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TIME-SCALE INTEGRAL INEQUALITIES

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Abstract

Some recent and classical integral inequalities are extended to the general time-scale calculus, including the inequalities of Steffensen, Iyengar, Čebyšev, and Hermite-Hadamard.

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1. Preliminaries on Time Scales

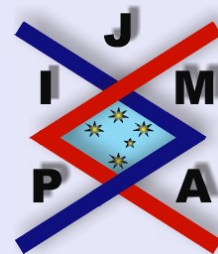
The unification and extension of continuous calculus, discrete calculus, q -calculus, and indeed arbitrary real-number calculus to time-scale calculus was first accomplished by Hilger in his Ph.D. thesis [8]. Since then, time-scale calculus has made steady inroads in explaining the interconnections that exist among the various calculi, and in extending our understanding to a new, more general and overarching theory. The purpose of this work is to illustrate this new understanding by extending some continuous and q -calculus inequalities and some of their applications, such as those by Steffensen, Hermite-Hadamard, Iyengar, and Čebyšev, to arbitrary time scales.

The following definitions will serve as a short primer on the time-scale calculus; they can be found in Agarwal and Bohner [1], Atici and Guseinov [3], and Bohner and Peterson [4]. A time scale \mathbb{T} is any nonempty closed subset of \mathbb{R} . Within that set, define the jump operators $\rho, \sigma : \mathbb{T} \rightarrow \mathbb{T}$ by

$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\} \quad \text{and} \quad \sigma(t) = \inf\{s \in \mathbb{T} : s > t\},$$

where $\inf \emptyset := \sup \mathbb{T}$ and $\sup \emptyset := \inf \mathbb{T}$. The point $t \in \mathbb{T}$ is left-dense, left-scattered, right-dense, right-scattered if $\rho(t) = t$, $\rho(t) < t$, $\sigma(t) = t$, $\sigma(t) > t$, respectively. If \mathbb{T} has a right-scattered minimum m , define $\mathbb{T}_\kappa := \mathbb{T} - \{m\}$; otherwise, set $\mathbb{T}_\kappa = \mathbb{T}$. If \mathbb{T} has a left-scattered maximum M , define $\mathbb{T}^\kappa := \mathbb{T} - \{M\}$; otherwise, set $\mathbb{T}^\kappa = \mathbb{T}$. The so-called graininess functions are $\mu(t) := \sigma(t) - t$ and $\nu(t) := t - \rho(t)$.

For $f : \mathbb{T} \rightarrow \mathbb{R}$ and $t \in \mathbb{T}_\kappa$, the nabla derivative [3] of f at t , denoted $f^\nabla(t)$, is the number (provided it exists) with the property that given any $\varepsilon > 0$, there



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is a neighborhood U of t such that

$$|f(\rho(t)) - f(s) - f^\nabla(t)[\rho(t) - s]| \leq \varepsilon|\rho(t) - s|$$

for all $s \in U$. Common special cases again include $\mathbb{T} = \mathbb{R}$, where $f^\nabla = f'$, the usual derivative; $\mathbb{T} = \mathbb{Z}$, where the nabla derivative is the backward difference operator, $f^\nabla(t) = f(t) - f(t - 1)$; q -difference equations with $0 < q < 1$ and $t > 0$,

$$f^\nabla(t) = \frac{f(t) - f(qt)}{(1 - q)t}.$$

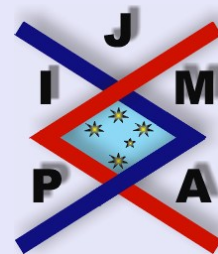
For $f : \mathbb{T} \rightarrow \mathbb{R}$ and $t \in \mathbb{T}^\kappa$, the delta derivative [4] of f at t , denoted $f^\Delta(t)$, is the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood U of t such that

$$|f(\sigma(t)) - f(s) - f^\Delta(t)[\sigma(t) - s]| \leq \varepsilon|\sigma(t) - s|$$

for all $s \in U$. For $\mathbb{T} = \mathbb{R}$, $f^\Delta = f'$, the usual derivative; for $\mathbb{T} = \mathbb{Z}$ the delta derivative is the forward difference operator, $f^\Delta(t) = f(t + 1) - f(t)$; in the case of q -difference equations with $q > 1$,

$$f^\Delta(t) = \frac{f(qt) - f(t)}{(q - 1)t}, \quad f^\Delta(0) = \lim_{s \rightarrow 0} \frac{f(s) - f(0)}{s}.$$

A function $f : \mathbb{T} \rightarrow \mathbb{R}$ is left-dense continuous or ld-continuous provided it is continuous at left-dense points in \mathbb{T} and its right-sided limits exist (finite) at right-dense points in \mathbb{T} . If $\mathbb{T} = \mathbb{R}$, then f is ld-continuous if and only if



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f is continuous. It is known from [3] or Theorem 8.45 in [4] that if f is ld-continuous, then there is a function F such that $F^\nabla(t) = f(t)$. In this case, we define

$$\int_a^b f(t)\nabla t = F(b) - F(a).$$

In the same way, from Theorem 1.74 in [4] we have that if g is right-dense continuous, there is a function G such that $G^\Delta(t) = g(t)$ and

$$\int_a^b g(t)\Delta t = G(b) - G(a).$$

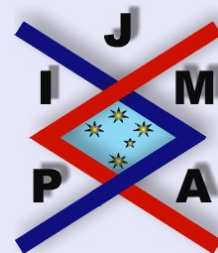
The following theorem is part of Theorem 2.7 in [3] and Theorem 8.47 in [4].

Theorem 1.1 (Integration by parts). *If $a, b \in \mathbb{T}$ and f^∇, g^∇ are left-dense continuous, then*

$$\int_a^b f(t)g^\nabla(t)\nabla t = (fg)(b) - (fg)(a) - \int_a^b f^\nabla(t)g(\rho(t))\nabla t$$

and

$$\int_a^b f(\rho(t))g^\nabla(t)\nabla t = (fg)(b) - (fg)(a) - \int_a^b f^\nabla(t)g(t)\nabla t.$$



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2. Taylor's Theorem Using Nabla Polynomials

The generalized polynomials for nabla equations [2] are the functions $\hat{h}_k : \mathbb{T}^2 \rightarrow \mathbb{R}$, $k \in \mathbb{N}_0$, defined recursively as follows: The function \hat{h}_0 is

$$(2.1) \quad \hat{h}_0(t, s) \equiv 1 \quad \text{for all } s, t \in \mathbb{T},$$

and, given \hat{h}_k for $k \in \mathbb{N}_0$, the function \hat{h}_{k+1} is

$$(2.2) \quad \hat{h}_{k+1}(t, s) = \int_s^t \hat{h}_k(\tau, s) \nabla \tau \quad \text{for all } s, t \in \mathbb{T}.$$

Note that the functions \hat{h}_k are all well defined, since each is ld-continuous. If for each fixed s we let $\hat{h}_k^\nabla(t, s)$ denote the nabla derivative of $\hat{h}_k(t, s)$ with respect to t , then

$$(2.3) \quad \hat{h}_k^\nabla(t, s) = \hat{h}_{k-1}(t, s) \quad \text{for } k \in \mathbb{N}, t \in \mathbb{T}_\kappa.$$

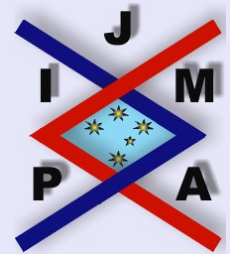
The above definition implies

$$\hat{h}_1(t, s) = t - s \quad \text{for all } s, t \in \mathbb{T}.$$

Obtaining an expression for \hat{h}_k for $k > 1$ is not easy in general, but for a particular given time scale it might be easy to find these functions; see [2] for some examples.

Theorem 2.1 (Taylor's Formula [2]). *Let $n \in \mathbb{N}$. Suppose f is $n + 1$ times nabla differentiable on $\mathbb{T}_{\kappa^{n+1}}$. Let $s \in \mathbb{T}_{\kappa^n}$, $t \in \mathbb{T}$, and define the functions \hat{h}_k by (2.1) and (2.2), i.e.,*

$$\hat{h}_0(t, s) \equiv 1 \quad \text{and} \quad \hat{h}_{k+1}(t, s) = \int_s^t \hat{h}_k(\tau, s) \nabla \tau \quad \text{for } k \in \mathbb{N}_0.$$



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Then we have

$$f(t) = \sum_{k=0}^n \hat{h}_k(t, s) f^{\nabla^k}(s) + \int_s^t \hat{h}_n(t, \rho(\tau)) f^{\nabla^{n+1}}(\tau) \nabla\tau.$$

We may also relate the functions \hat{h}_k as introduced in (2.1) and (2.2) (which we repeat below) to the functions h_k and g_k in the delta case [1, 4], and the functions \hat{g}_k in the nabla case, defined below.

Definition 2.1. For $t, s \in \mathbb{T}$ define the functions

$$h_0(t, s) = g_0(t, s) = \hat{h}_0(t, s) = \hat{g}_0(t, s) \equiv 1,$$

and given $h_n, g_n, \hat{h}_n, \hat{g}_n$ for $n \in \mathbb{N}_0$,

$$h_{n+1}(t, s) = \int_s^t h_n(\tau, s) \Delta\tau, g_{n+1}(t, s) = \int_s^t g_n(\sigma(\tau), s) \Delta\tau,$$

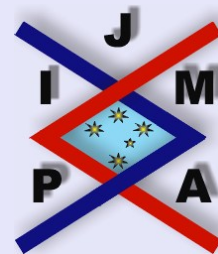
$$\hat{h}_{n+1}(t, s) = \int_s^t \hat{h}_n(\tau, s) \nabla\tau, \hat{g}_{n+1}(t, s) = \int_s^t \hat{g}_n(\rho(\tau), s) \nabla\tau.$$

The following theorem combines Theorem 9 of [2] and Theorem 1.112 of [4].

Theorem 2.2. Let $t \in \mathbb{T}_{\kappa}^{\kappa}$ and $s \in \mathbb{T}^{\kappa^n}$. Then

$$\hat{h}_n(t, s) = g_n(t, s) = (-1)^n h_n(s, t) = (-1)^n \hat{g}_n(s, t)$$

for all $n \geq 0$.



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3. Steffensen's inequality

For a q -difference equation version of the following result and most results in this paper, including proof techniques, see [7]. In fact, the presentation of the results to follow largely mirrors the organisation of [7].

Theorem 3.1 (Steffensen's Inequality (nabla)). *Let $a, b \in \mathbb{T}_{\kappa}^{\kappa}$ with $a < b$ and $f, g : [a, b] \rightarrow \mathbb{R}$ be nabla-integrable functions, with f of one sign and decreasing and $0 \leq g \leq 1$ on $[a, b]$. Assume $\ell, \gamma \in [a, b]$ such that*

$$b - \ell \leq \int_a^b g(t) \nabla t \leq \gamma - a \quad \text{if } f \geq 0, \quad t \in [a, b],$$

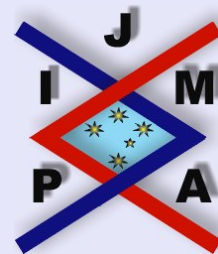
$$\gamma - a \leq \int_a^b g(t) \nabla t \leq b - \ell \quad \text{if } f \leq 0, \quad t \in [a, b].$$

Then

$$(3.1) \quad \int_{\ell}^b f(t) \nabla t \leq \int_a^b f(t) g(t) \nabla t \leq \int_a^{\gamma} f(t) \nabla t.$$

Proof. The proof given in the q -difference case [7] can be extended to general time scales. As in [7], we prove only the case in (3.1) where $f \geq 0$ for the left inequality; the proofs of the other cases are similar. After subtracting within the left inequality,

$$\begin{aligned} & \int_a^b f(t) g(t) \nabla t - \int_{\ell}^b f(t) \nabla t \\ &= \int_a^{\ell} f(t) g(t) \nabla t + \int_{\ell}^b f(t) g(t) \nabla t - \int_{\ell}^b f(t) \nabla t \end{aligned}$$



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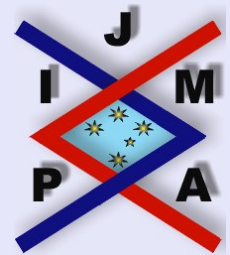
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$$\begin{aligned}
&= \int_a^\ell f(t)g(t)\nabla t - \int_\ell^b f(t)(1-g(t))\nabla t \\
&\geq \int_a^\ell f(t)g(t)\nabla t - f(\ell) \int_\ell^b (1-g(t))\nabla t \\
&= \int_a^\ell f(t)g(t)\nabla t - (b-\ell)f(\ell) + f(\ell) \int_\ell^b g(t)\nabla t \\
&\geq \int_a^\ell f(t)g(t)\nabla t - f(\ell) \int_a^b g(t)\nabla t + f(\ell) \int_\ell^b g(t)\nabla t \\
&= \int_a^\ell f(t)g(t)\nabla t - f(\ell) \left(\int_a^b g(t)\nabla t - \int_\ell^b g(t)\nabla t \right) \\
&= \int_a^\ell f(t)g(t)\nabla t - f(\ell) \int_a^\ell g(t)\nabla t \\
&= \int_a^\ell (f(t) - f(\ell))g(t)\nabla t \geq 0
\end{aligned}$$

since f is decreasing and g is nonnegative. □

Note that in the theorem above, we could easily replace the nabla integrals with delta integrals under the same hypotheses and get a completely analogous result. The following theorem more closely resembles the theorem in the continuous case; the proof is identical to that above and is omitted.

Theorem 3.2 (Steffensen's Inequality II). *Let $a, b \in \mathbb{T}_\kappa^\kappa$ and $f, g : [a, b] \rightarrow \mathbb{R}$ be nabla-integrable functions, with f decreasing and $0 \leq g \leq 1$ on $[a, b]$.*



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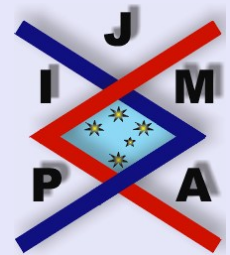
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Assume $\lambda := \int_a^b g(t) \nabla t$ such that $b - \lambda, a + \lambda \in \mathbb{T}$. Then

$$(3.2) \quad \int_{b-\lambda}^b f(t) \nabla t \leq \int_a^b f(t) g(t) \nabla t \leq \int_a^{a+\lambda} f(t) \nabla t.$$



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4. Taylor's Remainder

Suppose f is $n+1$ times nabla differentiable on $\mathbb{T}_{\kappa^{n+1}}$. Using Taylor's Theorem, Theorem 2.1, we define the remainder function by $\hat{R}_{-1,f}(\cdot, s) := f(s)$, and for $n > -1$,

$$(4.1) \quad \hat{R}_{n,f}(t, s) := f(s) - \sum_{j=0}^n \hat{h}_j(s, t) f^{\nabla j}(t) = \int_t^s \hat{h}_n(s, \rho(\tau)) f^{\nabla n+1}(\tau) \nabla \tau.$$

Lemma 4.1. *The following identity involving nabla Taylor's remainder holds:*

$$\int_a^b \hat{h}_{n+1}(t, \rho(s)) f^{\nabla n+1}(s) \nabla s = \int_a^t \hat{R}_{n,f}(a, s) \nabla s + \int_t^b \hat{R}_{n,f}(b, s) \nabla s.$$

Proof. Proceed by mathematical induction on n . For $n = -1$,

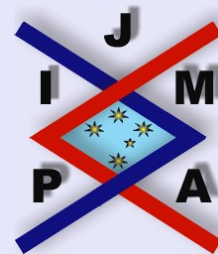
$$\int_a^b \hat{h}_0(t, \rho(s)) f^{\nabla 0}(s) \nabla s = \int_a^b f(s) \nabla s = \int_a^t f(s) \nabla s + \int_t^b f(s) \nabla s.$$

Assume the result holds for $n = k - 1$:

$$\int_a^b \hat{h}_k(t, \rho(s)) f^{\nabla k}(s) \nabla s = \int_a^t \hat{R}_{k-1,f}(a, s) \nabla s + \int_t^b \hat{R}_{k-1,f}(b, s) \nabla s.$$

Let $n = k$. By Corollary 11 in [2], for fixed $t \in \mathbb{T}$ we have

$$(4.2) \quad \hat{h}_{k+1}^{\nabla s}(t, s) = -\hat{h}_k(t, \rho(s)).$$



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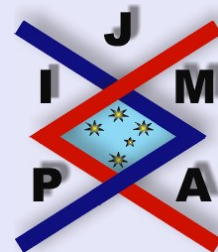
Thus using the nabla integration by parts rule, Theorem 1.1, we have

$$\begin{aligned} & \int_a^b \hat{h}_{k+1}(t, \rho(s)) f^{\nabla^{k+1}}(s) \nabla s \\ &= \int_a^b \hat{h}_k(t, \rho(s)) f^{\nabla^k}(s) \nabla s + \hat{h}_{k+1}(t, b) f^{\nabla^k}(b) - \hat{h}_{k+1}(t, a) f^{\nabla^k}(a). \end{aligned}$$

By the induction assumption and the definition of \hat{h}_{k+1} ,

$$\begin{aligned} \int_a^b \hat{h}_{k+1}(t, \rho(s)) f^{\nabla^{k+1}}(s) \nabla s &= \int_a^t \hat{R}_{k-1, f}(a, s) \nabla s + \int_t^b \hat{R}_{k-1, f}(b, s) \nabla s \\ &\quad + \hat{h}_{k+1}(t, b) f^{\nabla^k}(b) - \hat{h}_{k+1}(t, a) f^{\nabla^k}(a) \\ &= \int_a^t \hat{R}_{k-1, f}(a, s) \nabla s + \int_t^b \hat{R}_{k-1, f}(b, s) \nabla s \\ &\quad + \int_b^t \hat{h}_k(s, b) f^{\nabla^k}(b) \nabla s - \int_a^t \hat{h}_k(s, a) f^{\nabla^k}(a) \nabla s \\ &= \int_a^t \left[\hat{R}_{k-1, f}(a, s) - \hat{h}_k(s, a) f^{\nabla^k}(a) \right] \nabla s \\ &\quad + \int_t^b \left[\hat{R}_{k-1, f}(b, s) - \hat{h}_k(s, b) f^{\nabla^k}(b) \right] \nabla s \\ &= \int_a^t \hat{R}_{k, f}(a, s) \nabla s + \int_t^b \hat{R}_{k, f}(b, s) \nabla s. \end{aligned}$$

□



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Corollary 4.2. For $n \geq -1$,

$$\int_a^b \hat{h}_{n+1}(a, \rho(s)) f^{\nabla^{n+1}}(s) \nabla s = \int_a^b \hat{R}_{n,f}(b, s) \nabla s,$$

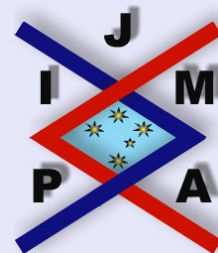
$$\int_a^b \hat{h}_{n+1}(b, \rho(s)) f^{\nabla^{n+1}}(s) \nabla s = \int_a^b \hat{R}_{n,f}(a, s) \nabla s.$$

Lemma 4.3. The following identity involving delta Taylor's remainder holds:

$$\int_a^b h_{n+1}(t, \sigma(s)) f^{\Delta^{n+1}}(s) \Delta s = \int_a^t R_{n,f}(a, s) \Delta s + \int_t^b R_{n,f}(b, s) \Delta s,$$

where

$$R_{n,f}(t, s) := f(s) - \sum_{j=0}^n h_j(s, t) f^{\Delta^j}(t).$$



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5. Applications of Steffensen's Inequality

In the following we generalize to arbitrary time scales some results from [7] by applying Steffensen's inequality, Theorem 3.1.

Theorem 5.1. *Let $f : [a, b] \rightarrow \mathbb{R}$ be an $n + 1$ times nabla differentiable function such that $f^{\nabla^{n+1}}$ is increasing and f^{∇^n} is monotonic (either increasing or decreasing) on $[a, b]$. Assume $\ell, \gamma \in [a, b]$ such that*

$$b - \ell \leq \frac{\hat{h}_{n+2}(b, a)}{\hat{h}_{n+1}(b, \rho(a))} \leq \gamma - a \quad \text{if } f^{\nabla^n} \text{ is decreasing,}$$

$$\gamma - a \leq \frac{\hat{h}_{n+2}(b, a)}{\hat{h}_{n+1}(b, \rho(a))} \leq b - \ell \quad \text{if } f^{\nabla^n} \text{ is increasing.}$$

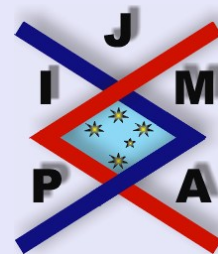
Then

$$f^{\nabla^n}(\gamma) - f^{\nabla^n}(a) \leq \frac{1}{\hat{h}_{n+1}(b, \rho(a))} \int_a^b \hat{R}_{n,f}(a, s) \nabla s \leq f^{\nabla^n}(b) - f^{\nabla^n}(\ell).$$

Proof. Assume f^{∇^n} is decreasing; the case where f^{∇^n} is increasing is similar and is omitted. Let $F := -f^{\nabla^{n+1}}$. Because f^{∇^n} is decreasing, $f^{\nabla^{n+1}} \leq 0$, so that $F \geq 0$ and decreasing on $[a, b]$. Define

$$g(t) := \frac{\hat{h}_{n+1}(b, \rho(t))}{\hat{h}_{n+1}(b, \rho(a))} \in [0, 1], \quad t \in [a, b], \quad n \geq -1.$$

Note that F, g satisfy the assumptions of Steffensen's inequality, Theorem 3.1;



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using (4.2),

$$\int_a^b g(t) \nabla t = \frac{1}{\hat{h}_{n+1}(b, \rho(a))} \int_a^b \hat{h}_{n+1}(b, \rho(t)) \nabla t = \frac{\hat{h}_{n+2}(b, a)}{\hat{h}_{n+1}(b, \rho(a))}.$$

Thus if

$$b - \ell \leq \frac{\hat{h}_{n+2}(b, a)}{\hat{h}_{n+1}(b, \rho(a))} \leq \gamma - a,$$

then

$$\int_{\ell}^b F(t) \nabla t \leq \int_a^b F(t) g(t) \nabla t \leq \int_a^{\gamma} F(t) \nabla t.$$

By Corollary 4.2 and the fundamental theorem of nabla calculus, this simplifies to

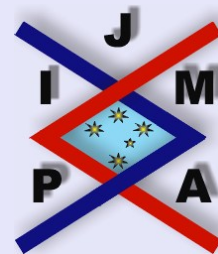
$$f^{\nabla n}(t)|_{t=a}^{\gamma} \leq \frac{1}{\hat{h}_{n+1}(b, \rho(a))} \int_a^b \hat{R}_{n,f}(a, s) \nabla s \leq f^{\nabla n}(t)|_{t=\ell}^b.$$

□

It is evident that an analogous result can be found for the delta integral case using the delta equivalent of Theorem 3.1.

Definition 5.1. A twice nabla-differentiable function $f : [a, b] \rightarrow \mathbb{R}$ is convex on $[a, b]$ if and only if $f^{\nabla 2} \geq 0$ on $[a, b]$.

The following corollary is the first Hermite-Hadamard inequality, derived from Theorem 5.1 with $n = 0$.



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Corollary 5.2. Let $f : [a, b] \rightarrow \mathbb{R}$ be convex and monotonic. Assume $\ell, \gamma \in [a, b]$ such that

$$\ell \geq b - \frac{\hat{h}_2(b, a)}{b - \rho(a)}, \quad \gamma \geq \frac{\hat{h}_2(b, a)}{b - \rho(a)} + a \quad \text{if } f \text{ is decreasing,}$$

$$\ell \leq b - \frac{\hat{h}_2(b, a)}{b - \rho(a)}, \quad \gamma \leq \frac{\hat{h}_2(b, a)}{b - \rho(a)} + a \quad \text{if } f \text{ is increasing.}$$

Then

$$f(\gamma) + \frac{\rho(a) - a}{b - \rho(a)} f(a) \leq \frac{1}{b - \rho(a)} \int_a^b f(t) \nabla t \leq \frac{b - a}{b - \rho(a)} f(a) + f(b) - f(\ell).$$

Another, slightly different, form of the first Hermite-Hadamard inequality is the following; this implies that for time scales with left-scattered points there are at least two inequalities of this type.

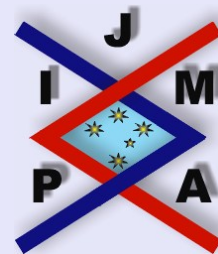
Theorem 5.3. Let $f : [a, b] \rightarrow \mathbb{R}$ be convex and monotonic. Assume $\ell, \gamma \in [a, b]$ such that

$$\ell \geq a + \frac{\hat{h}_2(b, a)}{b - a}, \quad \gamma \geq b - \frac{\hat{h}_2(b, a)}{b - a} \quad \text{if } f \text{ is decreasing,}$$

$$\ell \leq a + \frac{\hat{h}_2(b, a)}{b - a}, \quad \gamma \leq b - \frac{\hat{h}_2(b, a)}{b - a} \quad \text{if } f \text{ is increasing.}$$

Then

$$f(\gamma) \leq \frac{1}{b - a} \int_a^b f(\rho(t)) \nabla t \leq f(a) + f(b) - f(\ell).$$



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Proof. Assume f is decreasing and convex. Then $f^{\nabla^2} \geq 0$, $f^{\nabla} \leq 0$, and f^{∇} is increasing. Then $F := -f^{\nabla}$ is decreasing and satisfies $F \geq 0$. For $G := \frac{b-t}{b-a}$, $0 \leq G \leq 1$ and F, G satisfy the hypotheses of Theorem 3.1. Now the inequality expression

$$b - \ell \leq \int_a^b G(t) \nabla t \leq \gamma - a$$

takes the form

$$b - \ell \leq \frac{1}{b-a} \int_a^b (b-t) \nabla t \leq \gamma - a.$$

Concentrating on the left inequality,

$$\ell \geq b - \frac{1}{b-a} \int_a^b (b-t) \nabla t = b - \frac{1}{b-a} \int_a^b (b-a+a-t) \nabla t,$$

which simplifies to

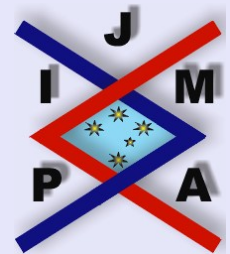
$$\ell \geq a + \frac{\hat{h}_2(b, a)}{b-a};$$

similarly,

$$\gamma \geq b - \frac{\hat{h}_2(b, a)}{b-a}.$$

Furthermore, note that $\int_r^s F(t) \nabla t = f(r) - f(s)$, and integration by parts yields

$$\int_a^b F(t) G(t) \nabla t = \frac{1}{b-a} \int_a^b (t-b) f^{\nabla}(t) \nabla t = f(a) - \frac{1}{b-a} \int_a^b f(\rho(t)) \nabla t.$$



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It follows that Steffensen's inequality takes the form

$$f(\ell) - f(b) \leq f(a) - \frac{1}{b-a} \int_a^b f(\rho(t)) \nabla t \leq f(a) - f(\gamma),$$

which can be rearranged to match the theorem's stated conclusion. □

Theorem 5.4. *Let $f : [a, b] \rightarrow \mathbb{R}$ be an $n + 1$ times nabla differentiable function such that*

$$m \leq f^{\nabla^{n+1}} \leq M$$

on $[a, b]$ for some real numbers $m < M$. Also, let $\ell, \gamma \in [a, b]$ such that

$$b - \ell \leq \frac{1}{M - m} [f^{\nabla^n}(b) - f^{\nabla^n}(a) - m(b - a)] \leq \gamma - a.$$

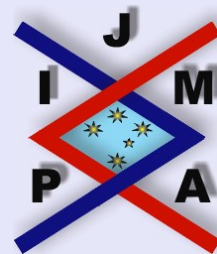
Then

$$\begin{aligned} m \hat{h}_{n+2}(b, a) + (M - m) \hat{h}_{n+2}(b, \ell) &\leq \int_a^b \hat{R}_{n,f}(a, t) \nabla t \\ &\leq M \hat{h}_{n+2}(b, a) + (m - M) \hat{h}_{n+2}(b, \gamma). \end{aligned}$$

Proof. Let

$$k(t) := \frac{1}{M - m} [f(t) - m \hat{h}_{n+1}(t, a)], \quad F(t) := \hat{h}_{n+1}(b, \rho(t)),$$

$$G(t) := k^{\nabla^{n+1}}(t) = \frac{1}{M - m} [f^{\nabla^{n+1}}(t) - m] \in [0, 1].$$



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Observe that F is nonnegative and decreasing, and

$$\int_a^b G(t) \nabla t = \frac{1}{M-m} [f^{\nabla n}(b) - f^{\nabla n}(a) - m(b-a)].$$

Since F, G satisfy the hypotheses of Theorem 3.1, we compute the various integrals given in (3.1). First, by (4.2),

$$\int_{\ell}^b F(t) \nabla t = \int_{\ell}^b \hat{h}_{n+1}(b, \rho(t)) \nabla t = -\hat{h}_{n+2}(b, t) \Big|_{t=\ell}^b = \hat{h}_{n+2}(b, \ell),$$

and

$$\int_a^{\gamma} F(t) \nabla t = -\hat{h}_{n+2}(b, t) \Big|_a^{\gamma} = \hat{h}_{n+2}(b, a) - \hat{h}_{n+2}(b, \gamma).$$

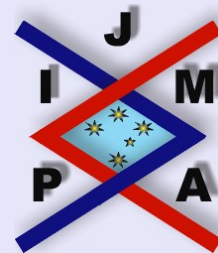
Moreover, using Corollary 4.2, we have

$$\begin{aligned} \int_a^b F(t)G(t) \nabla t &= \frac{1}{M-m} \int_a^b \hat{h}_{n+1}(b, \rho(t)) \left(f^{\nabla n+1}(t) - m \right) \nabla t \\ &= \frac{1}{M-m} \int_a^b \hat{R}_{n,f}(a, t) \nabla t + \frac{m}{M-m} \hat{h}_{n+2}(b, t) \Big|_a^b \\ &= \frac{1}{M-m} \int_a^b \hat{R}_{n,f}(a, t) \nabla t - \frac{m}{M-m} \hat{h}_{n+2}(b, a). \end{aligned}$$

Using Steffensen's inequality (3.1), we obtain

$$\hat{h}_{n+2}(b, \ell) \leq \frac{1}{M-m} \left[\int_a^b \hat{R}_{n,f}(a, t) \nabla t - m \hat{h}_{n+2}(b, a) \right] \leq \hat{h}_{n+2}(b, a) - \hat{h}_{n+2}(b, \gamma),$$

which yields the conclusion of the theorem. \square



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Theorem 5.5. Let $f : [a, b] \rightarrow \mathbb{R}$ be a nabla and delta differentiable function such that

$$m \leq f^\nabla, f^\Delta \leq M$$

on $[a, b]$ for some real numbers $m < M$.

(i) If there exist $\ell, \gamma \in [a, b]$ such that

$$b - \ell \leq \frac{1}{M - m} [f(b) - f(a) - m(b - a)] \leq \gamma - a,$$

then

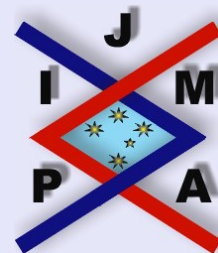
$$\begin{aligned} m\hat{h}_2(b, a) + (M - m)\hat{h}_2(b, \ell) &\leq \int_a^b f(t)\nabla t - (b - a)f(a) \\ &\leq M\hat{h}_2(b, a) + (m - M)\hat{h}_2(b, \gamma). \end{aligned}$$

(ii) If there exist $\ell, \gamma \in [a, b]$ such that

$$\gamma - a \leq \frac{1}{M - m} [f(b) - f(a) - m(b - a)] \leq b - \ell,$$

then

$$\begin{aligned} mh_2(a, b) + (M - m)h_2(a, \gamma) &\leq (b - a)f(b) - \int_a^b f(t)\Delta t \\ &\leq Mh_2(a, b) + (m - M)h_2(a, \ell). \end{aligned}$$



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Proof. The first part is just Theorem 5.4 with $n = 0$. For the second part, let

$$k(t) := \frac{1}{M - m} [f(t) - m(t - b)], \quad F(t) := h_1(a, \sigma(t)),$$

$$G(t) := k^\Delta(t) = \frac{1}{M - m} [f^\Delta(t) - m] \in [0, 1].$$

Clearly F is decreasing and nonpositive, and

$$\int_a^b G(t) \Delta t = \frac{1}{M - m} [f(b) - f(a) - m(b - a)] \in [\gamma - a, b - \ell].$$

Since F, G satisfy the hypotheses of Steffensen's inequality for delta integrals, we determine the corresponding integrals. First,

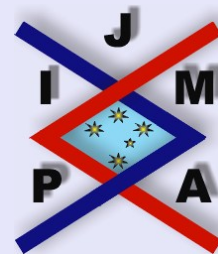
$$\int_\ell^b F(t) \Delta t = \int_\ell^b h_1(a, \sigma(t)) \Delta t = -h_2(a, t) \Big|_{t=\ell}^b = -h_2(a, b) + h_2(a, \ell),$$

and

$$\int_a^\gamma F(t) \Delta t = -h_2(a, t) \Big|_a^\gamma = -h_2(a, \gamma).$$

Moreover, using the formula for integration by parts for delta integrals,

$$\begin{aligned} \int_a^b F(t)G(t) \Delta t &= \int_a^b h_1(a, \sigma(t))k^\Delta(t) \Delta t \\ &= h_1(a, t)k(t) \Big|_a^b - \int_a^b h_1^\Delta(a, t)k(t) \Delta t \\ &= \frac{1}{M - m} \left[-(b - a)f(b) + \int_a^b f(t) \Delta t + mh_2(a, b) \right]. \end{aligned}$$



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Using Steffensen's inequality for delta integrals, we obtain

$$\begin{aligned} -h_2(a, b) + h_2(a, \ell) &\leq \frac{1}{M - m} \left[-(b - a)f(b) + \int_a^b f(t)\Delta t + mh_2(a, b) \right] \\ &\leq -h_2(a, \gamma), \end{aligned}$$

which yields the conclusion of (ii). \square

In [7], part (ii) of the above theorem also involved the equivalent of nabla derivatives for q -difference equations with $0 < q < 1$. However, the function used there, $F(t) = a - qt = a - \rho(t)$, is not of one sign on $[a, b]$, since $F(a) = a(1 - q) > 0$, $F(a/q) = 0$, and $F(a/q^2) = a(1 - 1/q) < 0$. For this reason we introduced a delta-derivative perspective in (ii) above and in the following.

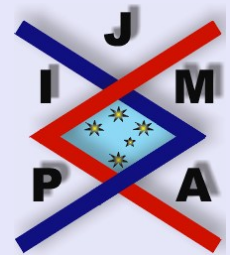
Corollary 5.6. *Let $f : [a, b] \rightarrow \mathbb{R}$ be a nabla and delta differentiable function such that*

$$m \leq f^\nabla, f^\Delta \leq M$$

on $[a, b]$ for some real numbers $m < M$. Assume there exist $\ell, \gamma \in [a, \rho(b)]$ such that

$$(5.1) \quad \rho(\gamma) - a \leq \frac{1}{M - m} [f(b) - f(a) - m(b - a)] \leq \gamma - a,$$

$$(5.2) \quad b - \ell \leq \frac{1}{M - m} [f(b) - f(a) - m(b - a)] \leq b - \rho(\ell).$$



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Then

$$\begin{aligned} 2mh_2(a, b) + (M - m) [h_2(\ell, b) + h_2(a, \rho(\gamma))] \\ \leq \int_a^b f(t) \nabla t - \int_a^b f(t) \Delta t + (b - a)(f(b) - f(a)) \\ \leq 2Mh_2(a, b) - (M - m) [h_2(\gamma, b) + h_2(a, \rho(\ell))]. \end{aligned}$$

Proof. By the previous theorem, Theorem 5.5,

$$\begin{aligned} m\hat{h}_2(b, a) + (M - m)\hat{h}_2(b, \ell) &\leq \int_a^b f(t) \nabla t - (b - a)f(a) \\ (5.3) \qquad \qquad \qquad &\leq M\hat{h}_2(b, a) + (m - M)\hat{h}_2(b, \gamma) \end{aligned}$$

using (i) and the fact that

$$b - \ell \leq \frac{1}{M - m} [f(b) - f(a) - m(b - a)] \leq \gamma - a;$$

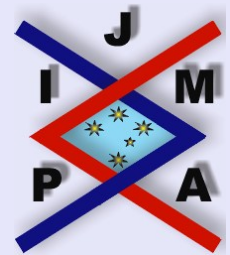
in like manner

$$\begin{aligned} mh_2(a, b) + (M - m)h_2(a, \rho(\gamma)) &\leq (b - a)f(b) - \int_a^b f(t) \Delta t \\ (5.4) \qquad \qquad \qquad &\leq Mh_2(a, b) + (m - M)h_2(a, \rho(\ell)) \end{aligned}$$

using (ii) and the fact that

$$\rho(\gamma) - a \leq \frac{1}{M - m} [f(b) - f(a) - m(b - a)] \leq b - \rho(\ell).$$

Add (5.3) to (5.4) and use Theorem 2.2 to arrive at the conclusion. \square



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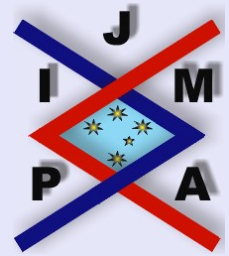


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Remark 1. If $\mathbb{T} = \mathbb{R}$, set $\lambda := b - \ell = \gamma - a$, so that $b - \gamma = \ell - a = b - a - \lambda$. Here the nabla and delta integrals of f on $[a, b]$ are identical, and $h_2(s, t) = (t - s)^2/2$, so the conclusion of the previous corollary, Corollary 5.6, is the known [7] inequality

$$m + \frac{(M - m)\lambda^2}{(b - a)^2} \leq \frac{f(b) - f(a)}{b - a} \leq M - \frac{(M - m)(b - a - \lambda)^2}{(b - a)^2}.$$

If $\mathbb{T} = \mathbb{Z}$, then $h_2(s, t) = (t - s)(t - s + 1)/2 = (t - s)^{\bar{2}}/2$ and

$$\int_a^b f(t) \nabla t - \int_a^b f(t) \Delta t = \sum_{t=a+1}^b f(t) - \sum_{t=a}^{b-1} f(t) = f(b) - f(a).$$

This time take $\lambda = b - \ell = \gamma - 1 - a$. The discrete conclusion of Corollary 5.6 is thus

$$m + \frac{(M - m)\lambda^{\bar{2}}}{(b - a)^{\bar{2}}} \leq \frac{f(b) - f(a)}{b - a} \leq M - \frac{(M - m)(b - a - \lambda - 1)^{\bar{2}}}{(b - a)^{\bar{2}}}.$$

Corollary 5.7 (Iyengar's Inequality). Let $f : [a, b] \rightarrow \mathbb{R}$ be a nabla and delta differentiable function such that

$$m \leq f^{\nabla}, f^{\Delta} \leq M$$

on $[a, b]$ for some real numbers $m < M$. Assume there exist $\ell, \gamma \in [a, \rho(b)]$ such

that (5.1), (5.2) are satisfied. Then

$$\begin{aligned} & (M - m) [h_2(\ell, b) + h_2(a, \rho(\ell)) - h_2(a, b)] \\ & \leq \int_a^b f(t) \nabla t + \int_a^b f(t) \Delta t - (b - a)(f(b) + f(a)) \\ & \leq (M - m) [h_2(a, b) - h_2(\gamma, b) - h_2(a, \rho(\gamma))]. \end{aligned}$$

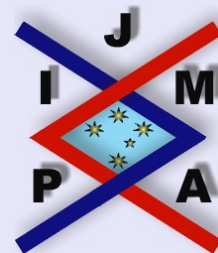
Proof. Subtract (5.4) from (5.3) and use Theorem 2.2 to arrive at the conclusion. \square

Remark 2. Again if $\mathbb{T} = \mathbb{R}$, then $\int_a^b f(t) \nabla t = \int_a^b f(t) \Delta t = \int_a^b f(t) dt$ and $h_2(t, s) = (t - s)^2/2$. Moreover, $\rho(\ell) = \ell$ and $\rho(\gamma) = \gamma$; set

$$\lambda = b - \ell = \gamma - a = \frac{1}{M - m} [f(b) - f(a) - m(b - a)].$$

This transforms the conclusion of Corollary 5.7 into a continuous calculus version,

$$\begin{aligned} & \left| \int_a^b f(t) dt - \frac{f(a) + f(b)}{2} (b - a) \right| \\ & \leq \frac{[f(b) - f(a) - m(b - a)] [M(b - a) + f(a) - f(b)]}{2(M - m)}. \end{aligned}$$



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6. Applications of Čebyšev's Inequality

Recently, Čebyšev's inequality on time scales for delta integrals was proven [9]. We repeat the statement of it here in the case of nabla integrals for completeness.

Theorem 6.1 (Čebyšev's inequality). *Let f and g be both increasing or both decreasing in $[a, b]$. Then*

$$\int_a^b f(t)g(t)\nabla t \geq \frac{1}{b-a} \int_a^b f(t)\nabla t \int_a^b g(t)\nabla t.$$

If one of the functions is increasing and the other is decreasing, then the above inequality is reversed.

The following is an application of Čebyšev's inequality, which extends a similar result in [7] to general time scales.

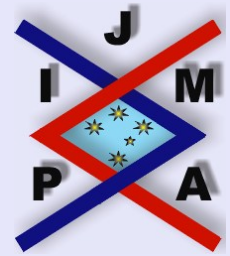
Theorem 6.2. *Assume that $f^{\nabla^{n+1}}$ is monotonic on $[a, b]$.*

(i) *If $f^{\nabla^{n+1}}$ is increasing, then*

$$\begin{aligned} 0 &\geq \int_a^b \hat{R}_{n,f}(a,t)\nabla t - \left[\frac{f^{\nabla^n}(b) - f^{\nabla^n}(a)}{b-a} \right] \hat{h}_{n+2}(b,a) \\ &\geq \left[f^{\nabla^{n+1}}(a) - f^{\nabla^{n+1}}(b) \right] \hat{h}_{n+2}(b,a). \end{aligned}$$

(ii) *If $f^{\nabla^{n+1}}$ is decreasing, then*

$$\begin{aligned} 0 &\leq \int_a^b \hat{R}_{n,f}(a,t)\nabla t - \left[\frac{f^{\nabla^n}(b) - f^{\nabla^n}(a)}{b-a} \right] \hat{h}_{n+2}(b,a) \\ &\leq \left[f^{\nabla^{n+1}}(a) - f^{\nabla^{n+1}}(b) \right] \hat{h}_{n+2}(b,a). \end{aligned}$$



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Proof. The situation for (ii) is analogous to that of (i). Assume (i), and set $F(t) := f^{\nabla^{n+1}}(t)$, $G(t) := \hat{h}_{n+1}(b, \rho(t))$. Then F is increasing by assumption, and G is decreasing, so that by Čebyšev's nabla inequality,

$$\int_a^b F(t)G(t)\nabla t \leq \frac{1}{b-a} \int_a^b F(t)\nabla t \int_a^b G(t)\nabla t.$$

By Corollary 4.2,

$$\int_a^b F(t)G(t)\nabla t = \int_a^b f^{\nabla^{n+1}}(t)\hat{h}_{n+1}(b, \rho(t))\nabla t = \int_a^b \hat{R}_{n,f}(a, t)\nabla t.$$

We also have

$$\int_a^b F(t)\nabla t = f^{\nabla^n}(b) - f^{\nabla^n}(a), \quad \int_a^b G(t)\nabla t = \int_a^b \hat{h}_{n+1}(b, \rho(t)) = \hat{h}_{n+2}(b, a).$$

Thus Čebyšev's inequality implies

$$\int_a^b \hat{R}_{n,f}(a, t)\nabla t \leq \frac{1}{b-a} [f^{\nabla^n}(b) - f^{\nabla^n}(a)] \hat{h}_{n+2}(b, a),$$

which subtracts to the left side of the inequality. Since $f^{\nabla^{n+1}}$ is increasing on $[a, b]$,

$$f^{\nabla^{n+1}}(a)\hat{h}_{n+2}(b, a) \leq \left[\frac{f^{\nabla^n}(b) - f^{\nabla^n}(a)}{b-a} \right] \hat{h}_{n+2}(b, a) \leq f^{\nabla^{n+1}}(b)\hat{h}_{n+2}(b, a),$$



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and we have

$$\int_a^b \hat{R}_{n,f}(a,t) \nabla t - \left[\frac{f^{\nabla n}(b) - f^{\nabla n}(a)}{b-a} \right] \hat{h}_{n+2}(b,a) \geq \int_a^b \hat{R}_{n,f}(a,t) \nabla t - f^{\nabla^{n+1}}(b) \hat{h}_{n+2}(b,a).$$

Now Corollary 4.2 and $f^{\nabla^{n+1}}$ is increasing imply that

$$f^{\nabla^{n+1}}(b) \int_a^b \hat{h}_{n+1}(b, \rho(t)) \nabla t \geq \int_a^b \hat{R}_{n,f}(a,t) \nabla t \geq f^{\nabla^{n+1}}(a) \int_a^b \hat{h}_{n+1}(b, \rho(t)) \nabla t,$$

which simplifies to

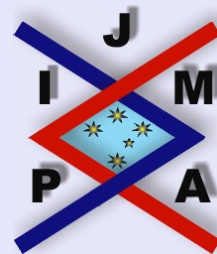
$$f^{\nabla^{n+1}}(b) \hat{h}_{n+2}(b,a) \geq \int_a^b \hat{R}_{n,f}(a,t) \nabla t \geq f^{\nabla^{n+1}}(a) \hat{h}_{n+2}(b,a).$$

This, together with the earlier lines give the right side of the inequality. \square

Theorem 6.3. Assume that $f^{\Delta^{n+1}}$ is monotonic on $[a, b]$ and the function g_k is as defined in Definition 2.1.

(i) If $f^{\Delta^{n+1}}$ is increasing, then

$$0 \leq (-1)^{n+1} \int_a^b R_{n,f}(b,t) \Delta t - \left[\frac{f^{\Delta n}(b) - f^{\Delta n}(a)}{b-a} \right] g_{n+2}(b,a) \leq \left[f^{\Delta^{n+1}}(b) - f^{\Delta^{n+1}}(a) \right] g_{n+2}(b,a).$$



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(ii) If $f^{\Delta^{n+1}}$ is decreasing, then

$$\begin{aligned} 0 &\geq (-1)^{n+1} \int_a^b R_{n,f}(b,t)\Delta t - \left[\frac{f^{\Delta^n}(b) - f^{\Delta^n}(a)}{b-a} \right] g_{n+2}(b,a) \\ &\geq \left[f^{\Delta^{n+1}}(b) - f^{\Delta^{n+1}}(a) \right] g_{n+2}(b,a). \end{aligned}$$

Proof. The situation for (ii) is analogous to that of (i). Assume (i), and set $F(t) := f^{\Delta^{n+1}}(t)$, $G(t) := (-1)^{n+1}h_{n+1}(a, \sigma(t))$. Then F and G are increasing, so that by Čebyšev's delta inequality,

$$\int_a^b F(t)G(t)\Delta t \geq \frac{1}{b-a} \int_a^b F(t)\Delta t \int_a^b G(t)\Delta t.$$

By Lemma 4.3 with $t = a$,

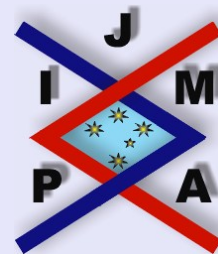
$$\begin{aligned} \int_a^b F(t)G(t)\Delta t &= (-1)^{n+1} \int_a^b f^{\Delta^{n+1}}(t)h_{n+1}(a, \sigma(t))\Delta t \\ &= (-1)^{n+1} \int_a^b R_{n,f}(b,t)\Delta t. \end{aligned}$$

We also have $\int_a^b F(t)\Delta t = f^{\Delta^n}(b) - f^{\Delta^n}(a)$, and, using Theorem 2.2,

$$\int_a^b G(t)\Delta t = (-1)^{n+1} \int_a^b h_{n+1}(a, \sigma(t))\Delta t = g_{n+2}(b,a).$$

Thus Čebyšev's inequality implies

$$(-1)^{n+1} \int_a^b R_{n,f}(b,t)\Delta t \geq \frac{1}{b-a} [f^{\Delta^n}(b) - f^{\Delta^n}(a)] g_{n+2}(b,a),$$



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which subtracts to the left side of the inequality. Since $f^{\Delta^{n+1}}$ is increasing on $[a, b]$,

$$f^{\Delta^{n+1}}(a)g_{n+2}(b, a) \leq \left[\frac{f^{\Delta^n}(b) - f^{\Delta^n}(a)}{b - a} \right] g_{n+2}(b, a) \leq f^{\Delta^{n+1}}(b)g_{n+2}(b, a),$$

and we have

$$\begin{aligned} & (-1)^{n+1} \int_a^b R_{n,f}(b, t) \Delta t - f^{\Delta^{n+1}}(a)g_{n+2}(b, a) \\ & \geq (-1)^{n+1} \int_a^b R_{n,f}(b, t) \Delta t - \left[\frac{f^{\Delta^n}(b) - f^{\Delta^n}(a)}{b - a} \right] g_{n+2}(b, a). \end{aligned}$$

Now Theorem 2.2 and Lemma 4.3 again with $t = a$ yield

$$(-1)^{n+1} \int_a^b R_{n,f}(b, t) \Delta t = \int_a^b g_{n+1}(\sigma(t), a) f^{\Delta^{n+1}}(t) \Delta t.$$

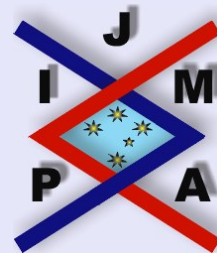
Since $f^{\Delta^{n+1}}$ is increasing,

$$\begin{aligned} f^{\Delta^{n+1}}(b) \int_a^b g_{n+1}(\sigma(t), a) \Delta t & \geq (-1)^{n+1} \int_a^b R_{n,f}(b, t) \Delta t \\ & \geq f^{\Delta^{n+1}}(a) \int_a^b g_{n+1}(\sigma(t), a) \Delta t, \end{aligned}$$

which simplifies to

$$f^{\Delta^{n+1}}(b)g_{n+2}(b, a) \geq (-1)^{n+1} \int_a^b R_{n,f}(b, t) \Delta t \geq f^{\Delta^{n+1}}(a)g_{n+2}(b, a).$$

This, together with the earlier lines give the right side of the inequality. \square



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Remark 3. If $\mathbb{T} = \mathbb{R}$, then combining Theorem 6.2 and Theorem 6.3 yields Theorem 3.1 in [6].

Remark 4. In Theorem 6.2 (i), if $n = 0$, we obtain

$$(6.1) \quad \int_a^b f(t) \nabla t \leq (b-a)f(a) + \frac{\hat{h}_2(b,a)}{b-a}(f(b) - f(a)).$$

Compare that with the following result.

Theorem 6.4. Assume that f is nabla convex on $[a, b]$; that is, $f^{\nabla^2} \geq 0$ on $[a, b]$. Then

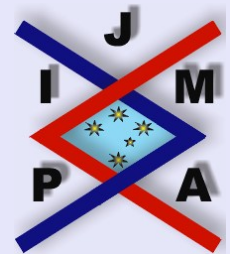
$$(6.2) \quad \int_a^b f(\rho(t)) \nabla t \leq (b-a)f(b) - \frac{\hat{h}_2(b,a)}{b-a}(f(b) - f(a)).$$

Proof. If $F := f^{\nabla}$ and $G(t) := t - a = \hat{h}_1(t, a)$, then both F and G are increasing functions. By Čebyšev's inequality on time scales, and the definition of \hat{h} in (2.2),

$$\int_a^b f^{\nabla}(t)(t-a) \nabla t \geq \frac{1}{b-a} \int_a^b f^{\nabla}(t) \nabla t \int_a^b \hat{h}_1(t, a) \nabla t.$$

Using nabla integration by parts on the left, and calculating the right yields the result. \square

The following result is a Hermite-Hadamard-type inequality for time scales; compare with Corollary 5.2.



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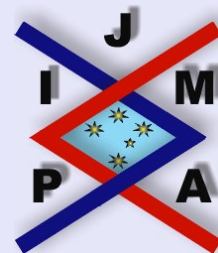
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Corollary 6.5. Let f be nabla convex on $[a, b]$. Then

$$\frac{1}{b-a} \int_a^b \frac{f(\rho(t)) + f(t)}{2} \nabla t \leq \frac{f(b) + f(a)}{2}.$$

Proof. Use (6.1), (6.2) and rearrange accordingly. □



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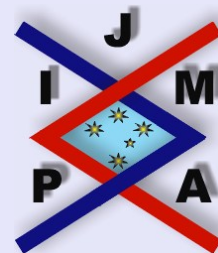
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