

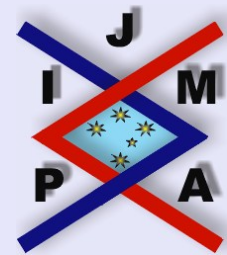
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GENERALIZED RELATIVE INFORMATION AND INFORMATION INEQUALITIES

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Abstract

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Abstract

In this paper, we have obtained bounds on Csiszár's f -divergence in terms of relative information of type s using Dragomir's [9] approach. The results obtained in particular lead us to bounds in terms of χ^2 -Divergence, Kullback-Leibler's relative information and Hellinger's discrimination.

2000 Mathematics Subject Classification: 94A17; 26D15.

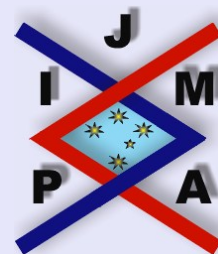
Key words: Relative information; Csiszár's f -divergence; χ^2 -divergence; Hellinger's discrimination; Relative information of type s ; Information inequalities.

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1. Introduction

Let

$$\Delta_n = \left\{ P = (p_1, p_2, \dots, p_n) \mid p_i > 0, \sum_{i=1}^n p_i = 1 \right\}, \quad n \geq 2,$$

be the set of complete finite discrete probability distributions.

The Kullback Leibler's [13] *relative information* is given by

$$(1.1) \quad K(P||Q) = \sum_{i=1}^n p_i \ln \left(\frac{p_i}{q_i} \right),$$

for all $P, Q \in \Delta_n$.

In Δ_n , we have taken all $p_i > 0$. If we take $p_i \geq 0, \forall i = 1, 2, \dots, n$, then in this case we have to suppose that $0 \ln 0 = 0 \ln \left(\frac{0}{0} \right) = 0$. From the *information theoretic* point of view we generally take all the logarithms with base 2, but here we have taken only natural logarithms.

We observe that the measure (1.1) is not symmetric in P and Q . Its symmetric version, famous as *J-divergence* (Jeffreys [12]; Kullback and Leiber [13]), is given by

$$(1.2) \quad J(P||Q) = K(P||Q) + K(Q||P) = \sum_{i=1}^n (p_i - q_i) \ln \left(\frac{p_i}{q_i} \right).$$

Let us consider the one parametric generalization of the measure (1.1), called *relative information of type s* given by

$$(1.3) \quad K_s(P||Q) = [s(s-1)]^{-1} \left[\sum_{i=1}^n p_i^s q_i^{1-s} - 1 \right], \quad s \neq 0, 1.$$



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In this case we have the following limiting cases

$$\lim_{s \rightarrow 1} K_s(P||Q) = K(P||Q),$$

and

$$\lim_{s \rightarrow 0} K_s(P||Q) = K(Q||P).$$

The expression (1.3) has been studied by Vajda [22]. Previous to it many authors studied its characterizations and applications (ref. Taneja [20] and on line book Taneja [21]).

We have some interesting particular cases of the measure (1.3).

(i) When $s = \frac{1}{2}$, we have

$$K_{1/2}(P||Q) = 4 [1 - B(P||Q)] = 4h(P||Q)$$

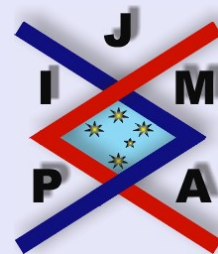
where

$$(1.4) \quad B(P||Q) = \sum_{i=1}^n \sqrt{p_i q_i},$$

is the famous as Bhattacharya's [1] *distance*, and

$$(1.5) \quad h(P||Q) = \frac{1}{2} \sum_{i=1}^n (\sqrt{p_i} - \sqrt{q_i})^2,$$

is famous as Hellinger's [11] *discrimination*.



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(ii) When $s = 2$, we have

$$K_2(P||Q) = \frac{1}{2}\chi^2(P||Q),$$

where

$$(1.6) \quad \chi^2(P||Q) = \sum_{i=1}^n \frac{(p_i - q_i)^2}{q_i} = \sum_{i=1}^n \frac{p_i^2}{q_i} - 1,$$

is the χ^2 -divergence (Pearson [16]).

(iii) When $s = -1$, we have

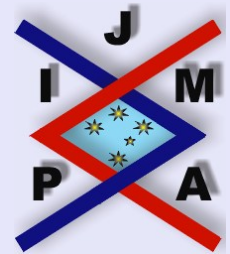
$$K_{-1}(P||Q) = \frac{1}{2}\chi^2(Q||P),$$

where

$$(1.7) \quad \chi^2(Q||P) = \sum_{i=1}^n \frac{(p_i - q_i)^2}{p_i} = \sum_{i=1}^n \frac{q_i^2}{p_i} - 1.$$

For simplicity, let us write the measures (1.3) in the unified way:

$$(1.8) \quad \Phi_s(P||Q) = \begin{cases} K_s(P||Q), & s \neq 0, 1, \\ K(Q||P), & s = 0, \\ K(P||Q), & s = 1. \end{cases}$$



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Summarizing, we have the following particular cases of the measures (1.8):

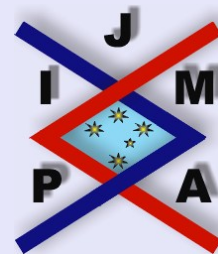
(i) $\Phi_{-1}(P||Q) = \frac{1}{2}\chi^2(Q||P)$.

(ii) $\Phi_0(P||Q) = K(Q||P)$.

(iii) $\Phi_{1/2}(P||Q) = 4[1 - B(P||Q)] = 4h(P||Q)$.

(iv) $\Phi_1(P||Q) = K(P||Q)$.

(v) $\Phi_2(P||Q) = \frac{1}{2}\chi^2(P||Q)$.



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2. Csiszár's f -Divergence and Information Bounds

Given a convex function $f : [0, \infty) \rightarrow \mathbb{R}$, the f -divergence measure introduced by Csiszár [4] is given by

$$(2.1) \quad C_f(p, q) = \sum_{i=1}^n q_i f\left(\frac{p_i}{q_i}\right),$$

where $p, q \in \mathbb{R}_+^n$.

The following two theorems can be seen in Csiszár and Körner [5].

Theorem 2.1. (*Joint convexity*). *If $f : [0, \infty) \rightarrow \mathbb{R}$ be convex, then $C_f(p, q)$ is jointly convex in p and q , where $p, q \in \mathbb{R}_+^n$.*

Theorem 2.2. (*Jensen's inequality*). *Let $f : [0, \infty) \rightarrow \mathbb{R}$ be a convex function. Then for any $p, q \in \mathbb{R}_+^n$, with $P_n = \sum_{i=1}^n p_i > 0, Q_n = \sum_{i=1}^n q_i > 0$, we have the inequality*

$$C_f(p, q) \geq Q_n f\left(\frac{P_n}{Q_n}\right).$$

The equality sign holds for strictly convex functions iff

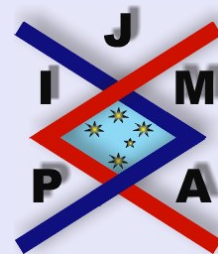
$$\frac{p_1}{q_1} = \frac{p_2}{q_2} = \dots = \frac{p_n}{q_n}.$$

In particular, for all $P, Q \in \Delta_n$, we have

$$C_f(P||Q) \geq f(1),$$

with equality iff $P = Q$.

In view of Theorems 2.1 and 2.2, we have the following result.



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Result 1. For all $P, Q \in \Delta_n$, we have

- (i) $\Phi_s(P||Q) \geq 0$ for any $s \in \mathbb{R}$, with equality iff $P = Q$.
- (ii) $\Phi_s(P||Q)$ is convex function of the pair of distributions $(P, Q) \in \Delta_n \times \Delta_n$ and for any $s \in \mathbb{R}$.

Proof. Take

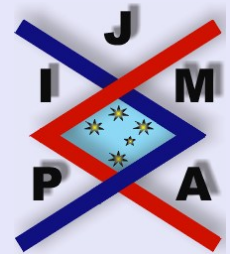
$$(2.2) \quad \phi_s(u) = \begin{cases} [s(s-1)]^{-1} [u^s - 1 - s(u-1)], & s \neq 0, 1; \\ u - 1 - \ln u, & s = 0; \\ 1 - u + u \ln u, & s = 1 \end{cases}$$

for all $u > 0$ in (2.1), we have

$$C_f(P||Q) = \Phi_s(P||Q) = \begin{cases} K_s(P||Q), & s \neq 0, 1; \\ K(Q||P), & s = 0; \\ K(P||Q), & s = 1. \end{cases}$$

Moreover,

$$(2.3) \quad \phi'_s(u) = \begin{cases} (s-1)^{-1} (u^{s-1} - 1), & s \neq 0, 1; \\ 1 - u^{-1}, & s = 0; \\ \ln u, & s = 1 \end{cases}$$



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and

$$(2.4) \quad \phi_s''(u) = \begin{cases} u^{s-2}, & s \neq 0, 1; \\ u^{-2}, & s = 0; \\ u^{-1}, & s = 1. \end{cases}$$

Thus we have $\phi_s''(u) > 0$ for all $u > 0$, and hence, $\phi_s(u)$ is strictly convex for all $u > 0$. Also, we have $\phi_s(1) = 0$. In view of Theorems 2.1 and 2.2 we have the proof of parts (i) and (ii) respectively. \square

For some studies on the measure (2.2) refer to Liese and Vajda [15], Österreicher [17] and Cerone et al. [3].

The following theorem summarizes some of the results studied by Dragomir [7], [8]. For simplicity we have taken $f(1) = 0$ and $P, Q \in \Delta_n$.

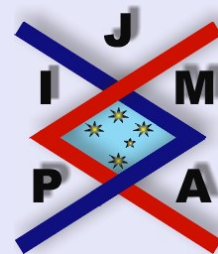
Theorem 2.3. *Let $f : \mathbb{R}_+ \rightarrow \mathbb{R}$ be differentiable convex and normalized i.e., $f(1) = 0$. If $P, Q \in \Delta_n$ are such that*

$$0 < r \leq \frac{p_i}{q_i} \leq R < \infty, \quad \forall i \in \{1, 2, \dots, n\},$$

for some r and R with $0 < r \leq 1 \leq R < \infty$, then we have the following inequalities:

$$(2.5) \quad 0 \leq C_f(P||Q) \leq \frac{1}{4} (R - r) (f'(R) - f'(r)),$$

$$(2.6) \quad 0 \leq C_f(P||Q) \leq \beta_f(r, R),$$



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and

$$\begin{aligned}
 (2.7) \quad 0 &\leq \beta_f(r, R) - C_f(P||Q) \\
 &\leq \frac{f'(R) - f'(r)}{R - r} [(R - 1)(1 - r) - \chi^2(P||Q)] \\
 &\leq \frac{1}{4} (R - r) (f'(R) - f'(r)),
 \end{aligned}$$

where

$$(2.8) \quad \beta_f(r, R) = \frac{(R - 1)f(r) + (1 - r)f(R)}{R - r},$$

and $\chi^2(P||Q)$ and $C_f(P||Q)$ are as given by (1.6) and (2.1) respectively.

In view of above theorem, we have the following result.

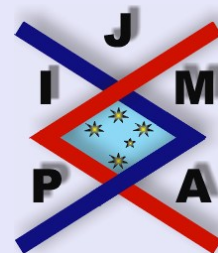
Result 2. Let $P, Q \in \Delta_n$ and $s \in \mathbb{R}$. If there exists r, R such that

$$0 < r \leq \frac{p_i}{q_i} \leq R < \infty, \quad \forall i \in \{1, 2, \dots, n\},$$

with $0 < r \leq 1 \leq R < \infty$, then we have

$$(2.9) \quad 0 \leq \Phi_s(P||Q) \leq \mu_s(r, R),$$

$$(2.10) \quad 0 \leq \Phi_s(P||Q) \leq \phi_s(r, R),$$



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and

$$(2.11) \quad \begin{aligned} 0 &\leq \phi_s(r, R) - \Phi_s(P||Q) \\ &\leq k_s(r, R) [(R-1)(1-r) - \chi^2(P||Q)] \\ &\leq \mu_s(r, R), \end{aligned}$$

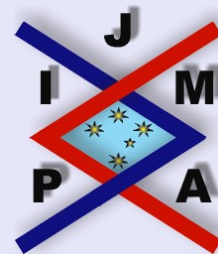
where

$$(2.12) \quad \mu_s(r, R) = \begin{cases} \frac{1}{4} \frac{(R-r)(R^{s-1}-r^{s-1})}{(s-1)}, & s \neq 1; \\ \frac{1}{4} (R-r) \ln \left(\frac{R}{r} \right), & s = 1 \end{cases}$$

$$(2.13) \quad \begin{aligned} \phi_s(r, R) &= \frac{(R-1)\phi_s(r) + (1-r)\phi_s(R)}{R-r} \\ &= \begin{cases} \frac{(R-1)(r^s-1) + (1-r)(R^s-1)}{(R-r)s(s-1)}, & s \neq 0, 1; \\ \frac{(R-1)\ln \frac{1}{r} + (1-r)\ln \frac{1}{R}}{(R-r)}, & s = 0; \\ \frac{(R-1)r \ln r + (1-r)R \ln R}{(R-r)}, & s = 1, \end{cases} \end{aligned}$$

and

$$(2.14) \quad k_s(r, R) = \frac{\phi'_s(R) - \phi'_s(r)}{R-r} = \begin{cases} \frac{R^{s-1} - r^{s-1}}{(R-r)(s-1)}, & s \neq 1; \\ \frac{\ln R - \ln r}{R-r}, & s = 1. \end{cases}$$



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Proof. The above result follows immediately from Theorem 2.3, by taking $f(u) = \phi_s(u)$, where $\phi_s(u)$ is as given by (2.2), then in this case we have $C_f(P||Q) = \Phi_s(P||Q)$. \square

Moreover,

$$\mu_s(r, R) = \frac{1}{4}(R - r)^2 k_s(r, R),$$

where

$$k_s(r, R) = \begin{cases} [L_{s-2}(r, R)]^{s-2}, & s \neq 1; \\ [L_{-1}(r, R)]^{-1} & s = 1, \end{cases}$$

and $L_p(a, b)$ is the famous (ref. Bullen, Mitrinović and Vasić [2]) *p-logarithmic mean* given by

$$L_p(a, b) = \begin{cases} \left[\frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)} \right]^{\frac{1}{p}}, & p \neq -1, 0; \\ \frac{b-a}{\ln b - \ln a}, & p = -1; \\ \frac{1}{e} \left[\frac{b^b}{a^a} \right]^{\frac{1}{b-a}}, & p = 0, \end{cases}$$

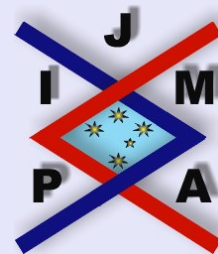
for all $p \in \mathbb{R}$, $a, b \in \mathbb{R}_+$, $a \neq b$.

We have the following corollaries as particular cases of Result 2.

Corollary 2.4. *Under the conditions of Result 2, we have*

$$(2.15) \quad 0 \leq \chi^2(Q||P) \leq \frac{1}{4}(R + r) \left(\frac{R - r}{rR} \right)^2,$$

$$(2.16) \quad 0 \leq K(Q||P) \leq \frac{(R - r)^2}{4Rr},$$



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$$(2.17) \quad 0 \leq K(P||Q) \leq \frac{1}{4}(R-r) \ln \left(\frac{R}{r} \right),$$

$$(2.18) \quad 0 \leq h(P||Q) \leq \frac{(R-r) (\sqrt{R} - \sqrt{r})}{8\sqrt{Rr}}$$

and

$$(2.19) \quad 0 \leq \chi^2(P||Q) \leq \frac{1}{2}(R-r)^2.$$

Proof. (2.15) follows by taking $s = -1$, (2.16) follows by taking $s = 0$, (2.17) follows by taking $s = 1$, (2.18) follows by taking $s = \frac{1}{2}$ and (2.19) follows by taking $s = 2$ in (2.9). \square

Corollary 2.5. *Under the conditions of Result 2, we have*

$$(2.20) \quad 0 \leq \chi^2(Q||P) \leq \frac{(R-1)(1-r)}{rR},$$

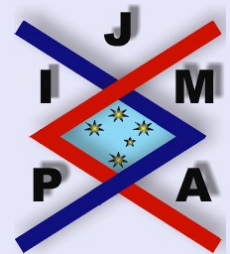
$$(2.21) \quad 0 \leq K(Q||P) \leq \frac{(R-1) \ln \frac{1}{r} + (1-r) \ln \frac{1}{R}}{R-r},$$

$$(2.22) \quad 0 \leq K(P||Q) \leq \frac{(R-1)r \ln r + (1-r)R \ln R}{R-r},$$

$$(2.23) \quad 0 \leq h(P||Q) \leq \frac{(\sqrt{R}-1)(1-\sqrt{r})}{\sqrt{R} + \sqrt{r}}$$

and

$$(2.24) \quad 0 \leq \chi^2(P||Q) \leq (R-1)(1-r).$$



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Proof. (2.20) follows by taking $s = -1$, (2.21) follows by taking $s = 0$, (2.22) follows by taking $s = 1$, (2.23) follows by taking $s = \frac{1}{2}$ and (2.24) follows by taking $s = 2$ in (2.10). \square

In view of (2.16), (2.17), (2.21) and (2.22), we have the following bounds on J -divergence:

$$(2.25) \quad 0 \leq J(P||Q) \leq \min \{t_1(r, R), t_2(r, R)\},$$

where

$$t_1(r, R) = \frac{1}{4}(R - r)^2 [(rR)^{-1} + (L_{-1}(r, R))^{-1}]$$

and

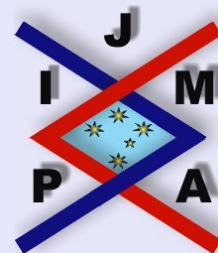
$$t_2(r, R) = (R - 1)(1 - r) (L_{-1}(r, R))^{-1}.$$

The expression $t_1(r, R)$ is due to (2.16) and (2.17) and the expression $t_2(r, R)$ is due to (2.21) and (2.22).

Corollary 2.6. *Under the conditions of Result 2, we have*

$$(2.26) \quad \begin{aligned} 0 &\leq \frac{(R - 1)(1 - r)}{rR} - \chi^2(Q||P) \\ &\leq \frac{R + r}{(rR)^2} [(R - 1)(1 - r) - \chi^2(P||Q)], \end{aligned}$$

$$(2.27) \quad \begin{aligned} 0 &\leq \frac{(R - 1) \ln \frac{1}{r} + (1 - r) \ln \frac{1}{R}}{R - r} - K(Q||P) \\ &\leq \frac{1}{rR} [(R - 1)(1 - r) - \chi^2(P||Q)], \end{aligned}$$



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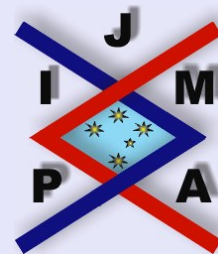
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$$(2.28) \quad 0 \leq \frac{(R-1)r \ln r + (1-r)R \ln R}{R-r} - K(P||Q) \\ \leq \frac{\ln R - \ln r}{R-r} [(R-1)(1-r) - \chi^2(P||Q)]$$

and

$$(2.29) \quad 0 \leq \frac{(\sqrt{R}-1)(1-\sqrt{r})}{(\sqrt{R}+\sqrt{r})} - h(P||Q) \\ \leq \frac{1}{2\sqrt{rR}(\sqrt{R}+\sqrt{r})} [(R-1)(1-r) - \chi^2(P||Q)].$$

Proof. (2.26) follows by taking $s = -1$, (2.27) follows by taking $s = 0$, (2.28) follows by taking $s = 1$, (2.29) follows by taking $s = \frac{1}{2}$ in (2.11). \square



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3. Main Results

In this section, we shall present a theorem generalizing the one obtained by Dragomir [9]. The results due to Dragomir [9] are limited only to χ^2 -divergence, while the theorem established here is given in terms of *relative information of type s* , that in particular lead us to bounds in terms of χ^2 -divergence, Kullback-Leibler's *relative information* and Hellinger's *discrimination*.

Theorem 3.1. *Let $f : I \subset \mathbb{R}_+ \rightarrow \mathbb{R}$ the generating mapping be normalized, i.e., $f(1) = 0$ and satisfy the assumptions:*

- (i) *f is twice differentiable on (r, R) , where $0 \leq r \leq 1 \leq R \leq \infty$;*
- (ii) *there exists the real constants m, M with $m < M$ such that*

$$(3.1) \quad m \leq x^{2-s} f''(x) \leq M, \quad \forall x \in (r, R), \quad s \in \mathbb{R}.$$

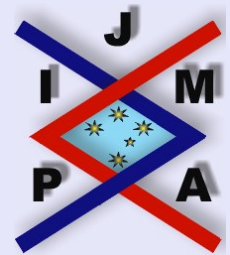
If $P, Q \in \Delta_n$ are discrete probability distributions satisfying the assumption

$$0 < r \leq \frac{p_i}{q_i} \leq R < \infty,$$

then we have the inequalities:

$$(3.2) \quad m [\phi_s(r, R) - \Phi_s(P||Q)] \leq \beta_f(r, R) - C_f(P||Q) \\ \leq M [\phi_s(r, R) - \Phi_s(P||Q)],$$

where $C_f(P||Q)$, $\Phi_s(P||Q)$, $\beta_f(r, R)$ and $\phi_s(r, R)$ are as given by (2.1), (1.8), (2.8) and (2.13) respectively.



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Proof. Let us consider the functions $F_{m,s}(\cdot)$ and $F_{M,s}(\cdot)$ given by

$$(3.3) \quad F_{m,s}(u) = f(u) - m\phi_s(u),$$

and

$$(3.4) \quad F_{M,s}(u) = M\phi_s(u) - f(u),$$

respectively, where m and M are as given by (3.1) and function $\phi_s(\cdot)$ is as given by (2.3).

Since $f(u)$ and $\phi_s(u)$ are normalized, then $F_{m,s}(\cdot)$ and $F_{M,s}(\cdot)$ are also normalized, i.e., $F_{m,s}(1) = 0$ and $F_{M,s}(1) = 0$. Moreover, the functions $f(u)$ and $\phi_s(u)$ are twice differentiable. Then in view of (2.4) and (3.1), we have

$$F_{m,s}''(u) = f''(u) - mu^{s-2} = u^{s-2} (u^{2-s} f''(u) - m) \geq 0$$

and

$$F_{M,s}''(u) = Mu^{s-2} - f''(u) = u^{s-2} (M - u^{2-s} f''(u)) \geq 0,$$

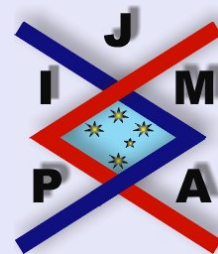
for all $u \in (r, R)$ and $s \in \mathbb{R}$. Thus the functions $F_{m,s}(\cdot)$ and $F_{M,s}(\cdot)$ are convex on (r, R) .

We have seen above that the real mappings $F_{m,s}(\cdot)$ and $F_{M,s}(\cdot)$ defined over \mathbb{R}_+ given by (3.3) and (3.4) respectively are normalized, twice differentiable and convex on (r, R) . Applying the r.h.s. of the inequality (2.6), we have

$$(3.5) \quad C_{F_{m,s}}(P||Q) \leq \beta_{F_{m,s}}(r, R),$$

and

$$(3.6) \quad C_{F_{M,s}}(P||Q) \leq \beta_{F_{M,s}}(r, R),$$



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respectively.

Moreover,

$$(3.7) \quad C_{F_{m,s}}(P||Q) = C_f(P||Q) - m\Phi_s(P||Q),$$

and

$$(3.8) \quad C_{F_{M,s}}(P||Q) = M\Phi_s(P||Q) - C_f(P||Q).$$

In view of (3.5) and (3.7), we have

$$C_f(P||Q) - m\Phi_s(P||Q) \leq \beta_{F_{m,s}}(r, R),$$

i.e.,

$$C_f(P||Q) - m\Phi_s(P||Q) \leq \beta_f(r, R) - m\phi_s(r, R)$$

i.e.,

$$m[\phi_s(r, R) - \Phi_s(P||Q)] \leq \beta_f(r, R) - C_f(P||Q).$$

Thus, we have the l.h.s. of the inequality (3.2).

Again in view of (3.6) and (3.8), we have

$$M\Phi_s(P||Q) - C_f(P||Q) \leq \beta_{F_{M,s}}(r, R),$$

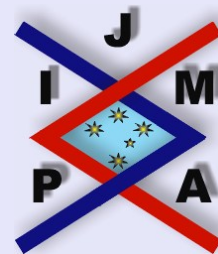
i.e.,

$$M\Phi_s(P||Q) - C_f(P||Q) \leq M\phi_s(r, R) - \beta_f(r, R),$$

i.e.,

$$\beta_f(r, R) - C_f(P||Q) \leq M[\phi_s(r, R) - \Phi_s(P||Q)].$$

Thus we have the r.h.s. of the inequality (3.2). □



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Remark 3.1. For similar kinds of results in comparing the f -divergence with Kullback-Leibler relative information see the work by Dragomir [10]. The case of Hellinger discrimination is discussed in Dragomir [6].

We shall now present some particular case of the Theorem 3.1.

3.1. Information Bounds in Terms of χ^2 -Divergence

In particular for $s = 2$, in Theorem 3.1, we have the following proposition:

Proposition 3.2. Let $f : I \subset \mathbb{R}_+ \rightarrow \mathbb{R}$ the generating mapping be normalized, i.e., $f(1) = 0$ and satisfy the assumptions:

- (i) f is twice differentiable on (r, R) , where $0 < r \leq 1 \leq R < \infty$;
- (ii) there exists the real constants m, M with $m < M$ such that

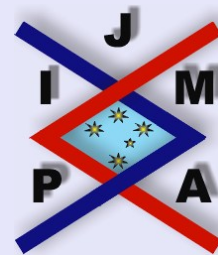
$$(3.9) \quad m \leq f''(x) \leq M, \quad \forall x \in (r, R).$$

If $P, Q \in \Delta_n$ are discrete probability distributions satisfying the assumption

$$0 < r \leq \frac{p_i}{q_i} \leq R < \infty,$$

then we have the inequalities:

$$(3.10) \quad \begin{aligned} \frac{m}{2} [(R-1)(1-r) - \chi^2(P||Q)] \\ \leq \beta_f(r, R) - C_f(P||Q) \\ \leq \frac{M}{2} [(R-1)(1-r) - \chi^2(P||Q)], \end{aligned}$$



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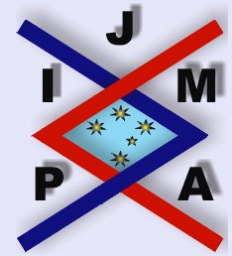


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where $C_f(P||Q)$, $\beta_f(r, R)$ and $\chi^2(P||Q)$ are as given by (2.1), (2.8) and (1.6) respectively.

The above proposition was obtained by Dragomir in [9]. As a consequence of the above Proposition 3.2, we have the following result.

Result 3. Let $P, Q \in \Delta_n$ and $s \in \mathbb{R}$. Let there exist r, R ($0 < r \leq 1 \leq R < \infty$) such that

$$0 < r \leq \frac{p_i}{q_i} \leq R < \infty, \quad \forall i \in \{1, 2, \dots, n\},$$

then in view of Proposition 3.2, we have

$$(3.11) \quad \begin{aligned} \frac{R^{s-2}}{2} [(R-1)(1-r) - \chi^2(P||Q)] \\ \leq \phi_s(r, R) - \Phi_s(P||Q) \\ \leq \frac{r^{s-2}}{2} [(R-1)(1-r) - \chi^2(P||Q)], \quad s \leq 2 \end{aligned}$$

and

$$(3.12) \quad \begin{aligned} \frac{r^{s-2}}{2} [(R-1)(1-r) - \chi^2(P||Q)] \\ \leq \phi_s(r, R) - \Phi_s(P||Q) \\ \leq \frac{R^{s-2}}{2} [(R-1)(1-r) - \chi^2(P||Q)], \quad s \geq 2. \end{aligned}$$

Proof. Let us consider $f(u) = \phi_s(u)$, where $\phi_s(u)$ is as given by (2.2), then according to expression (2.4), we have

$$\phi_s''(u) = u^{s-2}.$$

Now if $u \in [r, R] \subset (0, \infty)$, then we have

$$R^{s-2} \leq \phi_s''(u) \leq r^{s-2}, \quad s \leq 2,$$

or accordingly, we have

$$(3.13) \quad \phi_s''(u) \begin{cases} \leq r^{s-2}, & s \leq 2; \\ \geq r^{s-2}, & s \geq 2 \end{cases}$$

and

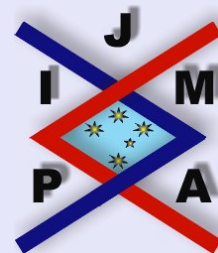
$$(3.14) \quad \phi_s''(u) \begin{cases} \leq R^{s-2}, & s \geq 2; \\ \geq R^{s-2}, & s \leq 2, \end{cases}$$

where r and R are as defined above. Thus in view of (3.9), (3.13) and (3.14), we have the proof. \square

In view of Result 3, we have the following corollary.

Corollary 3.3. *Under the conditions of Result 3, we have*

$$(3.15) \quad \begin{aligned} & \frac{1}{R^3} [(R-1)(1-r) - \chi^2(P||Q)] \\ & \leq \frac{(R-1)(1-r)}{rR} - \chi^2(Q||P) \\ & \leq \frac{1}{r^3} [(R-1)(1-r) - \chi^2(P||Q)], \end{aligned}$$



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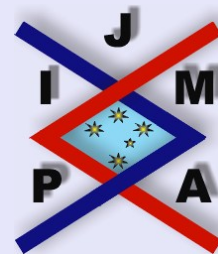
$$\begin{aligned}
 (3.16) \quad & \frac{1}{2R^2} [(R-1)(1-r) - \chi^2(P||Q)] \\
 & \leq \frac{(R-1) \ln \frac{1}{r} + (1-r) \ln \frac{1}{R}}{R-r} - K(Q||P) \\
 & \leq \frac{1}{2r^2} [(R-1)(1-r) - \chi^2(P||Q)],
 \end{aligned}$$

$$\begin{aligned}
 (3.17) \quad & \frac{1}{2R} [(R-1)(1-r) - \chi^2(P||Q)] \\
 & \leq \frac{(R-1)r \ln r + (1-r)R \ln R}{R-r} - K(P||Q) \\
 & \leq \frac{1}{2r} [(R-1)(1-r) - \chi^2(P||Q)]
 \end{aligned}$$

and

$$\begin{aligned}
 (3.18) \quad & \frac{1}{8\sqrt{R^3}} [(R-1)(1-r) - \chi^2(P||Q)] \\
 & \leq \frac{(\sqrt{R}-1)(1-\sqrt{r})}{\sqrt{R}+\sqrt{r}} - h(P||Q) \\
 & \leq \frac{1}{8\sqrt{r^3}} [(R-1)(1-r) - \chi^2(P||Q)].
 \end{aligned}$$

Proof. (3.15) follows by taking $s = -1$, (3.16) follows by taking $s = 0$, (3.17) follows by taking $s = 1$, (3.18) follows by taking $s = \frac{1}{2}$ in Result 3. While for $s = 2$, we have equality sign. \square



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Proposition 3.4. Let $f : I \subset \mathbb{R}_+ \rightarrow \mathbb{R}$ the generating mapping be normalized, i.e., $f(1) = 0$ and satisfy the assumptions:

- (i) f is twice differentiable on (r, R) , where $0 < r \leq 1 \leq R < \infty$;
- (ii) there exists the real constants m, M such that $m < M$ and

$$(3.19) \quad m \leq x^3 f''(x) \leq M, \quad \forall x \in (r, R).$$

If $P, Q \in \Delta_n$ are discrete probability distributions satisfying the assumption

$$0 < r \leq \frac{p_i}{q_i} \leq R < \infty,$$

then we have the inequalities:

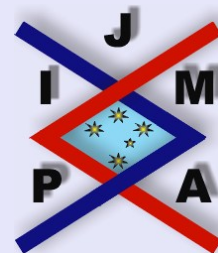
$$(3.20) \quad \begin{aligned} & \frac{m}{2} \left[\frac{(R-1)(1-r)}{rR} - \chi^2(Q||P) \right] \\ & \leq \beta_f(r, R) - C_f(P||Q) \\ & \leq \frac{m}{2} \left[\frac{(R-1)(1-r)}{rR} - \chi^2(Q||P) \right], \end{aligned}$$

where $C_f(P||Q)$, $\beta_f(r, R)$ and $\chi^2(Q||P)$ are as given by (2.1), (2.8) and (1.7) respectively.

As a consequence of above proposition, we have the following result.

Result 4. Let $P, Q \in \Delta_n$ and $s \in \mathbb{R}$. Let there exist r, R ($0 < r \leq 1 \leq R < \infty$) such that

$$0 < r \leq \frac{p_i}{q_i} \leq R < \infty, \quad \forall i \in \{1, 2, \dots, n\},$$



then in view of Proposition 3.4, we have

$$(3.21) \quad \begin{aligned} & \frac{R^{s+1}}{2} \left[\frac{(R-1)(1-r)}{rR} - \chi^2(Q||P) \right] \\ & \leq \phi_s(r, R) - \Phi_s(P||Q) \\ & \leq \frac{r^{s+1}}{2} \left[\frac{(R-1)(1-r)}{rR} - \chi^2(Q||P) \right], \quad s \leq -1 \end{aligned}$$

and

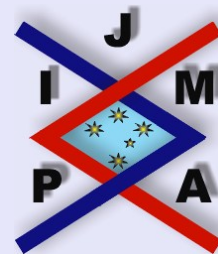
$$(3.22) \quad \begin{aligned} & \frac{r^{s+1}}{2} \left[\frac{(R-1)(1-r)}{rR} - \chi^2(Q||P) \right] \\ & \leq \phi_s(r, R) - \Phi_s(P||Q) \\ & \leq \frac{R^{s+1}}{2} \left[\frac{(R-1)(1-r)}{rR} - \chi^2(Q||P) \right], \quad s \geq -1. \end{aligned}$$

Proof. Let us consider $f(u) = \phi_s(u)$, where $\phi_s(u)$ is as given by (2.2), then according to expression (2.4), we have

$$\phi_s''(u) = u^{s-2}.$$

Let us define the function $g : [r, R] \rightarrow \mathbb{R}$ such that $g(u) = u^3 \phi_s''(u) = u^{s+1}$, then we have

$$(3.23) \quad \sup_{u \in [r, R]} g(u) = \begin{cases} R^{s+1}, & s \geq -1; \\ r^{s+1}, & s \leq -1 \end{cases}$$



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and

$$(3.24) \quad \inf_{u \in [r, R]} g(u) = \begin{cases} r^{s+1}, & s \geq -1; \\ R^{s+1}, & s \leq -1. \end{cases}$$

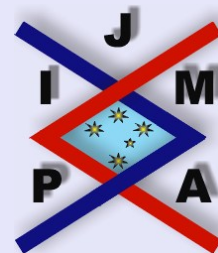
In view of (3.23), (3.24) and Proposition 3.4, we have the proof of the result. \square

In view of Result 4, we have the following corollary.

Corollary 3.5. *Under the conditions of Result 4, we have*

$$(3.25) \quad \begin{aligned} & \frac{r}{2} \left[\frac{(R-1)(1-r)}{rR} - \chi^2(Q||P) \right] \\ & \leq \frac{(R-1) \ln \frac{1}{r} + (1-r) \ln \frac{1}{R}}{R-r} - K(Q||P) \\ & \leq \frac{R}{2} \left[\frac{(R-1)(1-r)}{rR} - \chi^2(Q||P) \right], \end{aligned}$$

$$(3.26) \quad \begin{aligned} & \frac{r^2}{2} \left[\frac{(R-1)(1-r)}{rR} - \chi^2(Q||P) \right] \\ & \leq \frac{(R-1)r \ln r + (1-r)R \ln R}{R-r} - K(P||Q) \\ & \leq \frac{R^2}{2} \left[\frac{(R-1)(1-r)}{rR} - \chi^2(Q||P) \right], \end{aligned}$$



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$$\begin{aligned}
 (3.27) \quad & \frac{\sqrt{r^3}}{8} \left[\frac{(R-1)(1-r)}{rR} - \chi^2(Q||P) \right] \\
 & \leq \frac{(\sqrt{R}-1)(1-\sqrt{r})}{\sqrt{R}+\sqrt{r}} - h(P||Q) \\
 & \leq \frac{\sqrt{R^3}}{8} \left[\frac{(R-1)(1-r)}{rR} - \chi^2(Q||P) \right]
 \end{aligned}$$

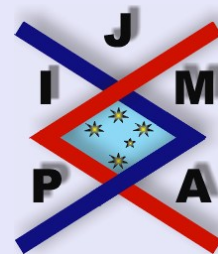
and

$$\begin{aligned}
 (3.28) \quad & r^3 \left[\frac{(R-1)(1-r)}{rR} - \chi^2(Q||P) \right] \\
 & \leq (R-1)(1-r) - \chi^2(P||Q) \\
 & \leq R^3 \left[\frac{(R-1)(1-r)}{rR} - \chi^2(Q||P) \right].
 \end{aligned}$$

Proof. (3.25) follows by taking $s = 0$, (3.26) follows by taking $s = 1$, (3.27) follows by taking $s = \frac{1}{2}$ and (3.28) follows by taking $s = 2$ in Result 4. While for $s = -1$, we have equality sign. \square

3.2. Information Bounds in Terms of Kullback-Leibler Relative Information

In particular for $s = 1$, in the Theorem 3.1, we have the following proposition (see also Dragomir [10]).



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Proposition 3.6. Let $f : I \subset \mathbb{R}_+ \rightarrow \mathbb{R}$ the generating mapping be normalized, i.e., $f(1) = 0$ and satisfy the assumptions:

- (i) f is twice differentiable on (r, R) , where $0 < r \leq 1 \leq R < \infty$;
- (ii) there exists the real constants m, M with $m < M$ such that

$$(3.29) \quad m \leq x f''(x) \leq M, \quad \forall x \in (r, R).$$

If $P, Q \in \Delta_n$ are discrete probability distributions satisfying the assumption

$$0 < r \leq \frac{p_i}{q_i} \leq R < \infty,$$

then we have the inequalities:

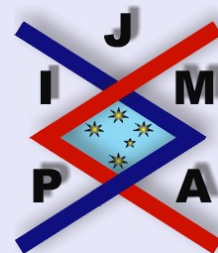
$$(3.30) \quad m \left[\frac{(R-1)r \ln r + (1-r)R \ln R}{R-r} - K(P||Q) \right] \\ \leq \beta_f(r, R) - C_f(P||Q) \\ \leq M \left[\frac{(R-1)r \ln r + (1-r)R \ln R}{R-r} - K(P||Q) \right],$$

where $C_f(P||Q)$, $\beta_f(r, R)$ and $K(P||Q)$ as given by (2.1), (2.8) and (1.1) respectively.

In view of the above proposition, we have the following result.

Result 5. Let $P, Q \in \Delta_n$ and $s \in \mathbb{R}$. Let there exist r, R ($0 < r \leq 1 \leq R < \infty$) such that

$$0 < r \leq \frac{p_i}{q_i} \leq R < \infty, \quad \forall i \in \{1, 2, \dots, n\},$$



then in view of Proposition 3.6, we have

$$(3.31) \quad r^{s-1} \left[\frac{(R-1)r \ln r + (1-r)R \ln R}{R-r} - K(P||Q) \right] \\ \leq \phi_s(r, R) - \Phi_s(P||Q) \\ \leq R^{s-1} \left[\frac{(R-1)r \ln r + (1-r)R \ln R}{R-r} - K(P||Q) \right], \quad s \geq 1$$

and

$$(3.32) \quad R^{s-1} \left[\frac{(R-1)r \ln r + (1-r)R \ln R}{R-r} - K(P||Q) \right] \\ \leq \phi_s(r, R) - \Phi_s(P||Q) \\ \leq r^{s-1} \left[\frac{(R-1)r \ln r + (1-r)R \ln R}{R-r} - K(P||Q) \right], \quad s \leq 1.$$

Proof. Let us consider $f(u) = \phi_s(u)$, where $\phi_s(u)$ is as given by (2.2), then according to expression (2.4), we have

$$\phi_s''(u) = u^{s-2}.$$

Let us define the function $g : [r, R] \rightarrow \mathbb{R}$ such that $g(u) = \phi_s''(u) = u^{s-1}$, then we have

$$(3.33) \quad \sup_{u \in [r, R]} g(u) = \begin{cases} R^{s-1}, & s \geq 1; \\ r^{s-1}, & s \leq 1 \end{cases}$$



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and

$$(3.34) \quad \inf_{u \in [r, R]} g(u) = \begin{cases} r^{s-1}, & s \geq 1; \\ R^{s-1}, & s \leq 1. \end{cases}$$

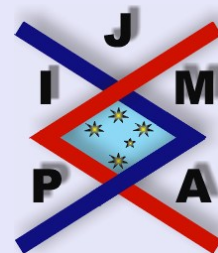
In view of (3.33), (3.34) and Proposition 3.6 we have the proof of the result. \square

In view of Result 5, we have the following corollary.

Corollary 3.7. *Under the conditions of Result 5, we have*

$$(3.35) \quad \begin{aligned} & \frac{2}{R^2} \left[\frac{(R-1)r \ln r + (1-r)R \ln R}{R-r} - K(P||Q) \right] \\ & \leq \frac{(R-1)(1-r)}{rR} - \chi^2(Q||P) \\ & \leq \frac{2}{r^2} \left[\frac{(R-1)r \ln r + (1-r)R \ln R}{R-r} - K(P||Q) \right], \end{aligned}$$

$$(3.36) \quad \begin{aligned} & \frac{1}{R} \left[\frac{(R-1)r \ln r + (1-r)R \ln R}{R-r} - K(P||Q) \right] \\ & \leq \frac{(R-1) \ln \frac{1}{r} + (1-r) \ln \frac{1}{R}}{R-r} - K(Q||P) \\ & \leq \frac{1}{r} \left[\frac{(R-1)r \ln r + (1-r)R \ln R}{R-r} - K(P||Q) \right], \end{aligned}$$



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$$\begin{aligned}
 (3.37) \quad & \frac{1}{4\sqrt{R}} \left[\frac{(R-1)r \ln r + (1-r)R \ln R}{R-r} - K(P||Q) \right] \\
 & \leq \frac{(\sqrt{R}-1)(1-\sqrt{r})}{\sqrt{R}+\sqrt{r}} - h(P||Q) \\
 & \leq \frac{1}{4\sqrt{r}} \left[\frac{(R-1)r \ln r + (1-r)R \ln R}{R-r} - K(P||Q) \right]
 \end{aligned}$$

and

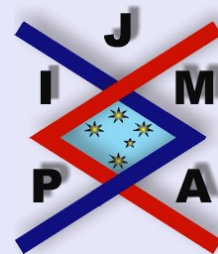
$$\begin{aligned}
 (3.38) \quad & 2r \left[\frac{(R-1)r \ln r + (1-r)R \ln R}{R-r} - K(P||Q) \right] \\
 & \leq (R-1)(1-r) - \chi^2(P||Q) \\
 & \leq 2R \left[\frac{(R-1)r \ln r + (1-r)R \ln R}{R-r} - K(P||Q) \right].
 \end{aligned}$$

Proof. (3.35) follows by taking $s = -1$, (3.36) follows by taking $s = 0$, (3.37) follows by taking $s = \frac{1}{2}$ and (3.38) follows by taking $s = 2$ in Result 5. For $s = 1$, we have equality sign. \square

In particular, for $s = 0$ in Theorem 3.1, we have the following proposition:

Proposition 3.8. *Let $f : I \subset \mathbb{R}_+ \rightarrow \mathbb{R}$ the generating mapping be normalized, i.e., $f(1) = 0$ and satisfy the assumptions:*

(i) *f is twice differentiable on (r, R) , where $0 < r \leq 1 \leq R < \infty$;*



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(ii) there exists the real constants m, M with $m < M$ such that

$$(3.39) \quad m \leq x^2 f''(x) \leq M, \quad \forall x \in (r, R).$$

If $P, Q \in \Delta_n$ are discrete probability distributions satisfying the assumption

$$0 < r \leq \frac{p_i}{q_i} \leq R < \infty,$$

then we have the inequalities:

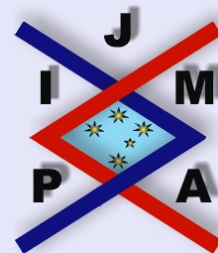
$$(3.40) \quad \begin{aligned} m & \left[\frac{(R-1) \ln \frac{1}{r} + (1-r) \ln \frac{1}{R}}{R-r} - K(Q||P) \right] \\ & \leq \beta_f(r, R) - C_f(P||Q) \\ & \leq M \left[\frac{(R-1) \ln \frac{1}{r} + (1-r) \ln \frac{1}{R}}{R-r} - K(Q||P) \right], \end{aligned}$$

where $C_f(P||Q)$, $\beta_f(r, R)$ and $K(Q||P)$ as given by (2.1), (2.8) and (1.1) respectively.

In view of Proposition 3.8, we have the following result.

Result 6. Let $P, Q \in \Delta_n$ and $s \in \mathbb{R}$. Let there exist r, R ($0 < r \leq 1 \leq R < \infty$) such that

$$0 < r \leq \frac{p_i}{q_i} \leq R < \infty, \quad \forall i \in \{1, 2, \dots, n\},$$



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then in view of Proposition 3.8, we have

$$(3.41) \quad \begin{aligned} & r^s \left[\frac{(R-1) \ln \frac{1}{r} + (1-r) \ln \frac{1}{R}}{R-r} - K(Q||P) \right] \\ & \leq \phi_s(r, R) - \Phi_s(P||Q) \\ & \leq R^s \left[\frac{(R-1) \ln \frac{1}{r} + (1-r) \ln \frac{1}{R}}{R-r} - K(Q||P) \right], \quad s \geq 0 \end{aligned}$$

and

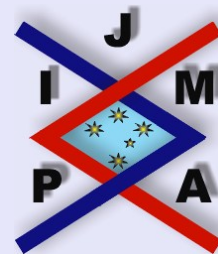
$$(3.42) \quad \begin{aligned} & R^s \left[\frac{(R-1) \ln \frac{1}{r} + (1-r) \ln \frac{1}{R}}{R-r} - K(Q||P) \right] \\ & \leq \phi_s(r, R) - \Phi_s(P||Q) \\ & \leq r^s \left[\frac{(R-1) \ln \frac{1}{r} + (1-r) \ln \frac{1}{R}}{R-r} - K(Q||P) \right], \quad s \leq 0. \end{aligned}$$

Proof. Let us consider $f(u) = \phi_s(u)$, where $\phi_s(u)$ is as given by (2.2), then according to expression (2.4), we have

$$\phi_s''(u) = u^{s-2}.$$

Let us define the function $g : [r, R] \rightarrow \mathbb{R}$ such that $g(u) = u^2 \phi_s''(u) = u^s$, then we have

$$(3.43) \quad \sup_{u \in [r, R]} g(u) = \begin{cases} R^s, & s \geq 0; \\ r^s, & s \leq 0 \end{cases}$$



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and

$$(3.44) \quad \inf_{u \in [r, R]} g(u) = \begin{cases} r^s, & s \geq 0; \\ R^s, & s \leq 0. \end{cases}$$

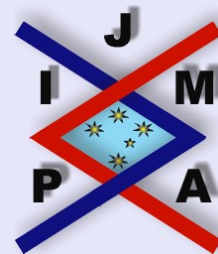
In view of (3.43), (3.44) and Proposition 3.8, we have the proof of the result. \square

In view of Result 6, we have the following corollary.

Corollary 3.9. *Under the conditions of Result 6, we have*

$$(3.45) \quad \begin{aligned} & \frac{2}{R} \left[\frac{(R-1) \ln \frac{1}{r} + (1-r) \ln \frac{1}{R}}{R-r} - K(Q||P) \right] \\ & \leq \frac{(R-1)(1-r)}{rR} - \chi^2(Q||P) \\ & \leq \frac{2}{r} \left[\frac{(R-1) \ln \frac{1}{r} + (1-r) \ln \frac{1}{R}}{R-r} - K(Q||P) \right], \end{aligned}$$

$$(3.46) \quad \begin{aligned} & r \left[\frac{(R-1) \ln \frac{1}{r} + (1-r) \ln \frac{1}{R}}{R-r} - K(Q||P) \right] \\ & \leq \frac{(R-1)r \ln r + (1-r)R \ln R}{R-r} - K(P||Q) \\ & \leq R \left[\frac{(R-1) \ln \frac{1}{r} + (1-r) \ln \frac{1}{R}}{R-r} - K(Q||P) \right], \end{aligned}$$



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$$\begin{aligned}
 (3.47) \quad & \frac{\sqrt{r}}{4} \left[\frac{(R-1) \ln \frac{1}{r} + (1-r) \ln \frac{1}{R}}{R-r} - K(Q||P) \right] \\
 & \leq \frac{(\sqrt{R}-1)(1-\sqrt{r})}{\sqrt{R}+\sqrt{r}} - h(P||Q) \\
 & \leq \frac{\sqrt{R}}{4} \left[\frac{(R-1) \ln \frac{1}{r} + (1-r) \ln \frac{1}{R}}{R-r} - K(Q||P) \right]
 \end{aligned}$$

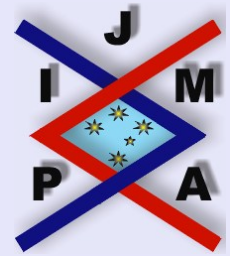
and

$$\begin{aligned}
 (3.48) \quad & 2r^2 \left[\frac{(R-1) \ln \frac{1}{r} + (1-r) \ln \frac{1}{R}}{R-r} - K(Q||P) \right] \\
 & \leq (R-1)(1-r) - \chi^2(P||Q) \\
 & \leq 2R^2 \left[\frac{(R-1) \ln \frac{1}{r} + (1-r) \ln \frac{1}{R}}{R-r} - K(Q||P) \right].
 \end{aligned}$$

Proof. (3.45) follows by taking $s = -1$, (3.46) follows by taking $s = 1$, (3.47) follows by taking $s = \frac{1}{2}$ and (3.48) follows by taking $s = 2$ in Result 6. For $s = 0$, we have equality sign. \square

3.3. Information Bounds in Terms of Hellinger's Discrimination

In particular, for $s = \frac{1}{2}$ in Theorem 3.1, we have the following proposition (see also Dragomir [6]).



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Proposition 3.10. Let $f : I \subset \mathbb{R}_+ \rightarrow \mathbb{R}$ the generating mapping be normalized, i.e., $f(1) = 0$ and satisfy the assumptions:

- (i) f is twice differentiable on (r, R) , where $0 < r \leq 1 \leq R < \infty$;
- (ii) there exists the real constants m, M with $m < M$ such that

$$(3.49) \quad m \leq x^{3/2} f''(x) \leq M, \quad \forall x \in (r, R).$$

If $P, Q \in \Delta_n$ are discrete probability distributions satisfying the assumption

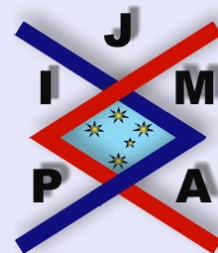
$$0 < r \leq \frac{p_i}{q_i} \leq R < \infty,$$

then we have the inequalities:

$$(3.50) \quad 4m \left[\frac{(\sqrt{R} - 1)(1 - \sqrt{r})}{\sqrt{R} + \sqrt{r}} - h(P||Q) \right] \\ \leq \beta_f(r, R) - C_f(P||Q) \\ \leq 4M \left[\frac{(\sqrt{R} - 1)(1 - \sqrt{r})}{\sqrt{R} + \sqrt{r}} - h(P||Q) \right],$$

where $C_f(P||Q)$, $\beta_f(r, R)$ and $h(P||Q)$ as given by (2.1), (2.8) and (1.5) respectively.

In view of Proposition 3.10, we have the following result.



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Result 7. Let $P, Q \in \Delta_n$ and $s \in \mathbb{R}$. Let there exist r, R ($0 < r \leq 1 \leq R < \infty$) such that

$$0 < r \leq \frac{p_i}{q_i} \leq R < \infty, \quad \forall i \in \{1, 2, \dots, n\},$$

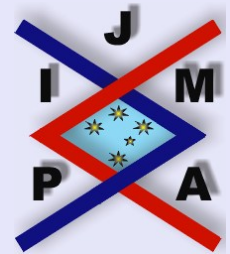
then in view of Proposition 3.10, we have

$$(3.51) \quad 4r^{\frac{2s-1}{2}} \left[\frac{(\sqrt{R}-1)(1-\sqrt{r})}{\sqrt{R}+\sqrt{r}} - h(P||Q) \right] \leq \phi_s(r, R) - \Phi_s(P||Q) \leq 4R^{\frac{2s-1}{2}} \left[\frac{(\sqrt{R}-1)(1-\sqrt{r})}{\sqrt{R}+\sqrt{r}} - h(P||Q) \right], \quad s \geq \frac{1}{2}$$

and

$$(3.52) \quad 4R^{\frac{2s-1}{2}} \left[\frac{(\sqrt{R}-1)(1-\sqrt{r})}{\sqrt{R}+\sqrt{r}} - h(P||Q) \right] \leq \phi_s(r, R) - \Phi_s(P||Q) \leq 4r^{\frac{2s-1}{2}} \left[\frac{(\sqrt{R}-1)(1-\sqrt{r})}{\sqrt{R}+\sqrt{r}} - h(P||Q) \right], \quad s \leq \frac{1}{2}.$$

Proof. Let the function $\phi_s(u)$ given by (3.29) be defined over $[r, R]$. Defining



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$g(u) = u^{3/2}\phi_s''(u) = u^{\frac{2s-1}{2}}$, obviously we have

$$(3.53) \quad \sup_{u \in [r, R]} g(u) = \begin{cases} R^{\frac{2s-1}{2}}, & s \geq \frac{1}{2}; \\ r^{\frac{2s-1}{2}}, & s \leq \frac{1}{2} \end{cases}$$

and

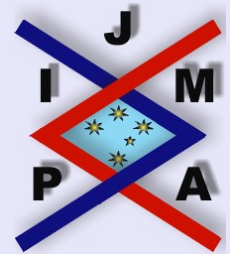
$$(3.54) \quad \inf_{u \in [r, R]} g(u) = \begin{cases} r^{\frac{2s-1}{2}}, & s \geq \frac{1}{2}; \\ R^{\frac{2s-1}{2}}, & s \leq \frac{1}{2}. \end{cases}$$

In view of (3.53), (3.54) and Proposition 3.10, we get the proof of the result. \square

In view of Result 7, we have the following corollary.

Corollary 3.11. *Under the conditions of Result 7, we have*

$$(3.55) \quad \begin{aligned} & \frac{8}{\sqrt{R^3}} \left[\frac{(\sqrt{R}-1)(1-\sqrt{r})}{\sqrt{R}+\sqrt{r}} - h(P||Q) \right] \\ & \leq \frac{(R-1)(1-r)}{rR} - \chi^2(Q||P) \\ & \leq \frac{8}{\sqrt{r^3}} \left[\frac{(\sqrt{R}-1)(1-\sqrt{r})}{\sqrt{R}+\sqrt{r}} - h(P||Q) \right], \end{aligned}$$



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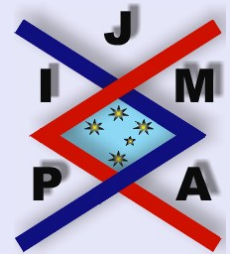
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$$\begin{aligned}
 (3.56) \quad & \frac{4}{\sqrt{R}} \left[\frac{(\sqrt{R}-1)(1-\sqrt{r})}{\sqrt{R}+\sqrt{r}} - h(P||Q) \right] \\
 & \leq \frac{(R-1)\ln\frac{1}{r} + (1-r)\ln\frac{1}{R}}{R-r} - K(Q||P) \\
 & \leq \frac{4}{\sqrt{r}} \left[\frac{(\sqrt{R}-1)(1-\sqrt{r})}{\sqrt{R}+\sqrt{r}} - h(P||Q) \right],
 \end{aligned}$$

$$\begin{aligned}
 (3.57) \quad & 4\sqrt{r} \left[\frac{(\sqrt{R}-1)(1-\sqrt{r})}{\sqrt{R}+\sqrt{r}} - h(P||Q) \right] \\
 & \leq \frac{(R-1)r\ln r + (1-r)R\ln R}{R-r} - K(P||Q) \\
 & \leq 4\sqrt{R} \left[\frac{(\sqrt{R}-1)(1-\sqrt{r})}{\sqrt{R}+\sqrt{r}} - h(P||Q) \right]
 \end{aligned}$$

and

$$\begin{aligned}
 (3.58) \quad & 8\sqrt{r^3} \left[\frac{(\sqrt{R}-1)(1-\sqrt{r})}{\sqrt{R}+\sqrt{r}} - h(P||Q) \right] \\
 & \leq (R-1)(1-r) - \chi^2(P||Q)
 \end{aligned}$$



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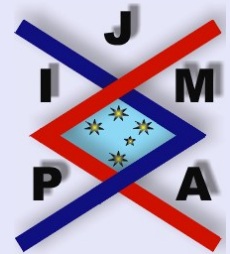
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$$\leq 8\sqrt{R^3} \left[\frac{(\sqrt{R} - 1)(1 - \sqrt{r})}{\sqrt{R} + \sqrt{r}} - h(P||Q) \right].$$

Proof. (3.55) follows by taking $s = -1$, (3.56) follows by taking $s = 0$, (3.57) follows by taking $s = 1$ and (3.58) follows by taking $s = 2$ in Result 7. For $s = \frac{1}{2}$, we have equality sign. \square



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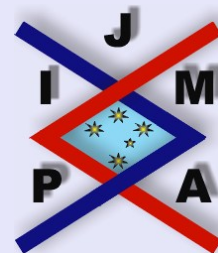
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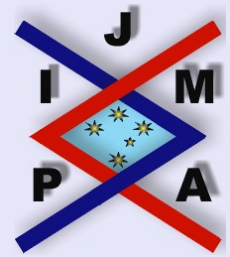
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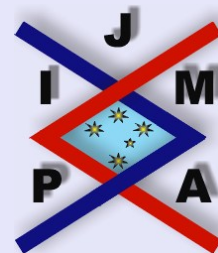
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