Journal of Inequalities in Pure and Applied Mathematics

http://jipam.vu.edu.au/

Volume 5, Issue 2, Article 31, 2004

STARLIKENESS AND CONVEXITY CONDITIONS FOR CLASSES OF FUNCTIONS DEFINED BY SUBORDINATION

R. AGHALARY, JAY M. JAHANGIRI, AND S.R. KULKARNI

UNIVERSITY OF URMIA, URMIA, IRAN.
raghalary@yahoo.com

KENT STATE UNIVERSITY, OHIO, USA.
jay@geauga.kent.edu

FERGUSSEN COLLEGE, PUNE, INDIA.
srkulkarni40@hotmail.com

Received 21 May, 2003; accepted 18 April, 2004
Communicated by H. Silverman

ABSTRACT. We consider the family $\mathcal{P}(1,b),\ b>0$, consisting of functions p analytic in the open unit disc U with the normalization $p(0)=1$ which have the disc formulation $|p-1|<b$ in U. Applying the subordination properties to certain choices of p using the functions $f_n(z)=z+\sum_{k=1+n}^{\infty}a_kz^k,\ n=1,2,...$, we obtain inclusion relations, sufficient starlikeness and convexity conditions, and coefficient bounds for functions in these classes. In some cases our results improve the corresponding results appeared in print.

Key words and phrases: Subordination, Hadamard product, Starlike, Convex.

2000 *Mathematics Subject Classification.* Primary 30C45.

1. INTRODUCTION

Let $\mathcal{A}$ denote the class of functions that are analytic in the open unit disc $U=\{z\in\mathcal{C}:|z|<1\}$ and let $\mathcal{A}_n$ be the subclass of $\mathcal{A}$ consisting of functions f_n of the form

(1.1) $$f_n(z)=z+\sum_{k=1+n}^{\infty}a_kz^k,\quad n=1,2,3,\dots\ .$$

The function $p\in\mathcal{A}$ and normalized by $p(0)=1$ is said to be in $\mathcal{P}(1,b)$ if

(1.2) $$|p(z)-1|<b,\quad b>0,\qquad z\in U.$$

The class $\mathcal{P}(1,b)$ which is defined using the disc formulation (1.2) was studied by Janowski [6] and has an alternative characterization in terms of subordination (see [5] or [14]), that is, for

ISSN (electronic): 1443-5756

069-03

$z \in U$, we have

$$p \in \mathcal{P}(1,b) \iff p(z) \prec 1 + bz. \tag{1.3}$$

For the functions ϕ and ψ in $\mathcal{A}$, we say that the ϕ is subordinate to ψ in U, denoted by $\phi \prec \psi$, if there exists a function $w(z)$ in $\mathcal{A}$ with $w(0) = 0$ and $|w(z)| < 1$, such that $\phi(z) = \psi(w(z))$ in U. For further references see Duren [3].

The family $\mathcal{P}(1,b)$ contains many interesting classes of functions which have close interrelations with different well-known classes of analytic univalent functions. For example, for $f_n \in \mathcal{A}_n$ if

$$\left(\frac{zf_n'}{f_n}\right) \in \mathcal{P}(1, 1-\alpha), \qquad 0 \le \alpha \le 1,$$

then f_n is starlike of order α in U and if

$$\left(1+\frac{zf_n''}{f_n'}\right) \in \mathcal{P}(1, 1-\alpha), \qquad 0 \le \alpha \le 1,$$

then f_n is convex of order α in U.

For $0 \le \alpha \le 1$ we let $\mathcal{S}^*(\alpha)$ be the class of functions $f_n \in \mathcal{A}_n$ which are starlike of order α in U, that is,

$$\mathcal{S}^*(\alpha) \equiv \left\{ f_n \in \mathcal{A}_n : \Re\left(\frac{zf_n'}{f_n}\right) \ge \alpha, \quad |z| < 1 \right\},$$

and let $\mathcal{K}(\alpha)$ be the class of functions $f_n \in \mathcal{A}_n$ which are convex of order α in U, that is,

$$\mathcal{K}(\alpha) \equiv \left\{ f_n \in \mathcal{A}_n : \Re\left(1+\frac{zf_n''}{f_n'}\right) \ge \alpha, \quad |z| < 1 \right\}.$$

Alexander [1] showed that f_n is convex in U if and only if zf_n' is starlike in U.

In this paper we investigate inclusion relations, starlikeness, convexity, and coefficient conditions on f_n and its related classes for two choices of $p(f_n)$ in $\mathcal{P}(1,b)$. In some cases, we improve the related known results appeared in the literature.

Define $\mathcal{F}(1,b)$ be the subclass of $\mathcal{P}(1,b)$ consisting of functions $p(f_1)$ so that

$$p(f_1(z)) = \frac{zf_1'(z)}{f_1(z)}\left(1+\frac{zf_1''(z)}{f_1'(z)}\right) \tag{1.4}$$

where $f_1 \in \mathcal{A}_1$ is given by (1.1).

For fixed $v > -1,\ n \ge 1$, and for $\lambda \ge 0$, define $\mathcal{M}_\lambda^v(1,b)$ be the subclass of $\mathcal{P}(1,b)$ consisting of functions $p(f_n)$ so that

$$p(f_n(z)) = (1-\lambda)\frac{D^v f_n(z)}{z} + \lambda(D^v f_n(z))' \tag{1.5}$$

where $f_n \in \mathcal{A}_n$ and $D^v f$ is the v-th order Ruscheweyh derivative [10].

The v-th order Ruscheweyh derivative D^v of a function f_n in $\mathcal{A}_n$ is defined by

$$D^v f_n(z) = \frac{z}{(1-z)^{1+v}} * f_n(z) = z + \sum_{k=1+n}^{\infty} B_k(v) a_k z^k, \tag{1.6}$$

where

$$B_k(v) = \frac{(1+v)(2+v)\cdots(v+k-1)}{(k-1)!}$$

and the operator " $*$ " stands for the convolution or Hadamard product of two power series

$$f(z) = \sum_{i=1}^{\infty} a_i z^i \quad \text{and} \quad g(z) = \sum_{i=1}^{\infty} b_i z^i$$

defined by

$$(f * g)(z) = f(z) * g(z) = \sum_{i=1}^{\infty} a_i b_i z^i.$$

2. THE FAMILY $\mathcal{F}(1, b)$

The class $\mathcal{F}(1, b)$ for certain values of b yields a sufficient starlikeness condition for the functions $f_1 \in \mathcal{A}_1$.

Theorem 2.1. *If* $0 < b \leq \frac{9}{4}$ *and* $p(f_1) \in \mathcal{F}(1, b)$ *then*

$$\frac{zf_1'}{f_1} \in \mathcal{P}\left(1, \frac{3 - \sqrt{9 - 4b}}{2}\right).$$

We need the following lemma, which is due to Jack [4].

Lemma 2.2. *Let* $w(z)$ *be analytic in* U *with* $w(0) = 0$. *If* $|w|$ *attains its maximum value on the circle* $|z| = r$ *at a points* z_0, *we can write* $z_0 w'(z_0) = kw(z_0)$ *for some real* $k,\ k \geq 1$.

Proof of Theorem 2.1. For $b_1 = \frac{3-\sqrt{9-4b}}{2}$ write $\frac{zf_1'(z)}{f_1(z)} = 1 + b_1 w(z)$. Obviously, w is analytic in U and $w(0) = 0$. The proof is complete if we can show that $|w| < 1$ in U. On the contrary, suppose that there exists $z_0 \in U$ such that $|w(z_0)| = 1$. Then, by Lemma 2.2, we must have $z_0 w'(z_0) = kw(z_0)$ for some real $k,\ k \geq 1$ which yields

$$\begin{aligned}\left|\frac{z_0 f_1'(z_0)}{f_1(z_0)}\left(1 + \frac{z_0 f_1''(z_0)}{f_1'(z_0)}\right) - 1\right| &= \left|(1 + b_1 w(z_0))^2 + b_1 z_0 w'(z_0) - 1\right| \\ &= \left|b_k + 2b_1 + b_1^2 w(z_0)\right| \\ &\geq 3b_1 - b_1^2 = b.\end{aligned}$$

This contradicts the hypothesis, and so the proof is complete. □

Corollary 2.3. *For* $0 < b \leq 2$ *let* $p(f_1) \in \mathcal{F}(1, b)$. *Then* $f_1 \in \mathcal{S}^*\left(\frac{-1+\sqrt{9-4b}}{2}\right)$.

Corollary 2.4. *If* $p(f_1) \in \mathcal{F}(1, b)$ *and* $0 < b \leq 2$, *then*

$$\left|\arg \frac{zf_1'(z)}{f_1(z)}\right| < \arcsin\left(\frac{3 - \sqrt{9 - 4b}}{2}\right).$$

It is not known if the above corollaries can be extended to the case when $b > 2$.

Corollary 2.5. *If* $\Re\left(\frac{f_1(z)}{zf_1'(z)+z^2 f_1''(z)}\right) > \frac{1}{2}$ *then* $f_1 \in \mathcal{S}^*\left(\frac{-1+\sqrt{5}}{2}\right)$.

Remark 2.6. For $0 < b < 2$, Theorem 2.1 is an improvement to Theorem 1 obtained by Obradović, Joshi, and Jovanović [8].

Corollary 2.7. *If* $p(f_1) \in \mathcal{F}(1, b)$ *then* f_1 *is convex in* U *for* $0 < b \leq 0.935449$.

Proof. For $p(f_1) \in \mathcal{F}(1, b)$ we can write $|\arg p(f_1)| < \arcsin b$. Therefore,

$$\left|\arg\left(1 + \frac{zf_1''(z)}{f_1'(z)}\right)\right| < \arcsin b + \arcsin\left(\frac{3 - \sqrt{9 - 4b}}{2}\right).$$

Now the proof is complete upon noting that the right hand side of the above inequality is less than $\frac{\pi}{2}$ for $b = 0.935449$. □

Remark 2.8. It is not known if the above Corollary 2.7 is sharp but it is an improvement to Corollary 2 obtained by Obradovic, Joshi, and Jovanovic [8].

Corollary 2.9. *If* $p(f_1) \in \mathcal{F}(1,b)$ *then* f_1 *is convex in the disc* $|z| < \frac{0.935449}{b}$ *for* $0.935449 \leq b \leq 1$.

Proof. We write $p(f_1) = 1 + bw(z)$ where w is a Schwarz function. Let $|z| \leq \rho$. Then $|w(z)| \leq \rho$ and so $|p(f_1) - 1| < b\rho$ for $|z| \leq \rho$. Upon choosing $b\rho = 0.935449$ it follows from the above Corollary 2.7 that $|\arg(1 + zf_1''/f_1')| < \pi/2$ for $|z| \leq \rho = 0.935449/b$. Therefore the proof is complete. ☐

In the following example we show that there exist functions f which are not necessarily starlike or univalent in U for $p(f_1) \in \mathcal{F}(1,b)$ if b is sufficiently large.

Example 2.1. For the spirallike function $g(z) = z/(1-z)^{1+i}$ we have

$$\Re\left(e^{-\frac{\pi}{4}i}\frac{zg'(z)}{g(z)}\right) = \frac{1}{\sqrt{2}}\left(\frac{1-|z|^2}{|1-z|^2}\right) > 0, \quad z \in U.$$

Since $\frac{zg'(z)}{g(z)} = \frac{1+iz}{1-z}$, we obtain

$$\Re\left(\frac{zg'(z)}{g(z)}\right) = \frac{1 - r(\cos\theta + \sin\theta)}{1 - 2r\cos\theta + r^2}$$

for $z = re^{i\theta}$. Thus $g(z)$ is not starlike for $|z| < t,\ \frac{1}{\sqrt{2}} < t < 1$. This means that $f(z) = \frac{g(rz)}{r}$ is not starlike in U. Now set

$$h(z) = \int_0^z \frac{g(\zeta)}{\zeta} d\zeta = i((1-z)^{-i} - 1)$$

and let $z_0 = \frac{e^{2\pi}-1}{e^{2\pi}+1} \approx 0.996$. Therefore, $h(z_0) = h(-z_0)$ and so h is not univalent in U. Consequently, $f(z) = \frac{h(z_0 z)}{z_0}$ is not univalent in U for sufficiently large values of b. On the other hand, $p(g) \in \mathcal{F}(1,b)$ for sufficiemtly large b, since,

$$|p(g(z)) - 1| = \left|\frac{1+3iz}{(1-z)^2} + \frac{z}{1-z} - 1\right| < b$$

for sufficiemtly large b.

The following theorem is the converse of Theorem 2.1 for a special case.

Theorem 2.10. *If* $\frac{zf_1'}{f_1} \in \mathcal{P}\left(1, \frac{3-\sqrt{5}}{2}\right)$ *then* $p(f_1) \in \mathcal{F}(1,1)$ *for* $|z| < r_0 = 0.7851$.

To prove our theorem, we need the following lemma due to Dieudonné [2].

Corollary 2.11. *Let* z_0 *and* w_0 *be given points in* U, *with* $z_0 \neq 0$. *Then for all functions* f *analytic and satisfying* $|f(z)| < 1$ *in* U, *with* $f(0) = 0$ *and* $f(z_0) = w_0$, *the region of values of* $f'(z_0)$ *is the closed disc*

$$\left|w - \frac{w_0}{z_0}\right| \leq \frac{|z_0|^2 - |w_0|^2}{|z_0|(1 - |z_0|^2)}.$$

Proof of Theorem 2.10. Write

$$q(z) = \frac{zf_1'(z)}{f_1(z)} = 1 + \left(\frac{3-\sqrt{5}}{2}\right) w(z),$$

where w is a Schwarz function. We need to find the largest disc $|z| < \rho$ for which

$$\left|\left[1 + \left(\frac{3-\sqrt{5}}{2}\right) w(z)\right]^2 + \left(\frac{3-\sqrt{5}}{2}\right) zw'(z) - 1\right|$$
$$= \left|\left(\frac{3-\sqrt{5}}{2}\right)^2 w^2(z) + (3-\sqrt{5})w(z) + \left(\frac{3-\sqrt{5}}{2}\right) zw'(z)\right| < 1.$$

For fixed $r = |z|$ and $R = |w(z)|$ we have $R \leq r$. Therefore, by Lemma 2.11, we obtain

$$|w'(z)| \leq \frac{R}{r} + \frac{r^2 - R^2}{r(1-r^2)}$$

and so

$$\begin{aligned}
|p(f_1) - 1| &= \left|\frac{zf_1'(z)}{f_1(z)}\left(1 + \frac{zf_1''(z)}{f_1'(z)}\right) - 1\right| \\
&= \left|\left(\frac{3-\sqrt{5}}{2}\right)^2 w^2(z) + (3-\sqrt{5})w(z) + \left(\frac{3-\sqrt{5}}{2}\right) zw'(z)\right| \\
&\leq t^2R^2 + 3tR + t\frac{r^2 - R^2}{1-r^2} \\
&= \frac{t}{1-r^2}\psi(R),
\end{aligned}$$

where

$$\psi(R) = R^2(t - tr^2 - 1) + 3R(1-r^2) + r^2 \text{ and } t = \frac{3-\sqrt{5}}{2}.$$

We note that $\psi(R)$ attains its maximum at $R_0 = \frac{3(1-r^2)}{2(1+tr^2-t)}$. So the theorem follows for $r_0 \approx 0.7851$ which is the root of the equation $\frac{t}{1-r^2}\psi(R_0) = 1$.

Letting z_0 and w_0 in Lemma 2.11 be so that $|z_0| = r_0$ and $|w_0| = \frac{3(1-r_0^2)}{2(1+tr_0^2-t)}$ we conclude that the bound given by Theorem 2.10 is sharp. □

3. THE FAMILY $\mathcal{M}_\lambda^v(1,b)$

We begin with stating and proving some properties of the family $\mathcal{M}_\lambda^v(1,b)$.

Theorem 3.1. *If $p(f_n) \in \mathcal{M}_\lambda^v(1,b)$ then*

$$\frac{D^v f_n(z)}{z} \in \mathcal{P}(1, \frac{b}{1+\lambda n}).$$

We need the following lemma, which is due to Miller and Mocanu [7].

Lemma 3.2. *Let $q(z) = 1 + q_n z^n + \cdots$ $(n \geq 1)$ be analytic in U and let $h(z)$ be convex univalent in U with $h(0) = 1$. If $q(z) + \frac{1}{c}zq'(z) \prec h(z)$ for $c > 0$, then*

$$q(z) \prec \frac{c}{n} z^{-c/n} \int_0^z h(t) t^{\frac{c}{n}-1} dt.$$

Proof of Theorem 3.1. For $p(f_n) \in \mathcal{M}_\lambda^v(1,b)$ set $q(z) = \frac{D^v f_n(z)}{z}$. Then we can write $q(z) + \lambda zq'(z) \prec 1 + bz$. Now, applying Lemma 3.2, we obtain

$$q(z) \prec +1 + \frac{b}{1+\lambda n} z.$$

Substituting back for $q(z)$ and choosing $w(z)$ to be analytic in U with $|w(z)| \leq |z|^n$, by the definition of subordination we have

$$\frac{D^v f_n(z)}{z} = 1 + \frac{b}{(1+\lambda n)} w(z).$$

Now the theorem follows using the necessary and sufficient condition (1.3). The estimates in Theorem 3.1 are sharp for $p(f_n)$ where f_n is given by

$$\frac{D^v f_n(z)}{z} = 1 + \frac{b}{(1+\lambda n)} z^n.$$

□

Corollary 3.3. *If* $p(f_n) \in \mathcal{M}_\lambda^v(1,b)$ *then*

$$\left|\frac{D^v f_n(z)}{z}\right| \leq 1 + \frac{b}{1+\lambda n}|z|^n.$$

Corollary 3.4. *If* $|f_n'(z) + \lambda z f_n''(z) - 1| < b$ *then*

$$f_n'(z) \prec 1 + \frac{b}{1+\lambda n} z.$$

Corollary 3.5. *If* $\left|(1-\lambda)\frac{f_n(z)}{z} + \lambda f_n'(z) - 1\right| < b$ *then*

$$\frac{f_n(z)}{z} \prec 1 + \frac{b}{1+\lambda n} z.$$

In the next two theorems we investigate the inclusion relations for classes of $\mathcal{M}_\lambda^v$.

Theorem 3.6. *For* $0 \leq \lambda_1 < \lambda$ *and* $v \geq 0$, *let* $b_1 = \frac{1+\lambda_1 n}{1+n\lambda} b$. *Then*

$$\mathcal{M}_\lambda^v(1,b) \subset \mathcal{M}_{\lambda_1}^v(1,b_1).$$

Proof. The case for $\lambda_1 = 0$ is trivial. For $\lambda_1 \neq 0$ suppose that $p(f_n) \in \mathcal{M}_\lambda^v(1,b)$. Therefore, we can write

$$\begin{aligned}(1-\lambda_1)\frac{D^v f_n(z)}{z} &+ \lambda_1 (D^v f_n(z))' \\ &= \frac{\lambda_1}{\lambda}\left[(1-\lambda)\frac{D^v f_n(z)}{z} + \lambda (D^v f_n(z))'\right] + \left(1 - \frac{\lambda_1}{\lambda}\right)\left(\frac{D^v f_n(z)}{z}\right).\end{aligned}$$

Now, by definition, $p(f_n) \in \mathcal{M}_{\lambda_1}^v(1,b_1)$ and so the proof is complete. □

Theorem 3.7. *Let* $v \geq 0$ *and* $b_1 = \frac{b(1+v)}{n+1+v}$. *Then*

$$\mathcal{M}_\lambda^{v+1}(1,b) \subset \mathcal{M}_\lambda^v(1,b_1).$$

Proof. For $f_n \in \mathcal{A}_n$ suppose that $p_1(f_n) \in \mathcal{M}_\lambda^{v+1}(1,b)$ where

$$p_1(f_n(z)) = (1-\lambda)\frac{D^{1+v} f_n(z)}{z} + \lambda (D^{v+1} f_n(z))'.$$

Set

$$p_2(f_n(z)) = (1-\lambda)\frac{D^v f_n(z)}{z} + \lambda (D^v f_n(z))'.$$

An elementary differentiation yields

$$\begin{aligned} p_1(f_n(z)) &= (1-\lambda)\frac{D^{1+v} f_n(z)}{z} + \lambda(D^{v+1} f_n(z))' \\ &= p_2(f_n(z)) + \frac{1}{1+v} z p_2'(f_n(z)). \end{aligned}$$

From this and Lemma 3.2, we conclude that $p_1(f_n) \in \mathcal{M}_\lambda^v(1, b_1)$. □

Corollary 3.8.

$$f_n'(z) + \lambda z f_n''(z) \in \mathcal{P}(1,b) \Longrightarrow (1-\lambda)\frac{f_n(z)}{z} + \lambda f_n'(z) \in \mathcal{P}\left(1, \frac{b}{1+n}\right).$$

Theorem 3.9. *For* $v \geq 0$ *and* $\lambda > 0$ *let* $b < 1 + \lambda n$. *If* $p(f_n) \in \mathcal{M}_\lambda^v(1, b)$ *then*

$$\left| \frac{z(D^v f_n(z))'}{D^v f_n(z)} - 1 \right| < \frac{b(2+\lambda n)}{\lambda[(1+\lambda n) - b]}.$$

Proof. First note that, we can write

$$\left|(1-\lambda)\frac{D^v f_n(z)}{z} + \lambda(D^v f_n(z))' - 1\right| < b\,; \quad \left|\frac{D^v f_n(z)}{z} - 1\right| < \frac{b}{1+\lambda n}.$$

For $b_1 = \frac{b(2+\lambda n)}{\lambda[(1+\lambda n)-b]}$ we define $w(z)$ by

$$1 + b_1 w(z) = \frac{[z(D^v f_n(z))']}{[D^v f_n(z)]}.$$

One can easily verify that $w(z)$ is analytic in U and $w(0) = 0$. To conclude the proof, it suffices to show that $|w(z)| < 1$ in U. If this is not the case, then by Lemma 2.2, there exists a point $z_0 \in U$ such that $|w(z_0)| = 1$ and $z_0 w'(z_0) = k w(z_0)$. Therefore

$$\begin{aligned} |p(f_n(z_0)) - 1| &= \left|(1-\lambda)\frac{D^v f(z_0)}{z_0} + \lambda(D^v f(z_0))' - 1\right| \\ &= \left|\frac{D^v f_n(z_0)}{z_0}\left[(1-\lambda) + \lambda\frac{z_0(D^v f_n(z_0))'}{D^v f_n(z_0)}\right] - 1\right| \\ &= \left|\lambda\left(\frac{z_0(D^v f_n(z_0))'}{D^v f_n(z_0)} - 1\right)\frac{D^v f_n(z_0)}{z_0} + \left(\frac{D^v f_n(z_0)}{z_0} - 1\right)\right| \\ &\geq \lambda b_1\left(1 - \frac{b}{1+n\lambda}\right) - \frac{b}{1+n\lambda} = b. \end{aligned}$$

This is a contradiction to the hypothesis and so $|w(z)| < 1$ in U. □

Corollary 3.10. i) *If* $f_n'(z) \in \mathcal{P}(1, \frac{1+n}{3+n})$ *then* $\frac{z f_n'(z)}{f_n(z)} \in \mathcal{P}(1,1)$.

ii) *If* $f_n'(z) + z f_n''(z) \in \mathcal{P}(1, \frac{1+n}{3+n})$ *then* $\frac{z f_n''(z)}{f_n'(z)} \in \mathcal{P}(1,1)$.

Theorem 3.11. *Let* $p(f_n) \in \mathcal{M}_\lambda^v(1, b)$ *for some* $\lambda > 0$. *If*

$$b = \begin{cases} \dfrac{\lambda(1+\lambda n)}{2+\lambda(n-1)}\,; & 0 < \lambda \leq \dfrac{(n-3)+\sqrt{n^2+2n+9}}{2n} \\ (1+\lambda n)\sqrt{\dfrac{2\lambda - 1}{\lambda^2 n^2 + 2\lambda(1+n)}}\,; & \dfrac{(n-3)+\sqrt{n^2+2n+9}}{2n} \leq \lambda \leq 1 \end{cases}$$

then

$$\Re\left(\frac{D^{v+1} f_n(z)}{D^v f_n(z)}\right) > \frac{v}{1+v}.$$

We need the following lemma, which is due to Ponnusamy and Singh [9].

Lemma 3.12. *Let* $0 < \lambda_1 < \lambda < 1$ *and let* Q *be analytic in* U *satisfying* $Q(z) \prec 1 + \lambda_1 z$*, and* $Q(0) = 1$*. If* $q(z)$ *is analytic in* $U,\ q(0) = 1$ *and satisfies*

$$Q(z)[c + (1 - c)q(z)] \prec 1 + \lambda z,$$

where

$$c = \begin{cases} \dfrac{1-\lambda}{1+\lambda_1}, & 0 < \lambda + \lambda_1 \le 1 \\ \\ \dfrac{1-(\lambda^2+\lambda_1^2)}{2(1-\lambda_1^2)}, & \lambda^2 + \lambda_1^2 \le 1 \le \lambda + \lambda_1 \end{cases}$$

then $\mathrm{Re}\{q(z)\} > 0,\ z \in U$.

Proof of Theorem 3.11. From Theorem 3.1 and the fact $0 < b < 1 < 1 + \lambda n$ we conclude that

$$\frac{D^v f_n(z)}{z} \prec 1 + b_1 z,\ \ 0 < b_1 = \frac{b}{1 + n\lambda} < b < 1.$$

On the other hand, we may write

$$\frac{D^v f_n(z)}{z}\left[(1-\lambda) + \lambda\left(\frac{z(D^v f_n(z))'}{D^v f_n(z)}\right)\right] \prec 1 + bz.$$

Letting $Q(z) = \frac{D^v f_n(z)}{z},\ q(z) = \frac{z(D^v f_n(z))'}{D^v f_n(z)}$, and $c = 1 - \lambda$, we see that all conditions in Lemma 3.12 are satisfied. This implies that $\mathrm{Re}\, q(z) > 0$ and so the proof is complete. ☐

Corollary 3.13. *Let* $p(f_n) \in \mathcal{M}_\lambda^v(1, b)$ *for some* $\lambda > 0$*. Then* $D^v f_n$ *is starlike in the disc*

$$|z| \le \begin{cases} \dfrac{\lambda(1+n\lambda)}{(2+\lambda(n-1))b} & \text{if } \ 0 < \lambda < \lambda_1 \text{ and } b_1 \le b \le 1 \\ \dfrac{(1+\lambda n)}{b}\sqrt{\dfrac{2\lambda-1}{\lambda^2 n^2 + 2\lambda(1+n)}} & \text{if } \ \lambda_1 \le \lambda \le 1 \text{ and } b_2 \le b \le 1, \end{cases}$$

where

$$\lambda_1 = \frac{(n-3) + \sqrt{n^2 + 2n + 9}}{2n},\ b_1 = \frac{\lambda(1+n\lambda)}{[2+\lambda(n-1)]}, \quad \textit{and}$$

$$b_2 = (1 + \lambda_n)\sqrt{\frac{2\lambda - 1}{\lambda^2 n^2 + 2\lambda(1+n)}}.$$

i) *If* $f_n' \in \mathcal{P}\left(1, \frac{(1+n)}{\sqrt{1+(1+n)^2}}\right)$ *then* f_n *is starlike in* U.

ii) *If* $f_n' + z f_n'' \in \mathcal{P}\left(1, \frac{(1+n)}{\sqrt{1+(1+n)^2}}\right)$ *then* f_n *is convex in* U.

If we let $\lambda = 1$ and $v = 0, 1$ in Corollary 3.13, then we obtain

Corollary 3.14. *Let* $\frac{(1+n)}{\sqrt{1+(1+n)^2}} \le b \le 1$ *and* $f_n \in \mathcal{A}_n$.

i) *If* $f_n' \in \mathcal{P}(1, b)$ *then* f *is starlike for* $|z| < \frac{(1+n)}{b\sqrt{1+(1+n)^2}}$.

ii) *If* $f_n' + z f_n'' \in \mathcal{P}(1, b)$ *then* f *is convex for* $|z| < \frac{1+n}{b\sqrt{1+(1+n)^2}}$.

4. COEFFICIENT BOUNDS

Sufficient coefficient conditions for $\mathcal{F}(1,b)$ and $\mathcal{M}^v_\lambda(1,b)$ are given next.

Theorem 4.1. *Let $p(f_1)$ be given by (1.4) for f_1 as in (1.1). If*

$$\sum_{k=2}^{\infty}(k^2+b-1)|a_k|<b, \tag{4.1}$$

then $p(f_1)\in\mathcal{F}(1,b)$.

Proof. We need to show that if (4.1) then $|p(f_1(z))-1|<b$. For $p(f_1)$ we can write

$$\begin{aligned}|p(f_1(z))-1| &= \left|\frac{zf_1'}{f_1}\left(1+\frac{zf_1''}{f_1'}\right)-1\right|\\ &= \left|\frac{\sum_{k=2}^{\infty}(k^2-1)a_kz^k}{z+\sum_{k=2}^{\infty}a_kz^k}\right|\\ &\leq \frac{\sum_{k=2}^{\infty}(k^2-1)|a_k||z|^{k-1}}{1-\sum_{k=2}^{\infty}|a_k||z|^{k-1}}\\ &< \frac{\sum_{k=2}^{\infty}(k^2-1)|a_k|}{1-\sum_{k=2}^{\infty}|a_k|}.\end{aligned}$$

The above right hand inequality is less than b by (4.1) and so $p(f_1)\in\mathcal{F}(1,b)$. □

Theorem 4.2. *Let $p(f_n)$ be given by (1.5) for f_n as in (1.1). If*

$$\sum_{k=1+n}^{\infty}(\lambda k-\lambda+1)B_k(v)|a_k|<b, \tag{4.2}$$

then $p(f_n)\in\mathcal{M}^v_\lambda(1,b)$.

Proof. Apply the Ruscheweyh derivative (1.6) to the function $f_n(z)$ and substitute in (1.5) to obtain

$$\begin{aligned}|p(f_n(z))-1| &= \left|(1-\lambda)\frac{D^vf_n(z)}{z}+\lambda(D^vf_n(z))'-1\right|\\ &= \left|\sum_{k=1+n}^{\infty}(\lambda k-\lambda+1)B_k(v)a_kz^{k-1}\right|\\ &< \sum_{k=1+n}^{\infty}(\lambda k-\lambda+1)B_k(v)|a_k|.\end{aligned}$$

Now this latter inequality is less than b by (4.2) and so $p(f_n)\in\mathcal{M}^v_\lambda(1,b)$. □

Next, by judiciously varying the arguments of the coefficients of the functions f_n given by (1.1), we shall show that the sufficient coefficient conditions (4.1) and (4.2) are also necessary for their respective classes with varying arguments.

A function f_n given by (1.1) is said to be in $\mathcal{V}(\theta_k)$ if $\arg(a_k)=\theta_k$ for all k. If, further, there exists a real number β such that $\theta_k+(k-1)\beta\equiv\pi(\operatorname{mod}2\pi)$ then f_n is said to be in $\mathcal{V}(\theta_k;\beta)$. The union of $\mathcal{V}(\theta_k;\beta)$ taken over all possible $\{\theta_k\}$ and all possible real β is denoted by $\mathcal{V}$. For more details see Silverman [13].

Some examples of functions in $\mathcal{V}$ are

- **i)** $\mathcal{T}\equiv\mathcal{V}(\pi;0)\subset\mathcal{V}$ where $\mathcal{T}$ is the class of analytic univalent functions with negative coefficients studied by Schild [11] and Silverman [12].

ii) Univalent functions of the form $z+\sum_{k=2}^{\infty}|a_k|e^{i\theta_k}z^k$ are in $\mathcal{V}(\theta_k;2\pi/k)\subset\mathcal{V}$ for $\theta_k=\pi-2(k-1)\pi/k$.

Note that the family $\mathcal{V}$ is rotationally invariant since $f_n\in\mathcal{V}(\theta_k;\beta)$ implies that

$$e^{-i\gamma}f_n(ze^{i\gamma})\in\mathcal{V}(\theta_k+(k-1)\gamma;\beta-\gamma).$$

Finally, we let

$$\mathcal{VF}(1,b)\equiv\mathcal{V}\cap\mathcal{F}(1,b)\quad\text{and}\quad\mathcal{VM}_\lambda^v(1,b)\equiv\mathcal{V}\cap\mathcal{M}_\lambda^v(1,b).$$

Theorem 4.3.

$$p(f_1)\in\mathcal{VF}(1,b)\iff\sum_{k=2}^{\infty}(k^2+b-1)|a_k|<b.$$

Proof. In light of Theorem 4.1, we only need to prove the "*only if*" part of the theorem. Suppose $p(f_1)\in\mathcal{VF}(1,b)$, then

$$|p(f_1)-1|=\left|\frac{\sum_{k=2}^{\infty}(k^2-1)a_kz^{k-1}}{1+\sum_{k=2}^{\infty}a_kz^{k-1}}\right|<b$$

or

$$\left|\sum_{k=2}^{\infty}(k^2-1)a_kz^{k-1}\right|<b\left|1+\sum_{k=2}^{\infty}a_kz^{k-1}\right|. \tag{4.3}$$

The condition (4.3) must hold for all values of z in U. Therefore, for $f_1\in\mathcal{V}(\theta_k;\beta)$ we set $z=re^{i\beta}$ in (4.3) and let $r\longrightarrow1^-$. Upon clearing the inequality (4.3) we obtain the condition

$$\sum_{k=2}^{\infty}(k^2-1)|a_k|<b\left(1-\sum_{k=2}^{\infty}|a_k|\right)$$

as required. □

Corollary 4.4. *If* $0<b\le1$ *and* $p(f_1)\in\mathcal{VF}(1,b)$ *then* f_1 *is convex in* U.

Corollary 4.5. *If* $1<b\le3$ *and* $p(f_1)\in\mathcal{VF}(1,b)$ *then* f_1 *is starlike in* U.

The above two corollaries can be justifed using Theorem 4.3 and the following lemma due to Silverman [12].

Lemma 4.6. *For* f_1 *of the form (1.1) and univalent in* U *we have*

i) *If* $\sum_{k=2}^{\infty}k^2|a_k|\le1$, *then* f_1 *is convex in* U.
ii) *If* $\sum_{k=2}^{\infty}k|a_k|\le1$, *then* f_1 *is starlike in* U.

Next, we show that the above sufficient coefficient condition (4.2) is also necessary for functions in $\mathcal{VM}_\lambda^v(1,b)$.

Theorem 4.7.

$$p(f_n)\in\mathcal{VM}_\lambda^v(1,b)\iff\sum_{k=1+n}^{\infty}(\lambda k-\lambda+1)B_k(v)|a_k|<b.$$

Proof. Suppose that $p(f_n)\in\mathcal{VM}_\lambda^v(1,b)$. Then, by (1.5), we have

$$|p(f_n(z))-1|=\left|(1-\lambda)\frac{D^vf_n(z)}{z}+\lambda(D^vf_n(z))'-1\right|<b.$$

On the other hand, for $f_n \in \mathcal{V}(\theta_k;\beta)$ we have

$$f_n(z) = z + \sum_{k=1+n}^{\infty} |a_k| e^{i\theta_k} z^k.$$

The condition required for $p(f_n) \in \mathcal{VM}_\lambda^v(1,b)$ must hold for all values of z in U. Setting $z = re^{i\beta}$ yields

$$\sum_{k=1+n}^{\infty} (\lambda k - \lambda + 1) B_k(v) |a_k| r^{k-1} < b.$$

The required coefficient condition follows upon letting $z \longrightarrow 1^-$. □

From the above Theorem 4.7 and Lemma 4.6.ii, we obtain

Corollary 4.8. *If $\lambda \geq 2b-1$ and $p(f_n) \in \mathcal{VM}_\lambda^v(1,b)$ then f is starlike in U.*

REFERENCES

[1] J.W. ALEXANDER, Functions which map the interior of the unit circle upon simple regions, *Ann. of Math. Soc.*, **17** (1915-1916), 12–22.

[2] J. DIEUDONNÉ, Recherches sur quelques problèms relatifs aux polynômes et aux functions bornées d'une variable complexe, *Ann. Sci. Ecole Norm. Sup.*, **48** (1931), 247–358.

[3] P.L. DUREN, *Univalent Functions*, Grundlehren der Mathematischen Wissenschaften 259, A series of Comprehensive Studies in Mathematics, Springer-Verlag New York Inc, 1983.

[4] I.S. JACK, Functions starlike and convex of order α, *J. London Math. Soc.*, **3** (1971), 469–474.

[5] J.M. JAHANGIRI, H. SILVERMAN AND E.M. SILVIA, Inclusion relations between classes of functions defined by subordination, *J. Math. Anal. Appl.*, **151** (1990), 318–329.

[6] W. JANOWSKI, Extremal problems for a family of functions with positive real part and for some related families, *Ann. Polon. Math.*, **23** (1970), 159–177.

[7] S.S. MILLER AND P.T. MOCANU, Differential subordination and univalent functions, *Michigan Math. J.*, **28** (1981), 157–171.

[8] M. OBRADOVIĆ, S.B. JOSHI AND I. JOVANOVIĆ, On certain sufficient conditions for starlikeness and convexity, *Indian J. Pure Appl. Math.*, **29** (1998), 271–275.

[9] S. PONNUSAMY AND V. SINGH, Convolution properties of some classes of analytic functions, *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov.*, (POMI) **226** (1996), 138–154.

[10] St. RUSCHEWEYH, New criteria for univalent functions, *Proc. Amer. Math. Soc.*, **49** (1975), 109–115.

[11] A. SCHILD, On a class of functions schlicht in the unit circle, *Proc. Amer. Math. Soc.*, **5** (1954), 115–120.

[12] H. SILVERMAN, Univalent functions with negative coefficients, *Proc. Amer. Math. Soc.*, **51** (1975), 109–116.

[13] H. SILVERMAN, Univalent functions with varying arguments, *Houston J. Math.*, **7** (1981), 283–287.

[14] H. SILVERMAN AND E. M. SILVIA, Subclasses of starlike functions subordinate to convex functions, *Canad. J. Math.*, **37** (1985), 48–61.