Journal of Inequalities in Pure and Applied Mathematics

http://jipam.vu.edu.au/

Volume 3, Issue 5, Article 81, 2002

SUFFICIENT CONDITIONS FOR STARLIKE FUNCTIONS OF ORDER α

V. RAVICHANDRAN, C. SELVARAJ, AND R. RAJALAKSMI

DEPARTMENT OF MATHEMATICS AND COMPUTER APPLICATIONS
SRI VENKATESWARA COLLEGE OF ENGINEERING
PENNALUR 602 105, INDIA.
vravi@svce.ac.in

DEPARTMENT OF MATHEMATICS
L N GOVERNMENT COLLEGE
PONNERI, 601 204, INDIA.

DEPARTMENT OF MATHEMATICS
LOYOLA COLLEGE
CHENNAI 600 034, INDIA.

Received 5 June, 2002; accepted 4 November, 2002
Communicated by H.M. Srivastava

Dedicated to the memory of Prof. K.S. Padmanabhan

ABSTRACT. In this paper, we obtain some sufficient conditions for an analytic function $f(z)$, defined on the unit disk $\triangle$, to be starlike of order α.

Key words and phrases: Starlike function of order α, Univalent function.

2000 *Mathematics Subject Classification.* 30C45.

1. INTRODUCTION

Let $\mathcal{A}_n$ be the class of all functions $f(z) = z + a_{n+1}z^{n+1} + \cdots$ which are analytic in $\triangle = \{z; |z| < 1\}$ and let $\mathcal{A}_1 = \mathcal{A}$. A function $f(z) \in \mathcal{A}$ is starlike of order α, if

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \quad 0 \le \alpha < 1,$$

for all $z \in \triangle$. The class of all starlike functions of order α is denoted by $S^*(\alpha)$. We write $S^*(0)$ simply as S^*. Recently, Li and Owa [3] proved the following:

ISSN (electronic): 1443-5756

The authors are thankful to the referee for his comments and suggestions.
067-02

Theorem 1.1. *If $f(z) \in \mathcal{A}$ satisfies*

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\left(\alpha\frac{zf''(z)}{f'(z)}+1\right)\right\} > -\frac{\alpha}{2}, \quad z \in \triangle,$$

for some α ($\alpha \geq 0$), then $f(z) \in S^*$.

In fact, Lewandowski, Miller and Zlotkiewicz [1] and Ramesha, Kumar, and Padmanabhan [7] have proved a weaker form of the above theorem. If the number $-\alpha/2$ is replaced by $-\alpha^2(1-\alpha)/4$, $(0 \leq \alpha < 2)$ in the above condition, Li and Owa [3] have proved that $f(z)$ is in $S^*(\alpha/2)$.

Li and Owa [3] have also proved the following:

Theorem 1.2. *If $f(z) \in \mathcal{A}$ satisfies*

$$\left|\frac{zf''(z)}{f'(z)}\left(\frac{zf'(z)}{f(z)}-1\right)\right| < \rho, \quad z \in \triangle,$$

where $\rho = 2.2443697$, then $f(z) \in S^*$.

The above theorem with $\rho = 3/2$ and $\rho = 1/6$ were earlier proved by Li and Owa [2] and Obradovic [6] respectively.

In this paper, we obtain some sufficient conditions for functions to be starlike of order β. To prove our result, we need the following:

Lemma 1.3. [4] *Let Ω be a set in the complex plane $\mathcal{C}$ and suppose that Φ is a mapping from $\mathcal{C}^2 \times \triangle$ to $\mathcal{C}$ which satisfies $\Phi(ix, y; z) \notin \Omega$ for $z \in \triangle$, and for all real x, y such that $y \leq -n(1+x^2)/2$. If the function $p(z) = 1 + c_n z^n + \cdots$ is analytic in $\triangle$ and $\Phi(p(z), zp'(z); z) \in \Omega$ for all $z \in \triangle$, then $\operatorname{Re} p(z) > 0$.*

2. SUFFICIENT CONDITIONS FOR STARLIKENESS

In this section, we prove some sufficient conditions for function to be starlike of order β.

Theorem 2.1. *If $f(z) \in \mathcal{A}_n$ satisfies*

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\left(\alpha\frac{zf''(z)}{f'(z)}+1\right)\right\} > \alpha\beta\left[\beta+\frac{n}{2}-1\right]+\left[\beta-\frac{\alpha n}{2}\right], \ \ z \in \triangle, \ 0 \leq \alpha, \beta \leq 1,$$

then $f(z) \in S^*(\beta)$.

Proof. Define $p(z)$ by

$$(1-\beta)p(z)+\beta = \frac{zf'(z)}{f(z)}.$$

Then $p(z) = 1 + c_n z^n + \cdots$ and is analytic in $\triangle$. A computation shows that

$$\frac{zf''(z)}{f'(z)} = \frac{(1-\beta)zp'(z) + [(1-\beta)p(z)+\beta]^2 - [(1-\beta)p(z)+\beta]}{(1-\beta)p(z)+\beta}$$

and hence

$$\begin{aligned}\frac{zf'(z)}{f(z)}\left(\alpha\frac{zf''(z)}{f'(z)}+1\right) &= \alpha(1-\beta)zp'(z)+\alpha(1-\beta)^2p^2(z)\\ &\qquad +(1-\beta)(1+2\alpha\beta-\alpha)p(z)+\beta[\alpha\beta+1-\alpha]\\ &= \Phi(p(z),zp'(z);z),\end{aligned}$$

where

$$\Phi(r,s;t)=\alpha(1-\beta)s+\alpha(1-\beta)^2r^2+(1-\beta)(1+2\alpha\beta-\alpha)r+\beta[\alpha\beta+1-\alpha].$$

For all real x and y satisfying $y\le -n(1+x^2)/2$, we have

$$\begin{aligned}\operatorname{Re}\Phi(ix,y;z) &= \alpha(1-\beta)y-\alpha(1-\beta)^2x^2+\beta[\alpha\beta+1-\alpha]\\ &\le -\frac{\alpha}{2}(1-\beta)n-\left[\frac{n\alpha}{2}(1-\beta)+\alpha(1-\beta)^2\right]x^2+\beta[\alpha\beta+1-\alpha]\\ &= -\frac{\alpha}{2}(1-\beta)n-\frac{\alpha(1-\beta)}{2}(n+2-2\beta)x^2+\beta(\alpha\beta+1-\alpha)\\ &\le \beta(\alpha\beta+1-\alpha)-\frac{\alpha}{2}(1-\beta)n\\ &= \alpha\beta\left(\beta+\frac{n}{2}-1\right)+\left(\beta-\frac{n\alpha}{2}\right).\end{aligned}$$

Let $\Omega=\left\{w;\operatorname{Re}w>\alpha\beta\left(\beta+\frac{n}{2}-1\right)+\left(\beta-\frac{n\alpha}{2}\right)\right\}$. Then $\Phi(p(z),zp'(z);z)\in\Omega$ and $\Phi(ix,y;z)\notin\Omega$ for all real x and $y\le -n(1+x^2)/2$, $z\in\triangle$. By an application of Lemma 1.3, the result follows. □

By taking $\beta=0$ and $n=1$ in the above theorem, we have the following:

Corollary 2.2. [3] *If $f(z)\in\mathcal{A}$ satisfies*

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\left(\alpha\frac{zf''(z)}{f'(z)}+1\right)\right\}>-\frac{\alpha}{2},\quad z\in\triangle,$$

for some α ($\alpha\ge 0$), then $f(z)\in S^*$.

If we take $\beta=\alpha/2$ and $n=1$, we get the following:

Corollary 2.3. [3] *If $f(z)\in\mathcal{A}$ satisfies*

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\left(\alpha\frac{zf''(z)}{f'(z)}+1\right)\right\}>-\frac{\alpha^2}{4}(1-\alpha),\quad z\in\triangle,$$

for some α ($0<\alpha\le 2$), then $f(z)\in S^*(\alpha/2)$.

In fact, in the proof of the above theorem, we have proved the following: If $p(z)=1+c_nz^n+\cdots$ is analytic in $\triangle$ and satisfies

$$\begin{aligned}&\operatorname{Re}(\alpha(1-\beta)zp'(z)+\alpha(1-\beta)^2p^2(z)+(1-\beta)(1+2\alpha\beta-\alpha)p(z)+\beta[\alpha\beta+1-\alpha])\\ &\qquad\qquad > \alpha\beta\left[\beta+\frac{n}{2}-1\right]+\left(\beta-\frac{\alpha n}{2}\right),\end{aligned}$$

then $\operatorname{Re}p(z)>0$. Using a method similar to the one used in the above theorem, we have the following:

Theorem 2.4. *Let $\alpha \geq 0$, $0 \leq \beta < 1$. If $f(z) \in \mathcal{A}_n$ satisfies*

$$\operatorname{Re}\left\{\frac{f(z)}{z}\left(\alpha\frac{zf'(z)}{f(z)} + 1 - \alpha\right)\right\} > -\frac{n}{2}\alpha(1-\beta) + \beta, \quad z \in \triangle,$$

then

$$\operatorname{Re}\frac{f(z)}{z} > \beta.$$

As a special case, we get the following: If $f(z) \in \mathcal{A}$ satisfies

$$\operatorname{Re}\{f'(z) + \alpha z f''(z)\} > -\frac{\alpha}{2}, \quad z \in \triangle,$$

$\alpha \geq 0$, then

$$\operatorname{Re} f'(z) > 0.$$

However, a sharp form of this result was proved by Nunokawa and Hoshino [5].

Theorem 2.5. *Let $0 \leq \beta < 1$, $a = (n/2 + 1 - \beta)^2$ and $b = (n/2 + \beta)^2$ satisfy $(a+b)\beta^2 < b(1 - 2\beta)$. Let t_0 be the positive real root of the equation*

$$2a(1-\beta)^2 t^2 + [3a\beta^2 + b(1-\beta)^2]t + [(a+2b)\beta^2 - (1-\beta)^2 b] = 0$$

and

$$\rho^2 = \frac{(1-\beta)^3(1+t_0)^2(at_0+b)}{\beta^2 + (1-\beta)^2 t_0}.$$

If $f(z) \in \mathcal{A}_n$ satisfies

$$\left|\frac{zf''(z)}{f'(z)}\left(\frac{zf'(z)}{f(z)} - 1\right)\right| \leq \rho, \quad z \in \triangle,$$

then $f(z) \in S^*(\beta)$.

Proof. Define $p(z)$ by

$$(1-\beta)p(z) + \beta = \frac{zf'(z)}{f(z)}.$$

Then $p(z) = 1 + c_n z^n + \cdots$ and is analytic in $\triangle$. A computation shows that

$$\frac{zf''(z)}{f'(z)} = \frac{(1-\beta)zp'(z) + [(1-\beta)p(z)+\beta]^2 - [(1-\beta)p(z)+\beta]}{(1-\beta)p(z)+\beta}$$

and hence

$$\begin{aligned}
&\frac{zf''(z)}{f'(z)}\left(\frac{zf'(z)}{f(z)} - 1\right)\\
&\qquad = \frac{(1-\beta)(p(z)-1)}{(1-\beta)p(z)+\beta}[(1-\beta)zp'(z) + [(1-\beta)p(z)+\beta]^2 - [(1-\beta)p(z)+\beta)]]\\
&\qquad \equiv \Phi(p(z), zp'(z); z).
\end{aligned}$$

Then, for all real x and y satisfying $y \le -n(1+x^2)/2$, we have

$$\begin{aligned}
&|\Phi(ix, y; z)|^2 \\
&\quad = \frac{(1-\beta)^2(1+x^2)}{\beta^2+(1-\beta)^2x^2}\left\{[(1-\beta)y-\beta+\beta^2-(1-\beta)^2x^2]^2\right. \\
&\qquad\qquad\qquad\qquad \left.+[2\beta(1-\beta)-(1-\beta)]^2x^2\right\} \\
&\quad = \frac{(1-\beta)^2(1+t)}{\beta^2+(1-\beta)^2t}\{[(1-\beta)y-\beta+\beta^2-(1-\beta)^2t]^2 \\
&\qquad\qquad\qquad\qquad + [2\beta(1-\beta)-(1-\beta)]^2t\} \\
&\quad \equiv g(t, y),
\end{aligned}$$

where $t = x^2 > 0$ and $y \le -n(1+t)/2$. Since

$$\frac{\partial g}{\partial y} = \frac{(1-\beta)^3(1+t)}{\beta^2+(1-\beta)^2t}[(1-\beta)y-\beta+\beta^2-(1-\beta)^2t]^2 < 0,$$

we have

$$g(t, y) \ge g\left(t, -\frac{n}{2}(1+t)\right) \equiv h(t).$$

Note that

$$h(t) = \frac{(1-\beta)^3(1+t)^2}{\beta^2+(1-\beta)^2t}\left[t\left(\frac{n}{2}+1-\beta\right)^2+\left(\frac{n}{2}+\beta\right)^2\right].$$

Also it is clear that $h'(-1) = 0$ and the other two roots of $h'(t) = 0$ are given by

$$2a(1-\beta)^2t^2+[3a\beta^2+b(1-\beta)^2]t+[(a+2b)\beta^2-(1-\beta)^2b] = 0,$$

where $a = (n/2+1-\beta)^2$ and $b = (n/2+\beta)^2$. Since t_0 is the positive root of this equation we have $h(t) \ge h(t_0)$ and hence

$$|\Phi(ix, y; z)|^2 \ge h(t_0).$$

Define $\Omega = \{w; |w| < \rho\}$. Then $\Phi(p(z), zp'(z); z) \in \Omega$ and $\Phi(ix, y; z) \notin \Omega$ for all real x and $y \le -n(1+x^2)/2$, $z \in \triangle$. Therefore by an application of Lemma 1.3, the result follows. $\square$

If we take $n = 1$, $\beta = 0$, we have $t_0 = \frac{\sqrt{73}-1}{36}$ and therefore we have the following:

Corollary 2.6. [3] *If $f(z) \in \mathcal{A}$ satisfies*

$$\left|\frac{zf''(z)}{f'(z)}\left(\frac{zf'(z)}{f(z)}-1\right)\right| < \rho, \quad z \in \triangle,$$

where $\rho^2 = \frac{827+73\sqrt{73}}{288}$, then $f(z) \in S^*$.

REFERENCES

[1] Z. LEWANDOWSKI, S.S. MILLER AND E. ZŁOTKIEWICZ, Generating functions for some classes of univalent functions, *Proc. Amer. Math. Soc.*, **56** (1976), 111–117.

[2] J.-L. LI AND S. OWA, Properties of the Salagean operator, *Georgian Math. J.*, **5(4)** (1998), 361–366.

[3] J.-L. LI AND S. OWA, Sufficient conditions for starlikeness, *Indian J. Pure Appl. Math.*, **33** (2002), 313–318.

[4] S.S. MILLER AND P.T. MOCANU, Differential subordinations and inequalities in the complex plane, *J. Differ. Equations*, **67** (1987), 199–211.

[5] M. NUNOKAWA AND S. HOSHINO, One criterion for multivalent functions, *Proc. Japan Acad., Ser. A*, **67** (1991), 35–37.

[6] M. OBRADOVIĆ, Ruscheweyh derivatives and some classes of univalent functions, in: *Current Topics in Analytic Function Theory*, (H.M. Srivastava and S. Owa, Editors), World Sci. Publishing, River Edge, NJ, 1992, pp. 220–233.

[7] C. RAMESHA, S. KUMAR AND K.S. PADMANABHAN, A sufficient condition for starlikeness, *Chinese J. Math.*, **23** (1995), 167–171.