10, p. 107]). Let $E \subseteq \mathbb{R}_+^n$, $f : E \rightarrow [0, +\infty)$. Then f is called an S -geometrically convex function, if for any $x, y \in E \subseteq \mathbb{R}_+^n$, $\ln x \succ \ln y$, we have

$$(2.1) \quad f(x) \geq f(y).$$

And f is called an S -geometrically concave function, if the inequality (2.1) is reversed.

Lemma 2.3 ([10, p. 108]). Let $E \subseteq \mathbb{R}_+^n$ be a symmetric logarithm convex set with a nonempty interior, $f : E \rightarrow [0, +\infty)$ be symmetric continuously differentiable on the interior of E and continuous on E . Then f is a S -geometrically convex function, if the following inequality

$$(2.2) \quad (\ln x_1 - \ln x_2) \left(x_1 \frac{\partial f}{\partial x_1} - x_2 \frac{\partial f}{\partial x_2} \right) \geq 0$$



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holds for all x in the interior of E . And f is S -geometrically concave function, if the inequality (2.2) is reversed.

Proof. Let $\ln E = \{\ln x | x \in E\}$, then $\ln E$ is a symmetric convex set and has a nonempty interior. Again, let $\varepsilon > 0$, $g(x) = f(x) + \varepsilon$ with $x \in E$, and $h(y) = \ln g(e^y)$ with $y \in \ln E = \{\ln x | x \in E\}$, then $g : E \rightarrow (0, +\infty)$, $h : \ln E \rightarrow (-\infty, +\infty)$. Further let $x = e^y$,

$$\begin{aligned} \frac{\partial h}{\partial y_1} &= \frac{\partial (\ln g(e^y))}{\partial y_1} \\ &= \frac{1}{g(e^y)} \cdot \frac{\partial (g(e^y))}{\partial y_1} \\ &= \frac{1}{g(x)} \cdot \frac{\partial (g(x))}{\partial x_1} \cdot e^{y_1} \\ &= \frac{x_1}{g(x)} \cdot \frac{\partial g}{\partial x_1}. \end{aligned}$$

Similarly,

$$\frac{\partial h}{\partial y_2} = \frac{x_2}{g(x)} \cdot \frac{\partial g}{\partial x_2}.$$

According to inequality 2.2,

$$\begin{aligned} (y_1 - y_2) \left(\frac{\partial h}{\partial y_1} - \frac{\partial h}{\partial y_2} \right) &= \frac{(\ln x_1 - \ln x_2)}{g(x)} \left(x_1 \frac{\partial g}{\partial x_1} - x_2 \frac{\partial g}{\partial x_2} \right) \\ &= \frac{(\ln x_1 - \ln x_2)}{g(x)} \left(x_1 \frac{\partial f}{\partial x_1} - x_2 \frac{\partial f}{\partial x_2} \right) \geq 0. \end{aligned}$$

Then by Lemma 2.1, we know that h is a S -convex function. For any $u, v \in E$ with



$\ln u \succ \ln v$, we have

$$h(\ln u) \geq h(\ln v), \quad \ln g(e^{\ln u}) \geq \ln g(e^{\ln v}),$$

and

$$g(u) \geq g(v), \quad f(u) \geq f(v).$$

So f is a S -geometrically convex function.

If the inequality (2.2) is reversed, we similarly have that f is a S -geometrically concave function.

The proof of Lemma 2.3 is completed. □

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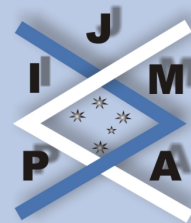
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3. The Proof of Theorem 1.2

Proof of Theorem 1.2. If $n = 2$, then $k = 2$.

$$f(x) = P_n(x, k-1) - P_n(x, k) = \frac{x_1 + x_2}{2} - \sqrt{x_1 x_2},$$

$$\frac{\partial f}{\partial x_1} = \frac{1}{2} - \frac{1}{2} \sqrt{\frac{x_2}{x_1}}, \quad x_1 \frac{\partial f}{\partial x_1} = \frac{1}{2} x_1 - \frac{1}{2} \sqrt{x_1 x_2},$$

$$(\ln x_1 - \ln x_2) \left(x_1 \frac{\partial f}{\partial x_1} - x_2 \frac{\partial f}{\partial x_2} \right) = (\ln x_1 - \ln x_2) \left(\frac{x_1 - x_2}{2} \right) \geq 0.$$

According to Lemma 2.3, if $n = 2$, Theorem 1.2 is true.

If $n \geq 3$, $k \geq 3$, Letting $\bar{x} = (x_3, x_4, \dots, x_n)$, $E_{n-2}(\bar{x}, 0) = 1$, we have

$$\begin{aligned} f(x) &= P_n(x, k-1) - P_n(x, k) \\ &= \left(\binom{n}{k-1}^{-1} E_n(x, k-1) \right)^{\frac{1}{k-1}} - \left(\binom{n}{k}^{-1} E_n(x, k) \right)^{\frac{1}{k}}, \end{aligned}$$

$$\begin{aligned} \frac{\partial f}{\partial x_1} &= \frac{1}{k-1} \cdot \binom{n}{k-1}^{-\frac{1}{k-1}} \cdot (E_n(x, k-1))^{\frac{1}{k-1}-1} \sum_{2 \leq i_1 < \dots < i_j \leq n} \prod_{j=1}^{k-2} x_{i_j} \\ &\quad - \frac{1}{k} \cdot \binom{n}{k}^{-\frac{1}{k}} \cdot (E_n(x, k))^{\frac{1}{k}-1} \sum_{2 \leq i_1 < \dots < i_j \leq n} \prod_{j=1}^{k-1} x_{i_j}, \end{aligned}$$



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$$x_1 \frac{\partial f}{\partial x_1} = \frac{1}{k-1} \cdot \binom{n}{k-1}^{-\frac{1}{k-1}} \cdot (E_n(x, k-1))^{\frac{1}{k-1}-1} [x_1 E_{n-2}(\bar{x}, k-2) + x_1 x_2 E_{n-2}(\bar{x}, k-3)] - \frac{1}{k} \cdot \binom{n}{k}^{-\frac{1}{k}} \cdot (E_n(x, k))^{\frac{1}{k}-1} [x_1 E_{n-2}(\bar{x}, k-1) + x_1 x_2 E_{n-2}(\bar{x}, k-2)],$$

and

$$x_2 \frac{\partial f}{\partial x_2} = \frac{1}{k-1} \cdot \binom{n}{k-1}^{-\frac{1}{k-1}} \cdot (E_n(x, k-1))^{\frac{1}{k-1}-1} [x_2 E_{n-2}(\bar{x}, k-2) + x_1 x_2 E_{n-2}(\bar{x}, k-3)] - \frac{1}{k} \cdot \binom{n}{k}^{-\frac{1}{k}} \cdot (E_n(x, k))^{\frac{1}{k}-1} [x_2 E_{n-2}(\bar{x}, k-1) + x_1 x_2 E_{n-2}(\bar{x}, k-2)].$$

So

$$(3.1) \quad (\ln x_1 - \ln x_2) \left(x_1 \frac{\partial f}{\partial x_1} - x_2 \frac{\partial f}{\partial x_2} \right) = (\ln x_1 - \ln x_2) \cdot \frac{1}{k-1} \cdot \binom{n}{k-1}^{-\frac{1}{k-1}} \cdot (E_n(x, k-1))^{\frac{1}{k-1}-1} \cdot (x_1 - x_2) \cdot E_{n-2}(\bar{x}, k-2) - (\ln x_1 - \ln x_2) \cdot \frac{1}{k} \cdot \binom{n}{k}^{-\frac{1}{k}} \cdot (E_n(x, k))^{\frac{1}{k}-1} \cdot (x_1 - x_2) \cdot E_{n-2}(\bar{x}, k-1).$$

On the other hand, by (1.2), we deduce

$$(x_1 + x_2) E_{n-2}(\bar{x}, k-1) \cdot E_{n-2}(\bar{x}, k-2) + x_1 x_2 [k E_{n-2}^2(\bar{x}, k-2) - (k-1) E_{n-2}(\bar{x}, k-3) \cdot E_{n-2}(\bar{x}, k-1)] \geq 0,$$



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so

$$\begin{aligned}
 & k \cdot [(x_1 + x_2) E_{n-2}(\bar{x}, k-1) + x_1 x_2 E_{n-2}(\bar{x}, k-2)] \cdot E_{n-2}(\bar{x}, k-2) \\
 & - (k-1) \cdot [(x_1 + x_2) E_{n-2}(\bar{x}, k-2) + x_1 x_2 E_{n-2}(\bar{x}, k-3)] \cdot E_{n-2}(\bar{x}, k-1) \geq 0, \\
 & k \cdot E_n(x, k) \cdot E_{n-2}(\bar{x}, k-2) - (k-1) \cdot E_n(x, k-1) \cdot E_{n-2}(\bar{x}, k-1) \geq 0.
 \end{aligned}$$

Again, according to (1.1), the following inequality holds.

$$\begin{aligned}
 & k \cdot P_n(x, k-1) \cdot E_n(x, k) \cdot E_{n-2}(\bar{x}, k-2) \\
 & - (k-1) \cdot P_n(x, k) \cdot E_n(x, k-1) \cdot E_{n-2}(\bar{x}, k-1) \geq 0,
 \end{aligned}$$

then

$$\begin{aligned}
 & \frac{1}{k-1} \cdot \binom{n}{k-1}^{-\frac{1}{k-1}} \cdot (E_n(x, k-1))^{\frac{1}{k-1}-1} \cdot E_{n-2}(\bar{x}, k-2) \\
 & - \frac{1}{k} \cdot \binom{n}{k}^{-\frac{1}{k}} \cdot (E_n(x, k))^{\frac{1}{k}-1} \cdot E_{n-2}(\bar{x}, k-1) \geq 0.
 \end{aligned}$$

Finally by (3.1), we can state that

$$(\ln x_1 - \ln x_2) \left(x_1 \frac{\partial f}{\partial x_1} - x_2 \frac{\partial f}{\partial x_2} \right) \geq 0,$$

Thus Theorem 1.2 holds by Lemma 2.3. □

Remark 3. If $n \geq 3, k = 2$, we know that f is neither a S-geometrically convex function nor a S-geometrically concave function.

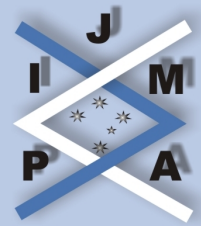
Remark 4. If $n = 2$, or $n \geq 3, 2 \leq k-1 < k \leq n$, and x logarithm majorizes y , according to Definition 2.3, we can state that

$$P_n(x, k-1) - P_n(x, k) \geq P_n(y, k-1) - P_n(y, k).$$

By Lemma 2.2 and

$$P_n(\bar{G}(x), k-1) - P_n(\bar{G}(x), k) = 0,$$

we know that Theorem 1.2 generalizes the Maclaurin-Inequality.



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