ON STARLIKENESS AND CONVEXITY OF ANALYTIC FUNCTIONS SATISFYING A DIFFERENTIAL INEQUALITY

SUKHWINDER SINGH

Department of Applied Sciences
Baba Banda Singh Bahadur Engineering College
Fatehgarh Sahib -140407 (Punjab), INDIA.

EMail: ss_billing@yahoo.co.in

SUSHMA GUPTA AND SUKHJIT SINGH

Department of Mathematics
Sant Longowal Institute of Engineering & Technology
Longowal-148106 (Punjab), INDIA.

EMail: sushmagupta1@yahoo.com sukhjit_d@yahoo.com

Received: 03 March, 2008

Accepted: 10 July, 2008

Communicated by: S.S. Dragomir

2000 AMS Sub. Class.: 30C45.

Key words: Multivalent function, Starlike function, Convex function, Multiplier transformation.

Abstract: In the present paper, the authors investigate a differential inequality defined by multiplier transformation in the open unit disk $E = \{z : |z| < 1\}$. As consequences, sufficient conditions for starlikeness and convexity of analytic functions are obtained.

vol. 9, iss. 3, art. 81, 2008

journal of inequalities in pure and applied mathematics

issn: 1443-5756

Contents

vol. 9, iss. 3, art. 81, 2008

journal of inequalities in pure and applied mathematics
issn: 1443-5756

1. Introduction

Let $\mathcal{A}_p$ denote the class of functions of the form $f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k$, $p \in \mathbb{N} = \{1, 2, \dots\}$, which are analytic in the open unit disc $E = \{z : |z| < 1\}$. We write $\mathcal{A}_1 = \mathcal{A}$. A function $f \in \mathcal{A}_p$ is said to be p-valent starlike of order α $(0 \leq \alpha < p)$ in E if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \qquad z \in E.$$

We denote by $S_p^*(\alpha)$, the class of all such functions. A function $f \in \mathcal{A}_p$ is said to be p-valent convex of order α $(0 \leq \alpha < p)$ in E if

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, \qquad z \in E.$$

Let $K_p(\alpha)$ denote the class of all those functions $f \in \mathcal{A}_p$ which are multivalently convex of order α in E. Note that $S_1^*(\alpha)$ and $K_1(\alpha)$ are, respectively, the usual classes of univalent starlike functions of order α and univalent convex functions of order α, $0 \leq \alpha < 1$, and will be denoted here by $S^*(\alpha)$ and $K(\alpha)$, respectively. We shall use S^* and K to denote $S^*(0)$ and $K(0)$, respectively which are the classes of univalent starlike (w.r.t. the origin) and univalent convex functions.

For $f \in \mathcal{A}_p$, we define the multiplier transformation $I_p(n, \lambda)$ as

$$(1.1) \qquad I_p(n,\lambda)f(z) = z^p + \sum_{k=p+1}^{\infty} \left(\frac{k+\lambda}{p+\lambda}\right)^n a_k z^k, \qquad (\lambda \geq 0, n \in \mathbb{Z}).$$

The operator $I_p(n, \lambda)$ has recently been studied by Aghalary et.al. [1]. Earlier, the operator $I_1(n, \lambda)$ was investigated by Cho and Srivastava [3] and Cho and Kim [2], whereas the operator $I_1(n, 1)$ was studied by Uralegaddi and Somanatha [11].

journal of inequalities in pure and applied mathematics
issn: 1443-5756

$I_1(n,0)$ is the well-known Sălăgean [10] derivative operator D^n, defined as: $D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k$, $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $f \in \mathcal{A}$.

A function $f \in \mathcal{A}_p$ is said to be in the class $S_n(p,\lambda,\alpha)$ for all z in E if it satisfies

$$\Re\left(\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)}\right) > \frac{\alpha}{p}, \tag{1.2}$$

for some $\alpha\,(0 \le \alpha < p, p \in \mathbb{N})$. We note that $S_0(1,0,\alpha)$ and $S_1(1,0,\alpha)$ are the usual classes $S^*(\alpha)$ and $K(\alpha)$ of starlike functions of order α and convex functions of order α, respectively.

In 1989, Owa, Shen and Obradovič [8] obtained a sufficient condition for a function $f \in \mathcal{A}$ to belong to the class $S_n(1,0,\alpha) = S_n(\alpha)$.

Recently, Li and Owa [4] studied the operator $I_1(n,0)$.

In the present paper, we investigate the differential inequality

$$\Re\left(\frac{(1-\alpha)I_p(n+1,\lambda)f(z) + \alpha I_p(n+2,\lambda)f(z)}{(1-\beta)I_p(n,\lambda)f(z) + \beta I_p(n+1,\lambda)f(z)}\right) > M(\alpha,\beta,\gamma,\lambda,p)$$

where α and β are real numbers and $M(\alpha,\beta,\gamma,\lambda,p)$ is a certain real number given in Section 2, for starlikeness and convexity of $f \in \mathcal{A}_p$. We obtain sufficient conditions for $f \in \mathcal{A}_p$ to be a member of $S_n(p,\lambda,\gamma)$, for some $\gamma\,(0 \le \gamma < p, p \in \mathbb{N})$. Many known results for starlikeness appear as corollaries to our main result and some new results regarding convexity of analytic functions are obtained.

journal of inequalities in pure and applied mathematics
issn: 1443-5756

2. Main Result

We shall make use of the following lemma of Miller and Mocanu to prove our result.

Lemma 2.1 ([6, 7]). *Let* Ω *be a set in the complex plane* $\mathbb{C}$ *and let* $\psi : \mathbb{C}^2 \times E \to \mathbb{C}$. *For* $u = u_1 + iu_2$, $v = v_1 + iv_2$, *assume that* ψ *satisfies the condition* $\psi(iu_2, v_1; z) \notin \Omega$, *for all* $u_2, v_1 \in \mathbb{R}$, *with* $v_1 \leq -(1 + u_2^2)/2$ *and for all* $z \in E$. *If the function* p, $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$, *is analytic in* E *and if* $\psi(p(z), zp'(z); z) \in \Omega$, *then* $\Re\, p(z) > 0$ *in* E.

We, now, state and prove our main theorem.

Theorem 2.2. *Let* $\alpha \geq 0$, $\beta \leq 1$, $\lambda \geq 0$ *and* $0 \leq \gamma < p$ *be real numbers such that* $\beta(1 - \frac{\gamma}{p}) < \frac{1}{2}$ *and* $\beta \leq \alpha$. *If* $f \in \mathcal{A}_p$ *satisfies the condition*

$$\Re\left(\frac{(1-\alpha)I_p(n+1,\lambda)f(z) + \alpha I_p(n+2,\lambda)f(z)}{(1-\beta)I_p(n,\lambda)f(z) + \beta I_p(n+1,\lambda)f(z)}\right) > M(\alpha,\beta,\gamma,\lambda,p), \tag{2.1}$$

then

$$\Re\left(\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)}\right) > \frac{\gamma}{p}$$

i.e., $f(z) \in S_n(p,\lambda,\gamma)$ *where,*

$$M(\alpha,\beta,\gamma,\lambda,p) = \frac{\frac{(1-\alpha)\gamma}{p} + \frac{\alpha\gamma^2}{p^2} - \frac{\alpha\left(1-\frac{\gamma}{p}\right)}{2(p+\lambda)}}{1 - \beta\left(1 - \frac{\gamma}{p}\right)}.$$

Proof. Since $0 \leq \gamma < p$, let us write $\mu = \frac{\gamma}{p}$. Thus, we have $0 \leq \mu < 1$.

Now we define,

$$\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)} = \mu + (1-\mu)r(z), \qquad z \in E. \tag{2.2}$$

Starlikeness and Convexity of Analytic Functions
Sukhwinder Singh, Sushma Gupta and Sukhjit Singh
vol. 9, iss. 3, art. 81, 2008

journal of inequalities in pure and applied mathematics
issn: 1443-5756

Therefore $r(z)$ is analytic in E and $r(0) = 1$.

Differentiating (2.2) logarithmically, we obtain

$$\frac{zI_p'(n+1,\lambda)f(z)}{I_p(n+1,\lambda)f(z)} - \frac{zI_p'(n,\lambda)f(z)}{I_p(n,\lambda)f(z)} = \frac{(1-\mu)zr'(z)}{\mu+(1-\mu)r(z)}, \qquad z \in E. \tag{2.3}$$

Using the fact that

$$zI_p'(n,\lambda)f(z) = (p+\lambda)I_p(n+1,\lambda)f(z) - \lambda I_p(n,\lambda)f(z).$$

Thus (2.3) reduces to

$$\frac{I_p(n+2,\lambda)f(z)}{I_p(n+1,\lambda)f(z)} = \mu + (1-\mu)r(z) + \frac{(1-\mu)zr'(z)}{(\lambda+p)[\mu+(1-\mu)r(z)]}.$$

Now, a simple calculation yields

$$\begin{aligned}
&\frac{(1-\alpha)I_p(n+1,\lambda)f(z)+\alpha I_p(n+2,\lambda)f(z)}{(1-\beta)I_p(n,\lambda)f(z)+\beta I_p(n+1,\lambda)f(z)} \\
&= \frac{(1-\alpha)+\alpha\left(\mu+(1-\mu)r(z)+\frac{(1-\mu)zr'(z)}{(\lambda+p)[\mu+(1-\mu)r(z)]}\right)}{(1-\beta)+\beta[\mu+(1-\mu)r(z)]}[\mu+(1-\mu)r(z)] \\
&= \frac{(1-\alpha)[\mu+(1-\mu)r(z)]+\alpha\left([\mu+(1-\mu)r(z)]^2+\frac{(1-\mu)zr'(z)}{(\lambda+p)}\right)}{(1-\beta)+\beta[\mu+(1-\mu)r(z)]} \\
&= \psi(r(z), zr'(z); z)
\end{aligned} \tag{2.4}$$

where,

$$\psi(u,v;z) = \frac{(1-\alpha)[\mu+(1-\mu)u]+\alpha\left((\mu+(1-\mu)u)^2+\frac{(1-\mu)v}{(\lambda+p)}\right)}{(1-\beta)+\beta[\mu+(1-\mu)u]}.$$

Starlikeness and Convexity of Analytic Functions
Sukhwinder Singh, Sushma Gupta and Sukhjit Singh
vol. 9, iss. 3, art. 81, 2008

journal of inequalities in pure and applied mathematics
issn: 1443-5756

Let $u = u_1 + iu_2$ and $v = v_1 + iv_2$, where u_1, u_2, v_1, v_2 are reals with $v_1 \leq -\frac{1+u_2^2}{2}$. Then, we have

$$\begin{aligned}
&\Re\,\psi(iu_2, v_1; z)\\
&= \frac{[(1-\alpha)\mu + \alpha\mu^2][1-\beta(1-\mu)]}{[1-\beta(1-\mu)]^2 + \beta^2(1-\mu)^2u_2^2}\\
&+ \frac{(1-\mu)^2[(1-\alpha)\beta - \alpha(1-\beta(1-\mu)) + 2\alpha\beta\mu]u_2^2 + \frac{\alpha(1-\mu)[1-\beta(1-\mu)]v_1}{p+\lambda}}{[1-\beta(1-\mu)]^2 + \beta^2(1-\mu)^2u_2^2}\\
&\leq \frac{\left[(1-\alpha)\mu + \alpha\mu^2 - \frac{\alpha(1-\mu)}{2(\lambda+p)}\right][1-\beta(1-\mu)]}{[1-\beta(1-\mu)]^2 + \beta^2(1-\mu)^2u_2^2}\\
&+ \frac{\left[(1-\mu)^2[(1-\alpha)\beta - \alpha(1-\beta(1-\mu)) + 2\alpha\beta\mu] - \frac{\alpha(1-\mu)[1-\beta(1-\mu)]}{2(p+\lambda)}\right]u_2^2}{[1-\beta(1-\mu)]^2 + \beta^2(1-\mu)^2u_2^2}\\
&= \frac{A + Bu_2^2}{[1-\beta(1-\mu)]^2 + \beta^2(1-\mu)^2u_2^2}\\
&= \phi(u_2), \quad \text{say}\\
&\leq \max \phi(u_2) \qquad (2.5)
\end{aligned}$$

where,

$$A = \left[(1-\alpha)\mu + \alpha\mu^2 - \frac{\alpha(1-\mu)}{2(\lambda+p)}\right][1-\beta(1-\mu)]$$

and

$$B = (1-\mu)^2[(1-\alpha)\beta - \alpha(1-\beta(1-\mu)) + 2\alpha\beta\mu] - \frac{\alpha(1-\mu)[1-\beta(1-\mu)]}{2(p+\lambda)}.$$

journal of inequalities in pure and applied mathematics
issn: 1443-5756

It can be easily verified that $\phi'(u_2) = 0$ implies that $u_2 = 0$. Under the given conditions, we observe that $\phi''(0) < 0$. Therefore,

$$\max \phi(u_2) = \phi(0) = M(\alpha, \beta, \gamma, \lambda, p). \tag{2.6}$$

Let

$$\Omega = \{w : \Re w > M(\alpha, \beta, \gamma, \lambda, p)\}.$$

Then from (2.1) and (2.4), we have $\psi(r(z), zr'(z); z) \in \Omega$ for all $z \in E$, but $\psi(iu_2, v_1; z) \notin \Omega$, in view of (2.5) and (2.6). Therefore, by Lemma 2.1 and (2.2), we conclude that

$$\Re\left(\frac{I_p(n+1, \lambda)f(z)}{I_p(n, \lambda)f(z)}\right) > \frac{\gamma}{p}.$$

□

journal of inequalities in pure and applied mathematics
issn: 1443-5756

3. Corollaries

By taking $p = 1$ and $\lambda = 0$ in Theorem 2.2. We have the following corollary.

Corollary 3.1. *Let $\alpha \geq 0$, $\beta \leq 1$ and $0 \leq \gamma < 1$ be real numbers such that $\beta(1-\gamma) < \frac{1}{2}$ and $\beta \leq \alpha$. If $f \in \mathcal{A}$ satisfies the condition*

$$\Re\left(\frac{(1-\alpha)D^{n+1}f(z) + \alpha D^{n+2}f(z)}{(1-\beta)D^{n}f(z) + \beta D^{n+1}f(z)}\right) > M(\alpha, \beta, \gamma, 0, 1),$$

then

$$\Re\frac{D^{n+1}f(z)}{D^{n}f(z)} > \gamma,$$

i.e. $f(z) \in S_n(\gamma)$, where,

$$M(\alpha, \beta, \gamma, 0, 1) = \frac{(1-\alpha)\gamma + \alpha\gamma^2 - \frac{\alpha(1-\gamma)}{2}}{1-\beta(1-\gamma)}.$$

By taking $p = 1, n = 0$ and $\lambda = 0$ in Theorem 2.2. We have the following corollary.

Corollary 3.2. *Let $\alpha \geq 0$, $\beta \leq 1$ and $0 \leq \gamma < 1$ be real numbers such that $\beta(1-\gamma) < \frac{1}{2}$ and $\beta \leq \alpha$. If $f \in \mathcal{A}$ satisfies the condition*

$$\Re\left(\frac{zf'(z) + \alpha z^2 f''(z)}{(1-\beta)f(z) + \beta zf'(z)}\right) > M(\alpha, \beta, \gamma, 0, 1),$$

then

$$\Re\frac{zf'(z)}{f(z)} > \gamma,$$

Starlikeness and Convexity of Analytic Functions
Sukhwinder Singh, Sushma Gupta and Sukhjit Singh
vol. 9, iss. 3, art. 81, 2008

journal of inequalities in pure and applied mathematics
issn: 1443-5756

i.e. $f(z) \in S^*(\gamma)$, *where,*

$$M(\alpha,\beta,\gamma,0,1) = \frac{(1-\alpha)\gamma + \alpha\gamma^2 - \frac{\alpha(1-\gamma)}{2}}{1-\beta(1-\gamma)}.$$

By taking $p = 1, n = 0, \lambda = 0$ and $\beta = 1$ in Theorem 2.2. We have the following corollary.

Corollary 3.3. *Let* $\alpha \geq 1$ *and* $\frac{1}{2} < \gamma < 1$ *be real numbers. If* $f \in \mathcal{A}$ *satisfies the condition*

$$\Re\left(1 + \alpha\frac{zf''(z)}{f'(z)}\right) > M(\alpha,1,\gamma,0,1),$$

then

$$\Re\frac{zf'(z)}{f(z)} > \gamma,$$

i.e. $f(z) \in S^*(\gamma)$, *where*

$$M(\alpha,1,\gamma,0,1) = 1 - \alpha(1-\gamma)\left(1 + \frac{1}{2\gamma}\right)$$

By taking $p = 1, n = 0, \lambda = 0$ and $\beta = 0$ in Theorem 2.2, we have the following result of Ravichandran et. al. [9].

Corollary 3.4. *Let* $\alpha \geq 0$ *and* $0 \leq \gamma < 1$ *be real numbers. If* $f \in \mathcal{A}$ *satisfies the condition*

$$\Re\frac{zf'(z)}{f(z)}\left(1 + \alpha\frac{zf''(z)}{f'(z)}\right) > M(\alpha,0,\gamma,0,1),$$

then

$$\Re\frac{zf'(z)}{f(z)} > \gamma,$$

journal of inequalities in pure and applied mathematics
issn: 1443-5756

i.e. $f(z) \in S^*(\gamma)$, *where,*

$$M(\alpha, 0, \gamma, 0, 1) = (1-\alpha)\gamma + \alpha\gamma^2 - \frac{\alpha(1-\gamma)}{2}.$$

Remark 1. In the case when $\gamma = \frac{\alpha}{2}$, Corollary 3.4 reduces to the result of Li and Owa [5].

By taking $p = 1, n = 0$ and $\lambda = 1$ in Theorem 2.2, we have the following corollary.

Corollary 3.5. *Let* $\alpha \geq 0$, $\beta \leq 1$ *and* $0 \leq \gamma < 1$ *be real numbers such that* $\beta(1-\gamma) < \frac{1}{2}$ *and* $\beta \leq \alpha$. *If* $f \in \mathcal{A}$ *satisfies the condition*

$$\Re \frac{1}{2}\left(\frac{(2-\alpha)f(z) + (2+\alpha)zf'(z) + \alpha z^2 f''(z)}{(2-\beta)f(z) + \beta z f'(z)}\right) > M(\alpha, \beta, \gamma, 1, 1),$$

then

$$\Re \frac{1}{2}\left(1 + \frac{zf'(z)}{f(z)}\right) > \gamma,$$

where,

$$M(\alpha, \beta, \gamma, 1, 1) = \frac{(1-\alpha)\gamma + \alpha\gamma^2 - \frac{\alpha(1-\gamma)}{4}}{1 - \beta(1-\gamma)}.$$

By taking $p = 1, n = 1$ and $\lambda = 0$ in Theorem 2.2, we have the following corollary.

Corollary 3.6. *Let* $\alpha \geq 0$, $\beta \leq 1$ *and* $0 \leq \gamma < 1$ *be real numbers such that* $\beta(1-\gamma) < \frac{1}{2}$ *and* $\beta \leq \alpha$. *If* $f \in \mathcal{A}$ *satisfies the condition*

$$\Re\left(\frac{zf'(z) + (2\alpha+1)z^2 f''(z) + \alpha z^3 f'''(z)}{zf'(z) + \beta z^2 f''(z)}\right) > M(\alpha, \beta, \gamma, 0, 1),$$

journal of inequalities in pure and applied mathematics
issn: 1443-5756

then

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right)>\gamma,$$

i.e. $f(z)\in K(\gamma)$, *where,*

$$M(\alpha,\beta,\gamma,0,1)=\frac{(1-\alpha)\gamma+\alpha\gamma^2-\frac{\alpha(1-\gamma)}{2}}{1-\beta(1-\gamma)}.$$

By taking $p=1, n=1, \lambda=0$ and $\beta=0$ in Theorem 2.2, we have the following corollary.

Corollary 3.7. *Let* $\alpha\geq 0$ *and* $0\leq\gamma<1$ *be real numbers. If* $f\in\mathcal{A}$ *satisfies the condition*

$$\Re\left(1+(2\alpha+1)\frac{zf''(z)}{f'(z)}+\alpha\frac{z^2f'''(z)}{f'(z)}\right)>M(\alpha,0,\gamma,0,1),$$

then

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right)>\gamma,$$

i.e., $f(z)\in K(\gamma)$, *where,*

$$M(\alpha,0,\gamma,0,1)=(1-\alpha)\gamma+\alpha\gamma^2-\frac{\alpha(1-\gamma)}{2}.$$

Remark 2. In the main result, the real number $M(\alpha,\beta,\gamma,\lambda,p)$ may not be the best possible as authors have not obtained the extremal function for it. The problem is still open for the best possible real number $M(\alpha,\beta,\gamma,\lambda,p)$.

journal of inequalities in pure and applied mathematics
issn: 1443-5756

References

[1] R. AGHALARY, R.M. ALI, S.B. JOSHI AND V. RAVICHANDRAN, Inequalities for analytic functions defined by certain linear operators, *International J. of Math. Sci.*, to appear.

[2] N.E. CHO AND T.H. KIM, Multiplier transformations and strongly close-to-convex functions, *Bull. Korean Math. Soc.*, **40** (2003), 399–410.

[3] N.E. CHO AND H.M. SRIVASTAVA, Argument estimates of certain analytic functions defined by a class of multiplier transformations, *Math. Comput. Modelling*, **37** (2003), 39–49.

[4] J.-L. LI AND S. OWA, Properties of the Sălăgean operator, *Georgian Math. J.*, **5**(4) (1998), 361–366.

[5] J.-L. LI AND S. OWA, Sufficient conditions for starlikeness, *Indian J. Pure Appl. Math.*, **33** (2002), 313–318.

[6] S.S. MILLER AND P.T. MOCANU, Differential subordinations and inequalities in the complex plane, *J. Diff. Eqns.*, **67** (1987), 199–211.

[7] S.S. MILLER AND P.T. MOCANU, *Differential Suordinations: Theory and Applications*, Series on Monographs and Textbooks in Pure and Applied Mathematics (No. 225), Marcel Dekker, New York and Basel, 2000.

[8] S. OWA, C.Y. SHEN AND M. OBRADOVIĆ, Certain subclasses of analytic functions, *Tamkang J. Math.*, **20** (1989), 105–115.

[9] V. RAVICHANDRAN, C. SELVARAJ AND R. RAJALAKSHMI, Sufficient conditions for starlike functions of order α, *J. Inequal. Pure and Appl. Math.*,

journal of inequalities in pure and applied mathematics
issn: 1443-5756

3(5) (2002), Art. 81. [ONLINE: http://jipam.vu.edu.au/article.php?sid=233].

[10] G.S. SĂLĂGEAN, Subclasses of univalent functions, *Lecture Notes in Math.*, **1013**, 362–372, *Springer-Verlag, Heidelberg*,1983.

[11] B.A. URALEGADDI AND C. SOMANATHA, Certain classes of univalent functions, in *Current Topics in Analytic Function Theory*, H.M. Srivastava and S. Owa (ed.), World Scientific, Singapore, (1992), 371–374.

journal of inequalities in pure and applied mathematics
issn: 1443-5756