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A UNIFIED TREATMENT OF CERTAIN SUBCLASSES OF PRESTARLIKE FUNCTIONS

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Abstract

In this paper we introduce and study some properties of a unified class $\mathcal{U}[\mu, \alpha, \beta, \gamma, \lambda, \eta, A, B]$ of prestarlike functions with negative coefficients in a unit disk U. These properties include growth and distortion, radii of convexity, radii of starlikeness and radii of close-to-convexity.

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Contents

1. Introduction

Let A denote the class of *normalized* analytic functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n\, z^n, \tag{1.1}$$

in the unit disk $U = \{z : |z| < 1\}$. Further let S denote the subclass of A consisting of analytic and univalent functions f in the unit disk U. A function f in S is said to be starlike of order α if and only if

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \tag{1.2}$$

for some α $(0 \leq \alpha < 1)$. We denote by $S^*(\alpha)$ the class of all starlike functions of order α. It is well-known that $S^*(\alpha) \subseteq S^*(0) \equiv S^*$.

Let the function

$$S_\alpha(z) = \frac{z}{(1-z)^{2(1-\alpha)}}, \qquad (z \in U; \quad 0 \leq \alpha < 1) \tag{1.3}$$

which is the extremal function for the class $S^*(\alpha)$. We also note that $S_\alpha(z)$ can be written in the form:

$$S_\alpha(z) = z + \sum_{n=2}^{\infty} |c_n(\alpha)| z^n, \tag{1.4}$$

where

$$c_n(\alpha) = \frac{\Pi_{j=2}^{n}(j-2\alpha)}{(n-1)!} \qquad (n \in \mathbf{N}\ \{1\},\ \mathbf{N} := \{1, 2, 3, \dots\}). \tag{1.5}$$

We note that $c_n(\alpha)$ is decreasing in α and satisfies

$$\lim_{n\to\infty} c_n(\alpha) = \begin{cases} \infty & \text{if } \alpha < \frac{1}{2}\,, \\ 1 & \text{if } \alpha = \frac{1}{2}\,, \\ 0 & \text{if } \alpha > \frac{1}{2}. \end{cases} \tag{1.6}$$

Also a function f in S is said to be convex of order α if and only if

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \tag{1.7}$$

for some α $(0 \le \alpha < 1)$. We denote by $K(\alpha)$ the class of all convex functions of order α. It is a fact that $f \in K(\alpha)$ if and only if $zf'(z) \in S^*(\alpha)$.

The well-known Hadamard product (or convolution) of two functions $f(z)$ given by (1.1) and $g(z)$ given by $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ is defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad (z \in U). \tag{1.8}$$

Let $\mathcal{R}[\mu, \alpha, \beta, \gamma, \lambda, A, B]$ denote the class of prestarlike functions satisfying the following condition

$$\left| \frac{\frac{zH_\lambda'(z)}{H_\lambda(z)} - 1}{2\gamma(B-A)\left(\frac{zH_\lambda'(z)}{H_\lambda(z)} - \mu\right) - B\left(\frac{zH_\lambda'(z)}{H_\lambda(z)} - 1\right)} \right| < \beta, \tag{1.9}$$

where $H_\lambda(z) = (1-\lambda)h(z) + \lambda z h'(z)$, $\lambda \geq 0$, $h = f * S_\alpha$, $0 < \beta \leq 1$, $0 \leq \mu < 1$, and

$$\frac{B}{2(B-A)} < \gamma \leq \begin{cases} \frac{B}{2(B-A)\mu}, & \mu \neq 0, \\ 1, & \mu = 0 \end{cases}$$

for fixed $-1 \leq A \leq B \leq 1$ and $0 < B \leq 1$.

We also note that a function f is a so-called α-prestarlike $(0 \leq \alpha < 1)$ function if, and only if, $h = f * S_\alpha \in S^*(\alpha)$ which was first introduced by Ruscheweyh [3], and was rigorously studied by Silverman and Silvia [4], Owa and Ahuja [5] and Uralegaddi and Sarangi [6]. Further, a function $f \in \mathcal{A}$ is in the class $\mathcal{C}[\mu, \alpha, \beta, \gamma, \lambda, A, B]$ if and only if, $zf'(z) \in \mathcal{R}[\mu, \alpha, \beta, \gamma, \lambda, A, B]$.

Let T denote the subclass of A consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad (a_n \geq 0). \tag{1.10}$$

Let us write

$$\mathcal{R}_{\mathcal{T}}[\mu, \alpha, \beta, \gamma, \lambda, A, B] = \mathcal{R}[\mu, \alpha, \beta, \gamma, \lambda, A, B] \cap T$$

and

$$\mathcal{C}_{\mathcal{T}}[\mu, \alpha, \beta, \gamma, \lambda, A, B] = \mathcal{C}[\mu, \alpha, \beta, \gamma, \lambda, A, B] \cap T$$

where T is the class of functions of the form (1.10) that are analytic and univalent in U. The idea of unifying the study of classes $\mathcal{R}_{\mathcal{T}}[\mu, \alpha, \beta, \gamma, \lambda, A, B]$

and $\mathcal{C}_{\mathcal{T}}[\mu, \alpha, \beta, \gamma, \lambda, A, B]$ thus, forming a new class $\mathcal{U}[\mu, \alpha, \beta, \gamma, \lambda, \eta, A, B]$ is somewhat or rather motivated from the work of [1] and [2].

In this paper, we will study the unified presentation of prestarlike functions belonging to $\mathcal{U}[\mu, \alpha, \beta, \gamma, \lambda, \eta, A, B]$ which include growth and distortion theorem, radii of convexity, radii of starlikeness and radii of close-to-convexity.

2. Coefficient Inequality

Our main tool in this paper is the following result, which can be easily proven, and the details are omitted.

Lemma 2.1. *Let the function f be defined by (1.10). Then $f \in \mathcal{R}_{\mathcal{T}}[\mu, \alpha, \beta, \gamma, \lambda, A, B]$ if and only if*

$$\sum_{n=2}^{\infty} \Lambda(n,\lambda) D[n,\beta,\gamma,A,B]|a_n| c_n(\alpha) \leq E[\beta,\gamma,\mu,A,B] \tag{2.1}$$

where

$$\Lambda(n,\lambda) = (1+(n-1)\lambda),$$
$$D[n,\beta,\gamma,A,B] = n-1+2\beta\gamma(n-\mu)(B-A)-B\beta(n-1),$$
$$E[\beta,\gamma,\mu,A,B] = 2\beta\gamma(1-\mu)(B-A).$$

The result is sharp.

Next, by observing that

$$f \in \mathcal{C}_{\mathcal{T}}[\mu,\alpha,\beta,\gamma,\lambda,A,B] \Leftrightarrow zf'(z) \in \mathcal{R}_{\mathcal{T}}[\mu,\alpha,\beta,\gamma,\lambda,A,B], \tag{2.2}$$

we gain the following Lemma 2.2.

Lemma 2.2. *Let the function f be defined by (1.10). Then $f \in \mathcal{C}_{\mathcal{T}}[\mu, \alpha, \beta, \gamma, \lambda, A, B]$ if and only if*

$$\sum_{n=2}^{\infty} n\Lambda(n,\lambda) D[n,\beta,\gamma,A,B]|a_n| c_n(\alpha) \leq E[\beta,\gamma,\mu,A,B] \tag{2.3}$$

where

$$\Lambda(n,\lambda) = (1+(n-1)\lambda),$$
$$D[n,\beta,\gamma,A,B] = n-1+2\beta\gamma(n-\mu)(B-A)-B\beta(n-1),$$
$$E[\beta,\gamma,\mu,A,B] = 2\beta\gamma(1-\mu)(B-A)$$

and $c_n(\alpha)$ given by (1.5).

In view of Lemma 2.1 and Lemma 2.2, we unified the classes $\mathcal{R}_{\mathcal{T}}[\mu,\alpha,\beta,\gamma,\lambda,A,B]$ and $\mathcal{C}_{\mathcal{T}}[\mu,\alpha,\beta,\gamma,\lambda,A,B]$ and so a new class $\mathcal{U}[\mu,\alpha,\beta,\gamma,\lambda,\eta,A,B]$ is formed. Thus we say that a function f defined by (1.10) belongs to $\mathcal{U}[\mu,\alpha,\beta,\gamma,\lambda,\eta,A,B]$ if and only if,

$$\sum_{n=2}^{\infty}(1-\eta+n\eta)\Lambda(n,\lambda)D[n,\beta,\gamma,A,B]|a_n|c_n(\alpha) \leq E[\beta,\gamma,\mu,A,B], \tag{2.4}$$

$$(0 \leq \alpha < 1;\ 0 < \beta \leq 1;\ \eta \geq 0;\ \lambda \geq 0;\ -1 \leq A \leq B \leq 1 \text{ and } 0 < B \leq 1),$$

where $\Lambda(n,\lambda)$, $D[n,\beta,\gamma,A,B]$, $E[\beta,\gamma,\mu,A,B]$ and $c_n(\alpha)$ are given in (Lemma 2.1 and Lemma 2.2) and given by (1.5), respectively.

Clearly, we obtain

$$\mathcal{U}[\mu,\alpha,\beta,\gamma,\lambda,\eta,A,B] = (1-\eta)\mathcal{R}_{\mathcal{T}}[\mu,\alpha,\beta,\gamma,A,B] + \eta\mathcal{C}_{\mathcal{T}}[\mu,\alpha,\beta,\gamma,A,B],$$

so that

$$\mathcal{U}[\mu,\alpha,\beta,\gamma,\lambda,0,A,B] = \mathcal{R}_{\mathcal{T}}[\mu,\alpha,\beta,\gamma,A,B],$$

and

$$\mathcal{U}[\mu,\alpha,\beta,\gamma,\lambda,1,A,B] = \mathcal{C}_{\mathcal{T}}[\mu,\alpha,\beta,\gamma,A,B].$$

3. Growth and Distortion Theorem

A distortion property for function f in the class $\mathcal{U}[\mu, \alpha, \beta, \gamma, \lambda, \eta, A, B]$ is given as follows:

Theorem 3.1. *Let the function f defined by (1.10) be in the class $\mathcal{U}[\mu, \alpha, \beta, \gamma, \lambda, \eta, A, B]$, then*

$$r - \frac{E[\beta, \gamma, \mu, A, B]}{2(1+\eta)\Lambda(2,\lambda)D[2, \beta, \gamma, A, B](1-\alpha)} r^2 \leq |f(z)| \leq r + \frac{E[\beta, \gamma, \mu, A, B]}{2(1+\eta)\Lambda(2,\lambda)D[2, \beta, \gamma, A, B](1-\alpha)} r^2, \tag{3.1}$$

$$(\eta \geq 0; \quad 0 \leq \alpha < 1; \quad 0 < \beta \leq 1; \quad z \in U)$$

and

$$1 - \frac{E[\beta, \gamma, \mu, A, B]}{(1+\eta)\Lambda(2,\lambda)D[2, \beta, \gamma, A, B](1-\alpha)} r \leq |f'(z)| \leq 1 + \frac{E[\beta, \gamma, \mu, A, B]}{(1+\eta)\Lambda(2,\lambda)D[2, \beta, \gamma, A, B](1-\alpha)} r, \tag{3.2}$$

$$(\eta \geq 0; \quad 0 \leq \alpha < 1; \quad 0 < \beta \leq 1; \quad z \in U).$$

The bounds in (3.1) and (3.2) are attained for the function f given by

$$f(z) = z - \frac{E[\beta, \gamma, \mu, A, B]}{2(1+\eta)\Lambda(2,\lambda)D[2, \beta, \gamma, A, B](1-\alpha)} z^2.$$

Proof. Observing that $c_n(\alpha)$ defined by (1.5) is nondecreasing for $(0 \le \alpha < 1)$, we find from (2.4) that

$$\sum_{n=2}^{\infty} |a_n| \le \frac{E[\beta,\gamma,\mu,A,B]}{2(1+\eta)\Lambda(2,\lambda)D[2,\beta,\gamma,A,B](1-\alpha)}. \tag{3.3}$$

Using (1.10) and (3.3), we readily have $(z \in U)$

$$\begin{aligned}|f(z)| &\ge |z| - \sum_{n=2}^{\infty} |a_n| c_n(\alpha)|z^n| \\ &\ge |z| - |z^2| \sum_{n=2}^{\infty} |a_n| c_n(\alpha), \\ &\ge r - \frac{E[\beta,\gamma,\mu,A,B]}{2(1+\eta)\Lambda(2,\lambda)D[2,\beta,\gamma,A,B](1-\alpha)} r^2, \qquad |z| = r < 1\end{aligned}$$

and

$$\begin{aligned}|f(z)| &\le |z| + \sum_{n=2}^{\infty} |a_n| c_n(\alpha)|z^n| \\ &\le |z| + |z^2| \sum_{n=2}^{\infty} |a_n| c_n(\alpha), \\ &\le r + \frac{E[\beta,\gamma,\mu,A,B]}{2(1+\eta)\Lambda(2,\lambda)D[2,\beta,\gamma,A,B](1-\alpha)} r^2, \qquad |z| = r < 1,\end{aligned}$$

which proves the assertion (3.1) of Theorem 3.1.

Also, from (1.10), we find for $z \in U$ that

$$\begin{aligned} |f'(z)| &\geq 1 - \sum_{n=2}^{\infty} n|a_n|c_n(\alpha)|z^{n-1}| \\ &\geq 1 - |z| \sum_{n=2}^{\infty} n|a_n|c_n(\alpha), \\ &\geq 1 - \frac{E[\beta,\gamma,\mu,A,B]}{(1+\eta)\Lambda(2,\lambda)D[2,\beta,\gamma,A,B](1-\alpha)} r, \qquad |z| = r < 1 \end{aligned}$$

and

$$\begin{aligned} |f'(z)| &\leq 1 + \sum_{n=2}^{\infty} n|a_n|c_n(\alpha)|z^{n-1}| \\ &\leq 1 + |z| \sum_{n=2}^{\infty} n|a_n|c_n(\alpha), \\ &\leq 1 + \frac{E[\beta,\gamma,\mu,A,B]}{2(1+\eta)\Lambda(2,\lambda)D[2,\beta,\gamma,A,B](1-\alpha)} r, \qquad |z| = r < 1, \end{aligned}$$

which proves the assertion (3.2) of Theorem 3.1. □

4. Radii Convexity and Starlikeness

The radii of convexity for class $\mathcal{U}[\mu, \alpha, \beta, \gamma, \lambda, \eta, A, B]$ is given by the following theorem.

Theorem 4.1. *Let the function f be in the class $\mathcal{U}[\mu, \alpha, \beta, \gamma, \lambda, \eta, A, B]$. Then the function f is convex of order ρ $(0 \le \rho < 1)$ in the disk $|z| < r_1(\mu, \alpha, \beta, \gamma, \lambda, \eta, A, B) = r_1$, where*

$$r_1 = \inf_n \left\{ \frac{2(1-\alpha)(1-\rho)\Lambda(n,\lambda)D[n,\beta,\gamma,A,B](1-\eta+n\eta)}{n(n-\rho)E[\beta,\gamma,\mu,A,B]} \right\}^{\frac{1}{n-1}}. \tag{4.1}$$

Proof. It sufficient to show that

$$\begin{aligned} \left|\frac{zf''(z)}{f'(z)}\right| &= \left|\frac{-\sum_{n=2}^{\infty} n(n-1)a_n z^{n-1}}{1-\sum_{n=2}^{\infty} na_n z^{n-1}}\right| \\ &\le \frac{\sum_{n=2}^{\infty} n(n-1)a_n|z|^{n-1}}{1-\sum_{n=2}^{\infty} na_n|z|^{1-n}} \end{aligned} \tag{4.2}$$

which implies that

$$\begin{aligned} (1-\rho) - \left|\frac{zf''(z)}{f'(z)}\right| &\ge (1-\rho) - \frac{\sum_{n=2}^{\infty} n(n-1)|a_n||z|^{n-1}}{1-\sum_{n=2}^{\infty} na_n z^{n-1}} \\ &= \frac{(1-\rho)-\sum_{n=2}^{\infty} n(n-\rho)a_n|z|^{n-1}}{1-\sum_{n=2}^{\infty} na_n|z|^{n-1}}. \end{aligned} \tag{4.3}$$

Hence from (4.1), if

$$|z|^{n-1} \le \frac{(1-\rho)}{n(n-\rho)} \cdot \frac{2(1-\alpha)\Lambda(n,\lambda)D[n,\beta,\gamma,A,B](1-\eta+n\eta)}{E[\beta,\gamma,\mu,A,B]}, \tag{4.4}$$

and according to (2.4)

$$1 - \rho - \sum_{n=2}^{\infty} n(n-\rho)a_n |z|^{n-1} > 1 - \rho - (1-\rho) = \rho. \tag{4.5}$$

Hence from (4.3), we obtain

$$\left| \frac{zf''(z)}{f'(z)} \right| < 1 - \rho$$

Therefore

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0,$$

which shows that f is convex in the disk $|z| < r_1(\mu, \alpha, \beta, \gamma, \lambda, \eta, \rho, A, B)$. □

By setting $\eta = 0$ and $\eta = 1$, we have the Corollary 4.2 and the Corollary 4.3, respectively.

Corollary 4.2. *Let the function f be in the class $\mathcal{R}_{\mathcal{T}}(\mu, \alpha, \beta, \gamma, \lambda, \rho, A, B)$. Then the function f is convex of order ρ $(0 \le \rho < 1)$ in the disk $|z| < r_2(\mu, \alpha, \beta, \gamma, \lambda, \rho, A, B) = r_2$, where*

$$r_2 = \inf_n \left\{ \frac{2(1-\alpha)(1-\rho)\Lambda(n,\lambda)D[n,\beta,\gamma,A,B]}{n(n-\rho)E[\beta,\gamma,\mu,A,B]} \right\}^{\frac{1}{n-1}}. \tag{4.6}$$

Corollary 4.3. *Let the function f be in the class $\mathcal{C}_{\mathcal{T}}(\mu, \alpha, \beta, \gamma, \lambda, \rho, A, B)$. Then the function f is convex of order ρ $(0 \le \rho < 1)$ in the disk $|z| < r_3(\mu, \alpha, \beta, \gamma, \lambda, \rho, A, B) = r_3$, where*

$$r_3 = \inf_n \left\{ \frac{2(1-\alpha)(1-\rho)\Lambda(n,\lambda)D[n,\beta,\gamma,A,B]}{(n-\rho)E[\beta,\gamma,\mu,A,B]} \right\}^{\frac{1}{n-1}}. \tag{4.7}$$

Theorem 4.4. *Let the function f be in the class $\mathcal{U}[\mu,\alpha,\beta,\gamma,\lambda,\eta,A,B]$. Then the function f is starlike of order ρ $(0\le\rho<1)$ in the disk $|z|<r_4(\mu,\alpha,\beta,\gamma,\lambda,\eta,A,B)=r_4$, where*

$$r_4=\inf_n\left\{\frac{2(1-\alpha)(1-\rho)\Lambda(n,\lambda)D[n,\beta,\gamma,A,B](1-\eta+n\eta)}{(n-\rho)E[\beta,\gamma,\mu,A,B]}\right\}^{\frac{1}{n-1}}. \tag{4.8}$$

Proof. It sufficient to show that

$$\left|\frac{zf'(z)}{f(z)}-1\right|<1-\rho$$

Using a similar method to Theorem 4.1 and making use of (2.4), we get (4.8). □

Letting $\eta=0$ and $\eta=1$, we have the Corollary 4.5 and the Corollary 4.6, respectively.

Corollary 4.5. *Let the function f be in the class $\mathcal{R}_{\mathcal{T}}(\mu,\alpha,\beta,\gamma,\lambda,\rho,A,B)$. Then the function f is starlike of order ρ $(0\le\rho<1)$ in the disk $|z|<r_5(\mu,\alpha,\beta,\gamma,\lambda,\rho,A,B)=r_5$, where*

$$r_5=\inf_n\left\{\frac{2(1-\alpha)(1-\rho)\Lambda(n,\lambda)D[n,\beta,\gamma,A,B]}{(n-\rho)E[\beta,\gamma,\mu,A,B]}\right\}^{\frac{1}{n-1}}. \tag{4.9}$$

Corollary 4.6. *Let the function f be in the class $\mathcal{C}_{\mathcal{T}}(\mu,\alpha,\beta,\gamma,\lambda,\rho,A,B)$. Then the function f is starlike of order ρ $(0\le\rho<1)$ in the disk $|z|<r_6(\mu,\alpha,\beta,\gamma,\lambda,\rho,A,B)=r_6$, where*

$$r_6=\inf_n\left\{\frac{2n(1-\alpha)(1-\rho)\Lambda(n,\lambda)D[n,\beta,\gamma,A,B]}{(n-\rho)E[\beta,\gamma,\mu,A,B]}\right\}^{\frac{1}{n-1}}. \tag{4.10}$$

Last, but not least we give the following result.

Theorem 4.7. *Let the function* f *be in the class* $\mathcal{U}[\mu, \alpha, \beta, \gamma, \lambda, \eta, A, B]$. *Then the function* f *is close-to-convex of order* ρ $(0 \leq \rho < 1)$ *in the disk* $|z| < r_7(\mu, \alpha, \beta, \gamma, \lambda, \eta, A, B) = r_7$, *where*

$$r_7 = \inf_n \left\{ \frac{2(1-\alpha)(1-\rho)\Lambda(n, \lambda)D[n, \beta, \gamma, A, B](1-\eta+n\eta)}{nE[\beta, \gamma, \mu, A, B]} \right\}^{\frac{1}{n-1}}. \tag{4.11}$$

Proof. It sufficient to show that

$$|f'(z) - 1| < 1 - \rho.$$

Using a similar technique to Theorem 4.1 and making use of (2.4), we get (4.11). □

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