A GEOMETRIC INEQUALITY INVOLVING A MOBILE POINT IN THE PLACE OF THE TRIANGLE

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Abstract: By using **Bottema's inequality** and several identities in triangles, we prove a weighted inequality concerning the distances between a mobile point P and three vertexes A, B, C of $\triangle ABC$. As an application, a conjecture with regard to Fermat's sum $PA + PB + PC$ is proved.

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1. Introduction and Main Results

For $\triangle ABC$, let a, b, c denote the side-lengths, A, B, C the angles, Δ the area, p the semi-perimeter, R the circumradius and r the inradius, respectively. In addition, supposing that P is a mobile point in the plane containing $\triangle ABC$, let PA, PB, PC denote the distances between P and A, B, C, respectively. We will customarily use the cyclic symbol, that is: $\sum f(a) = f(a) + f(b) + f(c)$, $\sum f(a,b) = f(a,b) + f(b,c) + f(c,a)$, $\prod f(a) = f(a)f(b)f(c)$, etc.

The following inequality can be easily proved by making use of **Bottema's inequality**:

$$(1.1)\qquad (PB+PC)\cos\frac{A}{2} + (PC+PA)\cos\frac{B}{2} + (PA+PB)\cos\frac{C}{2} \geq p \cdot \frac{p^2+2Rr+r^2}{4R^2}.$$

Here we choose to omit the details. From inequality (1.1) and the following known inequality (1.2) and identity (1.3) (see [3, 4, 6]):

$$(1.2)\qquad PA\cos\frac{A}{2} + PB\cos\frac{B}{2} + PC\cos\frac{C}{2} \geq p,$$

and

$$(1.3)\qquad q_1 = \cos^2\frac{B-C}{2} + \cos^2\frac{C-A}{2} + \cos^2\frac{A-B}{2} = \frac{p^2+4R^2+2Rr+r^2}{4R^2},$$

we easily get

$$(1.4)\qquad PA+PB+PC \geq p \cdot \frac{\cos^2\frac{B-C}{2} + \cos^2\frac{C-A}{2} + \cos^2\frac{A-B}{2}}{\cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2}}.$$

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Considering the refinement of inequality (1.4), Chu [2] posed a conjecture as follows.

Corollary 1.1. *For any $\triangle ABC$,*

$$PA+PB+PC \geq p\cdot \frac{\cos\frac{B-C}{2}+\cos\frac{C-A}{2}+\cos\frac{A-B}{2}}{\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}}. \tag{1.5}$$

The main object of this paper is to prove Conjecture 1.1, which is easily seen to follow from the following stronger result.

Theorem 1.2. *In $\triangle ABC$, we have*

$$\begin{aligned}(PB+PC)\cos\frac{A}{2}+(PC+PA)\cos\frac{B}{2}+(PA+PB)\cos\frac{C}{2} \\ \geq p\cdot\left[\cos\frac{B-C}{2}+\cos\frac{C-A}{2}+\cos\frac{A-B}{2}-1\right].\end{aligned} \tag{1.6}$$

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2. Preliminary Results

In order to prove our main result, we shall require the following four lemmas.

Lemma 2.1. *In* $\triangle ABC$*, we have that*

$$q_2 = \cos\frac{B-C}{2}\cdot\cos\frac{C-A}{2}\cdot\cos\frac{A-B}{2} = \frac{p^2+2Rr+r^2}{8R^2}, \tag{2.1}$$

$$\cos\frac{A}{2}\cdot\cos\frac{B}{2}\cdot\cos\frac{C}{2} = \frac{p}{4R}, \tag{2.2}$$

$$\begin{aligned} q_3 &= \sum \cos\frac{B}{2}\cos\frac{C}{2}\cos\frac{B-C}{2}\left(b^2+c^2-a^2\right) \\ &= \frac{p^4+2Rrp^2-r\left(2R+r\right)\left(4R+r\right)^2}{4R^2}, \end{aligned} \tag{2.3}$$

$$\begin{aligned} q_4 &= \frac{1}{2}\sum\left(\cos^2\frac{B}{2}+\cos^2\frac{C}{2}\right)\left(b^2+c^2-a^2\right) \\ &= \frac{\left(2R+3r\right)p^2-r\left(4R+r\right)^2}{2R}, \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} Q &= \sum\left(\cos\frac{B-C}{2}-\cos\frac{C-A}{2}\cos\frac{A-B}{2}\right) \\ &= q_1-3q_2-\frac{p\Delta^4\prod\left(b-c\right)^2}{a^2b^2c^2\prod\left(X+x\right)}, \end{aligned} \tag{2.5}$$

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where

$$X = a\sqrt{bc\,(p-b)\,(p-c)}, \qquad Y = b\sqrt{ca\,(p-c)\,(p-a)}, \qquad Z = c\sqrt{ab\,(p-a)\,(p-b)}, \tag{2.6}$$

and

$$x = (b+c)\,(p-b)\,(p-c)\,, \qquad y = (c+a)\,(p-c)\,(p-a)\,, \qquad z = (a+b)\,(p-a)\,(p-b)\,. \tag{2.7}$$

Proof. The proofs of identities (2.1) and (2.2) were given in [6]. Now, we present the proofs of identities (2.3) – (2.5). By utilizing the formulas

$$\cos\frac{A}{2} = \sqrt{\frac{p\,(p-a)}{bc}}, \qquad \sin\frac{A}{2} = \sqrt{\frac{(p-b)\,(p-c)}{bc}},$$
$$\cos\frac{B-C}{2} = \frac{b+c}{a}\sqrt{\frac{(p-b)\,(p-c)}{bc}},$$

and (see [5, pp.52])

$$\prod a = 4Rrp, \qquad \sum a = 2p \qquad \text{and} \qquad \sum bc = p^2 + 4Rr + r^2,$$

we get that

$$\begin{aligned} q_3 &= p\sum \frac{(b+c)\,(p-b)\,(p-c)}{a^2bc}\left(b^2+c^2-a^2\right) \\ &= \frac{p}{4a^2b^2c^2}\sum bc\,(b+c)\,(c+a-b)\,(a+b-c)\left(b^2+c^2-a^2\right) \end{aligned}$$

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$$= \frac{p}{4a^2b^2c^2}[6(ab+bc+ca)^2(a+b+c)^3 - 8(ab+bc+ca)^3(a+b+c) - (ab+bc+ca)(a+b+c)^5 - 2abc(ab+bc+ca)(a+b+c)^2 + 8abc(ab+bc+ca)^2 - abc(a+b+c)^4 - 4(a+b+c)a^2b^2c^2]$$

$$= \frac{p^4 + 2Rrp^2 - r(2R+r)(4R+r)^2}{4R^2}$$

and

$$\frac{1}{2}\sum\left(\cos^2\frac{B}{2} + \cos^2\frac{C}{2}\right)(b^2+c^2-a^2)$$
$$= \sum a^2\cos^2\frac{A}{2}$$
$$= \frac{p}{abc}\left(p\sum a^3 - \sum a^4\right)$$
$$= \frac{p}{abc}\left[\frac{5}{2}(ab+bc+ca)(a+b+c)^2 - 2(ab+bc+ca)^2 - \frac{1}{2}(a+b+c)^4 - \frac{5}{2}(a+b+c)abc\right]$$
$$= \frac{(2R+3r)p^2 - r(4R+r)^2}{2R}.$$

Thus, identities (2.3) and (2.4) hold true.

With (1.3), (2.1) and the formulas of half-angles, we obtain that

$$q_1 - 3q_2 = \frac{-p^2 + 8R^2 - 2Rr - r^2}{8R^2}$$
$$= \frac{1}{a^2b^2c^2}\sum x\left[bc(b+c) - (c+a)(a+b)(s-a)\right],$$

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and

$$Q = \sum \left(\cos \frac{B-C}{2} - \cos \frac{C-A}{2} \cos \frac{A-B}{2} \right)$$
$$= \frac{\sum X \left[bc\,(b+c) - (c+a)\,(a+b)\,(s-a) \right]}{a^2b^2c^2}.$$

It is easy to see that

$$X - x = \frac{\Delta^2\,(b-c)^2}{X+x}, \quad Y - y = \frac{\Delta^2\,(c-a)^2}{Y+y}, \quad \text{and} \quad Z - z = \frac{\Delta^2\,(a-b)^2}{Z+z}.$$

Then

$$a^2b^2c^2[Q - (q_1 - 3q_2)]$$
$$= \sum [bc(b+c) - (c+a)(a+b)(p-a)](X-x)$$
$$= \sum [bc(b+c) - (c+a)(a+b)(p-a)] \frac{\Delta^2(b-c)^2}{X+x}$$
$$= \sum p\Delta^2 \frac{(a-b)(a-c)(b-c)^2}{(X+x)}.$$

Therefore,

$$Q - (q_1 - 3q_2) = \sum p\Delta^2 \frac{(a-b)(a-c)(b-c)^2}{a^2b^2c^2(X+x)}$$
$$= \frac{p\Delta^2(a-b)(a-c)(b-c)}{a^2b^2c^2} \sum \frac{b-c}{X+x},$$

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where

$$\begin{aligned}
&\frac{b-c}{X+x}+\frac{c-a}{Y+y}+\frac{a-b}{Z+z}\\
&=\frac{(b-c)(Y-X+y-x)}{(X+x)(Y+y)}+\frac{(a-b)(Y-Z+y-z)}{(Z+z)(Y+y)}\\
&=\frac{p(p-c)(b-c)(b-a)}{(X+x)(Y+y)}+\frac{pabc(p-c)(b-c)(b-a)}{(X+x)(Y+y)(X+Y)}\\
&\quad+\frac{p(p-a)(a-b)(b-c)}{(Z+z)(Y+y)}+\frac{pabc(p-a)(a-b)(b-c)}{(Z+z)(Y+y)(Z+Y)}\\
&=\frac{p(b-c)(a-b)}{\prod(X+x)}\left[(p-a)(X+x)-(p-c)(Z+z)\right]\\
&\quad+\frac{pabc(b-c)(a-b)}{(X+Y)(Y+Z)\prod(X+x)}\\
&\quad\times\left[(p-a)(X+x)(X+Y)-(p-c)(Z+z)(Y+Z)\right]\\
&=\frac{-\Delta^2(b-c)(a-b)(a-c)}{\prod(X+x)}\\
&\quad+\frac{p(b-c)(a-b)(a-c)}{(Z+X)\prod(X+x)}\left[abc\prod(p-a)-ca(p-b)Y\right]\\
&\quad+\frac{pabc(b-c)(a-b)(a-c)}{(X+Y)(Z+Y)\prod(X+x)}\left[abc\prod(p-a)-Y\prod(p-a)\right.\\
&\quad+\frac{abc(Y-abc)\prod(p-a)}{Z+X}\\
&\quad\left.+\frac{abc(pb+ca)(p-b)\prod(p-a)-ca(p-b)Y\prod(p-a)}{Z+X}\right]
\end{aligned}$$

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$$= \frac{-\Delta^2(b-c)(a-b)(a-c)}{\prod(X+x)} + \frac{pabc(b-c)(a-b)(a-c)}{(X+Y)(Z+Y)\prod(X+x)}$$
$$\cdot \left\{ \left[\prod(p-a) - (p-b)\sqrt{ca(p-c)(p-a)} \right] (X+Y)(Y+Z) \right.$$
$$+ abc \prod(p-a) \left[Y - abc + pb(p-b) + ca(p-b) - (p-b)\sqrt{ca(p-c)(p-a)} \right]$$
$$\left. +(Z+X)(abc-Y)\prod(p-a) \right\}$$
$$= \frac{-\Delta^2(b-c)(a-b)(a-c)}{\prod(X+x)},$$

which implies the assertion (2.5). □

Lemma 2.2. *For any* $\triangle ABC$,

$$\sqrt{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\left(\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}\right)} \geq \frac{p}{2R}. \tag{2.8}$$

Proof. From **Euler's inequality** $R \geq 2r$, $abc = 4Rrp$, $a+b+c = 2p$ and the law of sines, we obtain that

$$\begin{aligned} 2R^2p \geq 4Rrp &\iff R^2(a+b+c) \geq abc \\ &\iff \sin A + \sin B + \sin C \geq 4\sin A \sin B \sin C. \end{aligned} \tag{2.9}$$

Taking

$$A \to \frac{\pi - A}{2}, \qquad B \to \frac{\pi - B}{2}, \qquad \text{and} \qquad C \to \frac{\pi - C}{2},$$

we easily get

$$\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2} \geq 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}. \tag{2.10}$$

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Inequality (2.8) follows immediately in view of (2.10) and (2.2). □

Lemma 2.3. *In $\triangle ABC$, we have*

$$\sum \cos\frac{B}{2}\cos\frac{C}{2}\left(b^2+c^2-a^2\right) \geq \frac{p^4+2Rrp^2-r\left(2R+r\right)\left(4R+r\right)^2}{4R^2}. \tag{2.11}$$

Proof. By employing (2.3) and the formulas of half-angles, inequality (2.11) is equivalent to

$$\begin{aligned}\sum \cos\frac{B}{2}\cos\frac{C}{2}\left(b^2+c^2-a^2\right) \\ \geq \sum \cos\frac{B}{2}\cos\frac{C}{2}\cos\frac{B-C}{2}\left(b^2+c^2-a^2\right),\end{aligned} \tag{2.12}$$

or

$$\sum \frac{\left(b^2+c^2-a^2\right)}{a^2bc}\left[a\sqrt{bc\left(s-b\right)\left(s-c\right)}-\left(b+c\right)\left(s-b\right)\left(s-c\right)\right] \geq 0, \tag{2.13}$$

that is

$$\sum \frac{\Delta^2}{abc}\cdot\frac{\left(b^2+c^2-a^2\right)}{a\left(X+x\right)}\left(b-c\right)^2 \geq 0, \tag{2.14}$$

where X, Y, Z and x, y, z are given, just as in the proof of Lemma 2.1, by (2.6) and (2.7), respectively.

Without loss of generality, we can assume that $a \geq b \geq c$ to obtain

$$a\left(X+x\right) \geq b\left(Y+y\right) \geq c\left(Z+z\right),$$

and

$$\left(a-c\right)^2 \geq \left(b-c\right)^2,$$

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and thus

$$\frac{(b-c)^2}{a\,(X+x)} \le \frac{(c-a)^2}{b\,(Y+y)}.$$

Hence, in order to prove inequality (2.14), we only need to prove that

$$\frac{\Delta^2}{abc}\left[\frac{(b^2+c^2-a^2)}{a(X+x)}(b-c)^2+\frac{(c^2+a^2-b^2)}{b(Y+y)}(c-a)^2\right]\ge 0. \tag{2.15}$$

We readily arrive at the following result for $a \ge b \ge c$,

$$\begin{aligned}&\frac{\Delta^2}{abc}\left[\frac{(b^2+c^2-a^2)}{a(X+x)}(b-c)^2+\frac{(c^2+a^2-b^2)}{b(Y+y)}(c-a)^2\right]\\ &\quad\ge \frac{\Delta^2}{abc}\left(b^2+c^2-a^2+c^2+a^2-b^2\right)\frac{(c-a)^2}{b\,(Y+y)}\\ &\quad= 2c^2\cdot\frac{\Delta^2}{abc}\cdot\frac{(c-a)^2}{b\,(Y+y)}\ge 0.\end{aligned}$$

This shows that the inequality (2.15) or (2.11) holds true. The proof of Lemma 2.3 is thus complete. □

Lemma 2.4 (Bottema's inequality, see [1, pp. 118, Theorem 12.56]). *Let Δ' denote the area of $\triangle A'B'C'$, and a', b', c' the side-lengths of $\triangle A'B'C'$, respectively. Then*

$$\begin{aligned}&(a'PA+b'PB+c'PC)^2\\ &\quad\ge\frac{1}{2}[a'^2(b^2+c^2-a^2)+b'^2(c^2+a^2-b^2)+c'^2(a^2+b^2-c^2)]+8\Delta\Delta'.\end{aligned} \tag{2.16}$$

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3. The Proof of Theorem 1.2

Proof. It is easy to show that

$$a' = \cos\frac{B}{2} + \cos\frac{C}{2}, \qquad b' = \cos\frac{C}{2} + \cos\frac{A}{2}, \qquad \text{and}$$
$$c' = \cos\frac{A}{2} + \cos\frac{B}{2}$$

are three side-lengths of a certain triangle. By using **Bottema's inequality** (2.16), in order to prove inequality (1.6), we only need to prove that

$$8\Delta\sqrt{\prod\cos\frac{A}{2}\sum\cos\frac{A}{2}} + \frac{1}{2}\sum\left(\cos\frac{B}{2} + \cos\frac{C}{2}\right)^2 (b^2 + c^2 - a^2)$$
$$\geq p^2\left[\sum\cos\frac{B-C}{2} - 1\right]^2$$

or

$$8\Delta\sqrt{\prod\cos\frac{A}{2}\sum\cos\frac{A}{2}} + q_4 + \sum\cos\frac{B}{2}\cos\frac{C}{2}\left(b^2 + c^2 - a^2\right) + 2p^2Q \geq p^2\left(q_1 + 1\right). \tag{3.1}$$

With identities (1.3), (2.4), (2.5), together with Lemma 2.2 and Lemma 2.3, in order to prove inequality (3.1), we only need to prove that

$$8\Delta\cdot\frac{p}{2R} + \frac{(2R+3r)\,p^2 - r\,(4R+r)^2}{2R} + \frac{p^4 + 2Rrp^2 - r\,(2R+r)\,(4R+r)^2}{4R^2}$$

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$$+2p^2\left[\frac{-p^2+8R^2-2Rr-r^2}{8R^2}-\frac{p\Delta^4\prod(b-c)^2}{a^2b^2c^2\prod(X+x)}\right] \geq p^2\left(\frac{p^2+4R^2+2Rr+r^2}{4R^2}+1\right)$$

or

$$\frac{-p^4+(4R^2+20Rr-2r^2)\,p^2-r\,(4R+r)^3}{4R^2} \geq \frac{2p^3\Delta^4\prod(b-c)^2}{a^2b^2c^2\prod(X+x)}. \tag{3.2}$$

From the known identities (see [5])

$$\Delta = rp \text{ and}$$
$$(b-c)^2(c-a)^2(a-b)^2 = 4r^2[-p^4+(4R^2+20Rr-2r^2)p^2-r(4R+r)^3],$$

inequality (3.2) is equivalent to

$$\prod(X+x) \geq 2r^4p^5. \tag{3.3}$$

For $X \geq x$, and with the following two known identities (see [5, pp.53])

$$\prod(b+c) = 2p(p^2+2Rr+r^2), \qquad \prod(p-a) = r^2p,$$

we obtain

$$\prod(X+x) \geq 8\prod x = 8\prod(b+c)\prod(p-a)^2$$
$$= 16r^4p^3(p^2+2Rr+r^2) > 16r^4p^5 > 2r^4p^5.$$

Therefore, inequality (3.3) holds. This completes the proof of Theorem 1.2. □

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4. Remarks

Remark 1. From inequalities (1.2) and (1.6), it is easy to see that inequality (1.5) holds.

Remark 2. In view of

$$\sum \cos\frac{B-C}{2} \geq \sum \cos^2\frac{B-C}{2} = \frac{p^2+4R^2+2Rr+r^2}{4R^2}$$
$$\iff \sum \cos\frac{B-C}{2} - 1 \geq \frac{p^2+2Rr+r^2}{4R^2},$$

it follows that inequality (1.6) is a refinement of inequality (1.1).

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