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ON SOME NEW MEAN VALUE INEQUALITIES

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ABSTRACT. In this paper, using the arithmetic-geometric mean inequality, we obtain some new mean value inequalities. Finally, some applications are given, they are extension of Hölder's inequalities.

Key words and phrases: Mean value inequality, Hölder's inequality, Continuous positive function, Extension.

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1. INTRODUCTION AND MAIN RESULTS

Let $a > 0$, $b > 0$ and $t \in (0, 1)$. It is well-known that the following arithmetic-geometric mean inequality holds

$$a^t b^{1-t} \le ta + (1-t)b. \tag{1.1}$$

The arithmetic-geometric mean inequality is a classical inequality with many applications. Also, there exist extensive works devoted to generalizing or improving the arithmetic-geometric mean inequality. In this respect, we refer the reader to [1] – [7] and the references cited therein for updated results.

In this paper, by (1.1), we obtain some new mean value inequalities. Finally, some applications are given.

In this paper, we agree

$$\sum_{i=q+1}^{q} b_i = 0, \qquad (b_i \in \mathbb{R}, \quad q \in \mathbb{N}).$$

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Theorem 1.1. *Let* $x_i > 0 \ \ (i = 1, 2, \dots, n;\ n \geq 2)$ *and* $t \in (0, 1)$.

(1) *For the following*

$$B(k) \triangleq \frac{1}{n^2}\left[k\sum_{i=1}^{k} x_i + \left(\sum_{i=1}^{n} x_i^t\right)\left(\sum_{i=k+1}^{n} x_i^{1-t}\right) + \left(\sum_{i=k+1}^{n} x_i^t\right)\left(\sum_{i=1}^{k} x_i^{1-t}\right)\right], \quad (k = 1, 2, \dots, n)$$

and

$$C(j) \triangleq \frac{1}{n^2}\left[(n-j+1)\sum_{i=j}^{n} x_i + \left(\sum_{i=1}^{n} x_i^t\right)\left(\sum_{i=1}^{j-1} x_i^{1-t}\right) + \left(\sum_{i=1}^{j-1} x_i^t\right)\left(\sum_{i=j}^{n} x_i^{1-t}\right)\right], \quad (j = 1, 2, \dots, n),$$

we have

$$\left(\frac{1}{n}\sum_{i=1}^{n} x_i^t\right)\left(\frac{1}{n}\sum_{i=1}^{n} x_i^{1-t}\right) = B(1) \leq B(2) \leq \cdots \leq B(k) \leq B(k+1) \leq \cdots \leq B(n) = \frac{1}{n}\sum_{i=1}^{n} x_i \tag{1.2}$$

and

$$\left(\frac{1}{n}\sum_{i=1}^{n} x_i^t\right)\left(\frac{1}{n}\sum_{i=1}^{n} x_i^{1-t}\right) = C(n) \leq C(n-1) \leq \cdots \leq C(j) \leq C(j-1) \leq \cdots \leq C(1) = \frac{1}{n}\sum_{i=1}^{n} x_i. \tag{1.3}$$

(2) *For* $1 \leq j < k < l \leq n \ (n \geq 3)$, *we have*

$$(k-j+1)\sum_{i=j}^{k} x_i + (l-k+1)\sum_{i=k}^{l} x_i + \left(\sum_{i=j}^{l} x_i^t\right)\left(\sum_{i=j}^{l} x_i^{1-t}\right) \leq (l-j+1)\sum_{i=j}^{l} x_i + \left(\sum_{i=j}^{k} x_i^t\right)\left(\sum_{i=j}^{k} x_i^{1-t}\right) + \left(\sum_{i=k}^{l} x_i^t\right)\left(\sum_{i=k}^{l} x_i^{1-t}\right). \tag{1.4}$$

Corollary 1.2. *Let* $x_i > 0 \ (i = 1, 2, \dots, n,\ n \geq 2)$ *and* p, q *be any two positive numbers.*

(1) *For*

$$D(k) \triangleq \frac{1}{n^2}\left[k\sum_{i=1}^{k} x_i^{p+q} + \left(\sum_{i=1}^{n} x_i^p\right)\left(\sum_{i=k+1}^{n} x_i^q\right) + \left(\sum_{i=k+1}^{n} x_i^p\right)\left(\sum_{i=1}^{k} x_i^q\right)\right], \quad (k = 1, 2, \dots, n)$$

and

$$E(j) \triangleq \frac{1}{n^2}\left[(n-j+1)\sum_{i=j}^{n} x_i^{p+q} + \left(\sum_{i=1}^{n} x_i^p\right)\left(\sum_{i=1}^{j-1} x_i^q\right) + \left(\sum_{i=1}^{j-1} x_i^p\right)\left(\sum_{i=j}^{n} x_i^q\right)\right], \quad (j = 1, 2, \ldots, n),$$

we have

$$\left(\frac{1}{n}\sum_{i=1}^{n} x_i^p\right)\left(\frac{1}{n}\sum_{i=1}^{n} x_i^q\right) = D(1) \le D(2) \le \cdots \le D(k) \le D(k+1) \le \cdots \le D(n) = \frac{1}{n}\sum_{i=1}^{n} x_i^{p+q} \tag{1.5}$$

and

$$\left(\frac{1}{n}\sum_{i=1}^{n} x_i^p\right)\left(\frac{1}{n}\sum_{i=1}^{n} x_i^q\right) = E(n) \le E(n-1) \le \cdots \le E(j) \le E(j-1) \le \cdots \le E(1) = \frac{1}{n}\sum_{i=1}^{n} x_i^{p+q}. \tag{1.6}$$

(2) *For* $1 \le j < k < l \le n$ $(n \ge 3)$, *we have*

$$(k-j+1)\sum_{i=j}^{k} x_i^{p+q} + (l-k+1)\sum_{i=k}^{l} x_i^{p+q} + \left(\sum_{i=j}^{l} x_i^p\right)\left(\sum_{i=j}^{l} x_i^q\right) \le (l-j+1)\sum_{i=j}^{l} x_i^{p+q} + \left(\sum_{i=j}^{k} x_i^p\right)\left(\sum_{i=j}^{k} x_i^q\right) + \left(\sum_{i=k}^{l} x_i^p\right)\left(\sum_{i=k}^{l} x_i^q\right). \tag{1.7}$$

2. PROOF OF THEOREM AND COROLLARY

Proof of Theorem 1.1. (1) Two equalities are clear in (1.2). To complete the proof of (1.2), we only need to prove that $B(k) \le B(k+1)$ $(1 \le k \le n-1)$. Indeed, from (1.1) we have

$$x_{k+1}^t \sum_{i=1}^{k} x_i^{1-t} = \sum_{i=1}^{k} x_{k+1}^t x_i^{1-t} \le \sum_{i=1}^{k} \left(t x_{k+1} + (1-t)x_i\right), \tag{2.1}$$

and

$$x_{k+1}^{1-t} \sum_{i=1}^{k} x_i^{t} = \sum_{i=1}^{k} x_{k+1}^{1-t} x_i^{t} \le \sum_{i=1}^{k} \left((1-t) x_{k+1} + t x_i\right). \tag{2.2}$$

Using (2.1) and (2.2), after a simple manipulation we get

$$x_{k+1}^t \sum_{i=1}^{k} x_i^{1-t} + x_{k+1}^{1-t} \sum_{i=1}^{k} x_i^{t} \le k x_{k+1} + \sum_{i=1}^{k} x_i. \tag{2.3}$$

For $k = 1, 2, \dots, n-1$, by (2.3) we get

$$\begin{aligned}
B(k) &= \frac{1}{n^2}\left[k\sum_{i=1}^{k} x_i + \left(\sum_{i=1}^{n} x_i^t\right)\left(\sum_{i=k+1}^{n} x_i^{1-t}\right) + \left(\sum_{i=k+1}^{n} x_i^t\right)\left(\sum_{i=1}^{k} x_i^{1-t}\right)\right] \\
&= \frac{1}{n^2}\left[k\sum_{i=1}^{k} x_i + x_{k+1} + \left(\sum_{i=1}^{n} x_i^t\right)\left(\sum_{i=k+2}^{n} x_i^{1-t}\right)\right. \\
&\qquad \left. + \left(\sum_{i=1}^{k} x_i^t + \sum_{i=k+2}^{n} x_i^t\right)x_{k+1}^{1-t} + \left(\sum_{i=k+2}^{n} x_i^t\right)\left(\sum_{i=1}^{k} x_i^{1-t}\right) + x_{k+1}^t\sum_{i=1}^{k} x_i^{1-t}\right] \\
&= \frac{1}{n^2}\left[k\sum_{i=1}^{k} x_i + x_{k+1} + x_{k+1}^t\sum_{i=1}^{k} x_i^{1-t} + x_{k+1}^{1-t}\sum_{i=1}^{k} x_i^t\right. \\
&\qquad \left. + \left(\sum_{i=1}^{n} x_i^t\right)\left(\sum_{i=k+2}^{n} x_i^{1-t}\right) + \left(\sum_{i=k+2}^{n} x_i^t\right)\left(\sum_{i=1}^{k+1} x_i^{1-t}\right)\right] \\
&\le \frac{1}{n^2}\left[k\sum_{i=1}^{k} x_i + x_{k+1} + kx_{k+1} + \sum_{i=1}^{k} x_i\right. \\
&\qquad \left. + \left(\sum_{i=1}^{n} x_i^t\right)\left(\sum_{i=k+2}^{n} x_i^{1-t}\right) + \left(\sum_{i=k+2}^{n} x_i^t\right)\left(\sum_{i=1}^{k+1} x_i^{1-t}\right)\right] \\
&= \frac{1}{n^2}\left[(k+1)\sum_{i=1}^{k+1} x_i + \left(\sum_{i=1}^{n} x_i^t\right)\left(\sum_{i=k+2}^{n} x_i^{1-t}\right) + \left(\sum_{i=k+2}^{n} x_i^t\right)\left(\sum_{i=1}^{k+1} x_i^{1-t}\right)\right] \\
&= B(k+1).
\end{aligned}$$

By same arguments of proof for (1.2), we can also get inequalities in (1.3).

(2) For $1 \le j < k < l \le n$, from (1.1) we have

$$\begin{aligned}
\left(\sum_{i=j}^{k-1} x_i^t\right)\left(\sum_{i=k+1}^{l} x_i^{1-t}\right) &= \sum_{i=j}^{k-1}\sum_{s=k+1}^{l} x_i^t x_s^{1-t} \\
&\le \sum_{i=j}^{k-1}\sum_{s=k+1}^{l} (tx_i + (1-t)x_s) \\
&= (l-k)\sum_{i=j}^{k-1} tx_i + (k-j)\sum_{i=k+1}^{l}(1-t)x_i
\end{aligned} \tag{2.4}$$

and

$$\left(\sum_{i=k+1}^{l} x_i^t\right)\left(\sum_{i=j}^{k-1} x_i^{1-t}\right) \le (l-k)\sum_{i=j}^{k-1}(1-t)x_i + (k-j)\sum_{i=k+1}^{l} tx_i. \tag{2.5}$$

Using (2.4) and (2.5), after a simple manipulation we have

$$\left(\sum_{i=j}^{k-1} x_i^t\right)\left(\sum_{i=k+1}^{l} x_i^{1-t}\right)+\left(\sum_{i=k+1}^{l} x_i^t\right)\left(\sum_{i=j}^{k-1} x_i^{1-t}\right) \leq (l-k)\sum_{i=j}^{k-1} x_i+(k-j)\sum_{i=k+1}^{l} x_i. \tag{2.6}$$

From (2.6) we obtain

$$\begin{aligned}
&(k-j+1)\sum_{i=j}^{k} x_i+(l-k+1)\sum_{i=k}^{l} x_i+\left(\sum_{i=j}^{l} x_i^t\right)\left(\sum_{i=j}^{l} x_i^{1-t}\right)\\
&= (k-j+1)\sum_{i=j}^{k} x_i+(l-k+1)\sum_{i=k+1}^{l} x_i+(l-k)x_k\\
&\qquad+\left(\sum_{i=j}^{k} x_i^t\right)\left(\sum_{i=j}^{k} x_i^{1-t}\right)+\left(\sum_{i=k+1}^{l} x_i^t\right)\left(\sum_{i=k+1}^{l} x_i^{1-t}\right)\\
&\qquad+x_k^t\sum_{i=k+1}^{l} x_i^{1-t}+x_k^{1-t}\sum_{i=k+1}^{l} x_i^t+x_k\\
&\qquad+\left(\sum_{i=j}^{k-1} x_i^t\right)\left(\sum_{i=k+1}^{l} x_i^{1-t}\right)+\left(\sum_{i=k+1}^{l} x_i^t\right)\left(\sum_{i=j}^{k-1} x_i^{1-t}\right)\\
&\leq (k-j+1)\sum_{i=j}^{k} x_i+(l-k+1)\sum_{i=k+1}^{l} x_i+(l-k)x_k\\
&\qquad+\left(\sum_{i=j}^{k} x_i^t\right)\left(\sum_{i=j}^{k} x_i^{1-t}\right)+\left(\sum_{i=k}^{l} x_i^t\right)\left(\sum_{i=k}^{l} x_i^{1-t}\right)\\
&\qquad+(l-k)\sum_{i=j}^{k-1} x_i+(k-j)\sum_{i=k+1}^{l} x_i\\
&= (l-j+1)\sum_{i=j}^{l} x_i+\left(\sum_{i=j}^{k} x_i^t\right)\left(\sum_{i=j}^{k} x_i^{1-t}\right)+\left(\sum_{i=k}^{l} x_i^t\right)\left(\sum_{i=k}^{l} x_i^{1-t}\right),
\end{aligned}$$

which implies (1.4).

This completes the proof of Theorem 1.1. □

Proof of Corollary 1.2. Replace t, $1-t$ and x_i in Theorem 1.1 by $\frac{p}{p+q}$, $\frac{q}{p+q}$ and x_i^{p+q}, respectively. We obtain Corollary 1.2. □

3. APPLICATIONS

Proposition 3.1. *Let* $x_{ir} > 0$ $(i = 1, 2, \ldots, n,\ n \geq 2;\ r = 1, 2, \ldots, m,\ m \geq 2)$ *and* $t \in (0, 1)$. *For*

$$F(k) \triangleq \frac{1}{n^2}\left[k\sum_{i=1}^{k}\left(\sum_{r=1}^{m}x_{ir}\right) + \left(\sum_{i=1}^{n}\left(\sum_{r=1}^{m}x_{ir}\right)^t\right)\left(\sum_{i=k+1}^{n}\left(\sum_{r=1}^{m}x_{ir}\right)^{1-t}\right)\right.$$
$$\left. + \left(\sum_{i=k+1}^{n}\left(\sum_{r=1}^{m}x_{ir}\right)^t\right)\left(\sum_{i=1}^{k}\left(\sum_{r=1}^{m}x_{ir}\right)^{1-t}\right)\right], \quad (k = 1, 2, \ldots, n)$$

and

$$G(h) \triangleq \frac{1}{n^2}\left[\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\left(\sum_{r=1}^{h}x_{ir}\right)^t\left(\sum_{r=1}^{h}x_{jr}\right)^{1-t} + \sum_{r=h+1}^{m}x_{ir}^t x_{jr}^{1-t}\right)\right], \quad (h = 1, 2, \ldots, m),$$

we have

$$\begin{aligned}(3.1)\qquad &\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\sum_{r=1}^{m}x_{ir}^t x_{jr}^{1-t}\right)\\ &= G(1) \leq G(2) \leq \cdots \leq G(h) \leq G(h+1) \leq \cdots \leq G(m)\\ &= \left(\frac{1}{n}\sum_{i=1}^{n}\left(\sum_{r=1}^{m}x_{ir}\right)^t\right)\left(\frac{1}{n}\sum_{i=1}^{n}\left(\sum_{r=1}^{m}x_{ir}\right)^{1-t}\right)\\ &= F(1) \leq F(2) \leq \cdots \leq F(k) \leq F(k+1) \leq \cdots \leq F(n)\\ &= \frac{1}{n}\sum_{i=1}^{n}\sum_{r=1}^{m}x_{ir}.\end{aligned}$$

Proof. For $x_{ir} > 0$, $x_{jr} > 0$ ($1 \leq i, j \leq n,\ r = 1, 2, \ldots, m$) and $t \in (0, 1)$. We write

$$P(i, j; h) \triangleq \left(\sum_{r=1}^{h}x_{ir}\right)^t\left(\sum_{r=1}^{h}x_{jr}\right)^{1-t} + \sum_{r=h+1}^{m}x_{ir}^t x_{jr}^{1-t} \quad (h = 1, 2, \ldots, m).$$

The first named author of this paper showed in [8] that the following chain of Hölder's inequalities holds

$$\begin{aligned}(3.2)\qquad \sum_{r=1}^{m}x_{ir}^t x_{jr}^{1-t} &= P(i, j; 1)\\ &\leq P(i, j; 2) \leq \cdots \leq P(i, j; h) \leq P(i, j; h+1) \leq \cdots \leq P(i, j; m)\\ &= \left(\sum_{r=1}^{m}x_{ir}\right)^t\left(\sum_{r=1}^{m}x_{jr}\right)^{1-t}.\end{aligned}$$

From the properties of inequality and (3.2), we have

$$\begin{aligned}(3.3)\qquad \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\sum_{r=1}^{m}x_{ir}^t x_{jr}^{1-t}\right) &= \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}P(i, j; 1)\\ &\leq \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}P(i, j; 2)\end{aligned}$$

$$\begin{aligned}
&\leq \cdots \leq \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}P(i,j;h)\\
&\leq \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}P(i,j;h+1) \leq \cdots\\
&\leq \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}P(i,j;m)\\
&= \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\sum_{r=1}^{m}x_{ir}\right)^{t}\left(\sum_{r=1}^{m}x_{jr}\right)^{1-t}\\
&= \left(\frac{1}{n}\sum_{i=1}^{n}\left(\sum_{r=1}^{m}x_{ir}\right)^{t}\right)\left(\frac{1}{n}\sum_{i=1}^{n}\left(\sum_{r=1}^{m}x_{ir}\right)^{1-t}\right).
\end{aligned}$$

It is easy to see that

$$G(h) = \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}P(i,j;h), \quad h = 1,2,\ldots,m. \tag{3.4}$$

(3.3) and (3.4) imply inequalities between the first equality and the second equality in (3.1).

Replacing x_i in (1.2) by $\sum_{r=1}^{m} x_{ir}$, we obtain inequalities between the third equality and the fourth equality in (3.1).

This completes the proof of Proposition 3.1. □

Proposition 3.2. *Let* $f_i : [a,b] \mapsto (0,+\infty)$ $(a<b)$ *be continuous functions* $(i = 1,2,\ldots,n,\ n \geq 2)$ *and* $t \in (0,1)$*. For*

$$\begin{aligned}
H(k) = \frac{1}{n^2}\Bigg[&k\sum_{i=1}^{k}\left(\int_a^b f_i(x)dx\right) + \left(\sum_{i=1}^{n}\left(\int_a^b f_i(x)dx\right)^{t}\right)\left(\sum_{i=k+1}^{n}\left(\int_a^b f_i(x)dx\right)^{1-t}\right)\\
&+\left(\sum_{i=k+1}^{n}\left(\int_a^b f_i(x)dx\right)^{t}\right)\left(\sum_{i=1}^{k}\left(\int_a^b f_i(x)dx\right)^{1-t}\right)\Bigg], \quad (k = 1,2,\ldots,n)
\end{aligned}$$

and any $y \in [a,b]$*, we have*

$$\begin{aligned}
&\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\int_a^b (f_i(x))^t(f_j(x))^{1-t}dx\right)\\
&\leq \frac{1}{n^2}\left[\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\left(\int_a^y f_i(x)dx\right)^{t}\left(\int_a^y f_j(x)dx\right)^{1-t} + \int_y^b (f_i(x))^t(f_j(x))^{1-t}dx\right)\right]\\
&\leq \left(\frac{1}{n}\sum_{i=1}^{n}\left(\int_a^b f_i(x)dx\right)^{t}\right)\left(\frac{1}{n}\sum_{i=1}^{n}\left(\int_a^b f_i(x)dx\right)^{1-t}\right)\\
&= H(1) \leq H(2) \leq \cdots \leq H(k) \leq H(k+1) \leq \cdots \leq H(n)\\
&= \frac{1}{n}\sum_{i=1}^{n}\int_a^b f_i(x)dx.
\end{aligned} \tag{3.5}$$

Proof. For $1 \le i, j \le n$, $t \in (0,1)$, $y \in [a,b]$ and continuous functions $f_i : [a,b] \mapsto (0,+\infty)$ ($i = 1,2,\ldots,n; n \ge 2$), in [8], Wang also obtained the following refinement for the integral form of Hölder's inequalities:

$$\begin{aligned}
&\int_a^b (f_i(x))^t (f_j(x))^{1-t} dx \\
&\le \left(\int_a^y f_i(x)dx\right)^t \left(\int_a^y f_j(x)dx\right)^{1-t} + \int_y^b (f_i(x))^t (f_j(x))^{1-t} dx \\
&\le \left(\int_a^b f_i(x)dx\right)^t \left(\int_a^b f_j(x)dx\right)^{1-t}.
\end{aligned} \tag{3.6}$$

Using the properties of inequality and (3.6), we have

$$\begin{aligned}
&\frac{1}{n^2}\sum_{i=1}^n\sum_{j=1}^n \left(\int_a^b (f_i(x))^t (f_j(x))^{1-t} dx\right) \\
&\le \frac{1}{n^2}\sum_{i=1}^n\sum_{j=1}^n \left(\left(\int_a^y f_i(x)dx\right)^t \left(\int_a^y f_j(x)dx\right)^{1-t} + \int_y^b (f_i(x))^t (f_j(x))^{1-t} dx\right) \\
&\le \frac{1}{n^2}\sum_{i=1}^n\sum_{j=1}^n \left(\int_a^b f_i(x)dx\right)^t \left(\int_a^b f_j(x)dx\right)^{1-t} \\
&= \left(\frac{1}{n}\sum_{i=1}^n \left(\int_a^b f_i(x)dx\right)^t\right) \left(\frac{1}{n}\sum_{i=1}^n \left(\int_a^b f_i(x)dx\right)^{1-t}\right),
\end{aligned}$$

which is two inequalities of left hand in (3.2).

Replacing x_i in (1.2) by $\int_a^b f_i(x)dx$, we obtain inequalities between the two equalities in (3.2).

This completes the proof of Proposition 3.2. □

Remark 3.3. (3.1) and (3.2) are extensions of Hölder's inequalities.

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