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RATE OF CONVERGENCE OF THE DISCRETE POLYA ALGORITHM FROM CONVEX SETS. A PARTICULAR CASE

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ABSTRACT. In this work we deal with best approximation in ℓ_p^n, $1 < p \le \infty$, $n \ge 2$. For $1 < p < \infty$, let h_p denote the best ℓ_p^n-approximation to $f \in \mathbb{R}^n$ from a closed, convex subset K of $\mathbb{R}^n$, $f \notin K$, and let h^* be a best uniform approximation to f from K. In case that $h^* - f = (\rho_1, \rho_2, \cdots, \rho_n)$, $|\rho_j| = \rho$ for $j = 1, 2, \cdots, n$, we show that the behavior of $\|h_p - h^*\|$ as $p \to \infty$ depends on a property of separation of the set K from the ℓ_∞^n-ball $\{x \in \mathbb{R}^n : \|x - f\| \le \rho\}$ at $h^* - f$.

Key words and phrases: Best uniform approximation, Rate of convergence, Polya Algorithm, Strong uniqueness.

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1. INTRODUCTION

Let $(w_1, w_2, \dots, w_n)$ be a fixed vector in $\mathbb{R}^n$, with $w_j > 0$, $j \in I_n := \{1, 2, \dots, n\}$, $n \ge 2$. For $x = (x(1), x(2), \dots, x(n)) \in \mathbb{R}^n$ we define

$$\|x\|_{p,w} := \left(\sum_{j=1}^{n} w_j |x(j)|^p \right)^{\frac{1}{p}}, \quad 1 \le p < \infty, \quad \text{and}$$
$$\|x\| := \max_{1 \le j \le n} |x(j)|.$$

Also we define $N := \sum_{j=1}^{n} w_j$.

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Throughout the paper, K will always be a nonempty, closed, convex subset of $\mathbb{R}^n$. For $f \in \mathbb{R}^n \backslash K$, we will say that $h_{p,w} \in K$, $1 \le p < \infty$, is a best $\ell^n_{p,w}$-approximation to f from K if

$$\|f - h_{p,w}\|_{p,w} \le \|f - h\|_{p,w} \quad \forall\, h \in K.$$

The existence of at least one best $\ell^n_{p,w}-$approximation to f from K is a known fact for $1 \le p < \infty$. Likewise, there always exists a best uniform approximation to f from K, i.e., an $h^* \in K$ that satisfies

$$\|f - h^*\| \le \|f - h\| \quad \forall\, h \in K.$$

We will henceforth assume $f = 0$ and $0 \notin K$. This causes no loss of generality, since all relevant properties are translation invariant. If $1 < p < \infty$, there is a unique best $\ell^n_{p,w}-$approximation. In this case, the next theorem [14] characterizes the best $\ell^n_{p,w}-$approximation to 0 from K.

Theorem 1.1 (Characterization of the best $\ell^n_{p,w}$-approximation)**.** *Let K be a closed, convex subset of $\mathbb{R}^n$, $0 \notin K$. Then $h_{p,w}$, $1 \le p < \infty$, is a best $\ell^n_{p,w}-$approximation to 0 from K if and only if for all $h \in K$,*

$$\sum_{j=1}^{n} w_j(h_{p,w}(j) - h(j))|h_{p,w}(j)|^{p-1}\mathrm{sgn}(h_{p,w}(j)) \le 0, \quad \textit{if } p > 1. \tag{1.1}$$

$$\sum_{j\in R(h_{1,w})} w_j\,(h_{1,w}(j) - h(j))\,\mathrm{sgn}(h_{1,w}(j)) \le \sum_{j\in Z(h_{1,w})} w_j\,|h(j)|, \quad \textit{if } p = 1, \tag{1.2}$$

where, if $g \in \mathbb{R}^n$, $Z(g) := \{j \in I_n : g(j) = 0\}$ and $R(g) := I_n \setminus Z(g)$.

It is also known [1, 6, 7] that if K is an affine subspace, then

$$\lim_{p\to\infty} h_{p,w} = h^*, \tag{1.3}$$

where in this case h^* is a particular best uniform approximation to 0 from K, called *strict* uniform approximation [12, 7] and whose definition is also valid in any closed, convex K. In [3, 8] it is proved that there exists a constant $M > 0$ such that $p\,\|h_{p,w} - h^*\| \le M$ for all $p > 1$. Moreover, from [13] it is deduced that there are constants $M_1, M_2 > 0$ and $0 \le a \le 1$, depending on K, such that

$$M_1\, a^p \le p\,\|h_{p,w} - h^*\| \le M_2\, a^p \quad \text{for all } p > 1.$$

In [2, 7] it is shown that if K is not an affine subspace, then $h_{p,w}$ does not necessarily converge to the strict uniform approximation, though (1.3) is always valid whenever h^* is the unique best uniform approximation to 0 from K. In [6, 7] we can find sufficient conditions on K under which (1.3) is satisfied. In any case, the convergence of $h_{p,w}$ as $p \to \infty$ to a best uniform approximation is known as the **Polya algorithm** [11]. The purpose of this paper is to study the behavior of $\|h_{p,w} - h^*\|$ as $p \to \infty$ when h^* is a best uniform approximation to 0 from K and h^* satisfies $|h^*(j)| = \rho > 0\; \forall\, j \in I_n$.

2. RELATION BETWEEN STRONG UNIQUENESS AND RATE OF CONVERGENCE

A useful concept in order to get a first general result on the rate of convergence of the Polya algorithm is strong uniqueness. It was established in 1963 by Newman and Shapiro [10] in the context of the uniform approximation to continuous functions by means of elements of a Haar space, although we could define it in any normed space.

Definition 2.1. Let $h^* \in K$ be a best uniform approximation to $0 \in \mathbb{R}^n$ from K. We say that h^* is *strongly unique* if there exists $\gamma > 0$ such that

$$\|h - h^*\| \le \gamma(\|h\| - \|h^*\|) \quad \forall\, h \in K. \tag{2.1}$$

It is obvious that if h^* is strongly unique, then h^* is the unique best uniform approximation to $0 \in \mathbb{R}^n$ from K.

Theorem 2.1. *If the best uniform approximation h^* to 0 from K is strongly unique, then $p\,\|h_{p,w} - h^*\|$ is bounded for all $p \geq 1$.*

Proof. We first note that for every $h \in K$,

$$m^{\frac{1}{p}}\,\|h\| \leq \|h\|_{p,w} \leq N^{\frac{1}{p}}\,\|h\|, \tag{2.2}$$

where $m := \min_{j \in I_n}\{w_j\}$.

Let $\gamma > 0$ satisfy (2.1). Then for any $p \geq 1$,

$$\|h_{p,w} - h^*\| \leq \gamma(\|h_{p,w}\| - \|h^*\|). \tag{2.3}$$

Applying (2.2) and the definition of best $\ell^n_{p,w}$-approximation, we have

$$\begin{aligned} \|h_{p,w}\| - \|h^*\| &\leq \frac{1}{m^{\frac{1}{p}}}\,\|h_{p,w}\|_{p,w} - \|h^*\| \\ &\leq \frac{1}{m^{\frac{1}{p}}}\|h^*\|_{p,w} - \|h^*\| \\ &\leq \left[\left(\frac{N}{m}\right)^{\frac{1}{p}} - 1\right]\|h^*\| \\ &\leq \frac{(N-m)\|h^*\|}{m\,p}. \end{aligned}$$

From (2.3) we finally conclude that

$$p\,\|h_{p,w} - h^*\| \leq \frac{\gamma(N-m)\|h^*\|}{m} \quad \text{for all } p \geq 1.$$

□

The above inequality improves the proposal in [4] and [5].

2.1. **The Particular Case $|h^*(j)| = \rho > 0$, $j = 1, 2, \ldots, n$.** We henceforth suppose that $h^* \in K$ is a best uniform approximation to 0 from K, where $|h^*(j)| = \rho > 0$ for all $j \in I_n$. Under these conditions we will analyze the behaviour of $\|h_{p,w} - h^*\|$ as $p \to \infty$. In Theorem 2.3, our main result, we will prove that the converse of Theorem 2.1 – which is generally not true – is valid in this particular case. Since $\{x \in \mathbb{R}^n : \|x\| \leq \rho\} \cap K = \{h \in K : \|h\| = \rho\}$, it is easy to see that there is a hyperplane $\left\{(x(1), x(2), \ldots, x(n)) : \sum_{j=1}^n a_j \operatorname{sgn}(h^*(j))\, x(j) = \rho\right\}$, with $0 \leq a_j \leq 1$, all $j \in I_n$, and $\sum_{j=1}^n a_j = 1$, that separates K from the ball $\{x \in \mathbb{R}^n : \|x\| \leq \rho\}$ at h^*, i.e., $\sum_{j=1}^n a_j \operatorname{sgn}(h^*(j))\, h(j) \geq \rho$ for all $h \in K$.

Definition 2.2. We will say that

$$\pi := \left\{(x(1), x(2), \ldots, x(n)) : \sum_{j=1}^n a_j \operatorname{sgn}(h^*(j))\, x(j) = \rho\right\}$$

is a hyperplane that *strongly separates K from the ball $\{x \in \mathbb{R}^n : \|x\| \leq \rho\}$ at h^**, or equivalently, that π is a *strongly separating hyperplane at h^**, if

$$0 < a_j < 1, \text{ all } j \in I_n, \quad \sum_{j=1}^n a_j = 1 \tag{2.4}$$

and

$$\sum_{j=1}^{n} a_j \operatorname{sgn}(h^*(j))\, h(j) \geq \rho \quad \forall\, h \in K. \tag{2.5}$$

In the proofs of Lemma 2.2 and Theorems 2.3 and 2.4 we will assume $h^*(j) = 1$ for all $j \in I_n$. This causes no loss of generality, since we can replace K by the closed, convex set

$$\left\{ \tilde{h} \in \mathbb{R}^n : \tilde{h}(j) = \frac{1}{\rho}\, h(j) \operatorname{sgn}(h^*(j)),\ j \in I_n,\ h \in K \right\}.$$

Lemma 2.2. *If $p_k \|h_{p_k,w} - h^*\|$ is bounded for $p_k \to \infty$, then there exists a strongly separating hyperplane at h^*.*

Proof. Since $\lim_{p_k \to \infty} h_{p_k,w}(j) = h^*(j) = 1$, all $j \in I_n$, we can suppose $h_{p_k,w}(j) > 0$, all $j \in I_n$ and, without loss of generality, all p_k. Then, for every p_k the formula of characterization (1.1) can be expressed in the form

$$\sum_{j=1}^{n} w_j (h_{p_k,w}(j) - h(j)) h_{p_k,w}^{p_k - 1}(j) \leq 0 \quad \forall\, h \in K.$$

Dividing by $\|h_{p_k,w}\|_{p_k,w}^{p_k}$, for every p_k we obtain

$$\sum_{j=1}^{n} w_j \left(\frac{h_{p_k,w}(j)}{\|h_{p_k,w}\|_{p_k,w}} \right)^{p_k} \frac{h(j)}{h_{p_k,w}(j)} \geq 1 \quad \forall\, h \in K. \tag{2.6}$$

Keeping in mind that

$$w_j h_{p_k,w}^{p_k}(j) \leq \|h_{p_k,w}\|_{p_k,w}^{p_k} \leq \|h^*\|_{p_k,w}^{p_k} = N, \quad j \in I_n,$$

and after passage to a subsequence, we can suppose that $h_{p_k,w}^{p_k}(j)$, all $j \in I_n$, and $\|h_{p_k,w}\|_{p_k,w}^{p_k}$ are convergent. Now, by hypothesis, $p_k\, |h_{p_k,w}(j) - 1|$ is bounded for all $j \in I_n$ and all p_k. Hence we get

$$\lim_{p_k \to \infty} h_{p_k,w}^{p_k}(j) = \lim_{p_k \to \infty} \operatorname{Exp}\left(p_k (h_{p_k,w}(j) - 1) \right) > 0, \quad \text{all } j \in I_n.$$

Writing

$$a_j = \lim_{p_k \to \infty} w_j \left(\frac{h_{p_k,w}(j)}{\|h_{p_k,w}\|_{p_k,w}} \right)^{p_k}, \quad j \in I_n,$$

we therefore deduce that $0 < a_j < 1$, all $j \in I_n$, and $\sum_{j=1}^{n} a_j = 1$. Taking limits as $p_k \to \infty$ in (2.6), we finally conclude that

$$\sum_{j=1}^{n} a_j\, h(j) \geq 1 \quad \forall\, h \in K.$$

Then $\left\{ (x(1), x(2), \ldots, x(n)) := \sum_{j=1}^{n} a_j x(j) = 1 \right\}$ is a strongly separating hyperplane at h^*. □

Theorem 2.3. *The following statements are equivalent:*

(a) *The best uniform approximation to 0 from K, h^*, is strongly unique.*
(b) *$p\, \|h_{p,w} - h^*\|$ is bounded for all $p \geq 1$.*
(c) *$p_k\, \|h_{p_k,w} - h^*\|$ is bounded for a sequence $p_k \to \infty$.*
(d) *There exists a strongly separating hyperplane at h^*.*

Proof. (a) $\Rightarrow$ (b) is Theorem 2.1. (b) $\Rightarrow$ (c) is obvious. (c) $\Rightarrow$ (d) is Lemma 2.2. To complete the theorem, we now prove (d) $\Rightarrow$ (a). Suppose that there is a strongly separating hyperplane π at $h^* = (1, 1, \dots, 1)$. Let $h \in K$. Observe that $\|h\| \geq 1$. Let I_n^+ denote the subset of indices j in I_n such that $h(j) \geq 1$, and let $I_n^- := I_n \backslash I_n^+$. For all $j \in I_n^+$ we have $|h(j)-1| = h(j)-1 \leq \|h\|-1$. On the other hand, if $j \in I_n^-$, then $|h(j) - 1| = 1 - h(j)$. Moreover, the inequality

$$\sum_{i \in I_n} a_i(h(i) - 1) \geq 0$$

implies

$$\begin{aligned} a_j(1 - h(j)) &\leq \sum_{i \in I_n, i \neq j} a_i(h(i) - 1) \\ &\leq \sum_{i \in I_n^+} a_i(h(i) - 1) \\ &\leq (\|h\| - 1) \sum_{i \in I_n^+} a_i \\ &\leq (\|h\| - 1)(1 - a_j). \end{aligned}$$

Thus, for all $j \in I_n$ we have

$$|h(j) - 1| \leq \left(\frac{1}{\min_{i \in I_n} a_i} - 1 \right) (\|h\| - 1) := \gamma(\|h\| - 1),$$

and so $\|h - h^*\| \leq \gamma(\|h\| - \|h^*\|)$. □

Our goal now is to show that, under the conditions of Theorem 2.3, either $h_{p,w} = h^*$ for all p or there exist constants $M_1, M_2 > 0$ such that

$$M_1 \leq p \, \|h_{p,w} - h^*\| \leq M_2 \text{ for all } p \geq 1.$$

On the other hand, if there exists no strongly separating hyperplane at h^*, then the following example in $\mathbb{R}^2$, where $\lim_{p \to \infty} h_{p,w} = h^*$, shows that the rate of convergence is as slow as we want.

Example 2.1. Let $\alpha : [1, +\infty) \to (0, 1]$ be a continuous strictly decreasing function such that $\alpha(1) = 1$ and $\lim\limits_{t \to \infty} \alpha(t) = 0$ and let $\beta : (0, 1] \to [1, +\infty)$ denote its inverse function, that will also be a strictly decreasing function. We define

$$f(x) := 1 + \int_x^1 (1 - t)^{\beta(1-t)} dt, \quad 0 \leq x \leq 1,$$

and let K be the convex hull of the set $\{(x, y) \in \mathbb{R}^2 : y = f(x), x \in [0, 1]\}$.

Observe that $h^* = (1, 1)$ is the unique best uniform approximation to $(0, 0)$ from K. Moreover, the function f is smooth, convex and $f'(1) = 0$. This implies that the strongly separating hyperplane at h^* does not exist.

Let $h_p = (1 - \varepsilon_p, 1 + \delta_p)$ be the best p-approximation to $(0, 0)$ from K, with $\varepsilon_p, \delta_p \downarrow 0$ as $p \to \infty$. Since the slopes of the curve $y = f(x)$ and the ℓ_p-ball coincide at h_p, we have

$$\frac{(1 - \varepsilon_p)^{p-1}}{(1 + \delta_p)^{p-1}} = \varepsilon_p^{\beta(\varepsilon_p)}$$

and therefore

$$\lim_{p \to \infty} \varepsilon_p^{\beta(\varepsilon_p)/(p-1)} = \lim_{p \to \infty} \frac{1 - \varepsilon_p}{1 + \delta_p} = 1. \tag{2.7}$$

If $\varepsilon_p \leq \alpha(p)$, then $\beta(\varepsilon_p) \geq \beta(\alpha(p)) = p$, which contradicts (2.7). Then, for p large, we have $\varepsilon_p > \alpha(p)$. This shows that the rate of convergence of h_p to h^* as $p \to \infty$ can be as slow as we want.

Theorem 2.4. *The following conditions are equivalent*

(a) $h_{p,w} = h^*$ *for all* $p \geq 1$,
(b) $h_{p_0,w} = h^*$ *for some* $p_0 \geq 1$,
(c) *the hyperplane*

$$\pi := \left\{ (x(1), x(2), \ldots, x(n)) : \sum_{j=1}^{n} \operatorname{sgn}(h^*(j)) \frac{w_j}{N} x(j) = \rho \right\} \tag{2.8}$$

is a strongly separating hyperplane at h^*.

Proof. (a) ⇒ (b) is obvious. (b) ⇒ (c) follows immediately from Theorem 1.1. Indeed, if $h_{p_0,w} = h^*$ for some $p_0 \geq 1$, then from (1.1) if $p_0 > 1$ or (1.2) if $p_0 = 1$, we have

$$\sum_{j=1}^{n} w_j (h^*(j) - h(j)) \operatorname{sgn}(h^*(j)) \leq 0 \quad \forall h \in K, \tag{2.9}$$

which is equivalent to the fact that π is a strongly separating hyperplane at h^*. Also from (1.1) and (1.2), the inequality (2.9) implies that $h_{p,w} = h^*$ for all $p \geq 1$ and so (c) ⇒ (a). □

Theorem 2.5. *Suppose that* $h_{p,w} \neq h^*$ *for some* $p \geq 1$ *and there exists a strongly separating hyperplane at* h^*. *Then there are constants* $M_1, M_2 > 0$ *such that*

$$M_1 \leq p \, \|h_{p,w} - h^*\| \leq M_2 \quad \textit{for all } p \geq 1.$$

Proof. Assume that there exists a strongly separating hyperplane at h^*, where $h^*(j) = 1$ for all $j \in I_n$. From Theorem 2.3, there is a constant $M_2 > 0$ such that

$$p \, \|h_{p,w} - h^*\| \leq M_2.$$

Therefore, to prove the theorem it is sufficient to show that $\inf_{p \geq 1}\{p \, \|h_{p,w} - h^*\|\} > 0$. Suppose the contrary. In order to get a contradiction, we only need to consider the two following exhaustive cases:

(1) There exists a sequence $p_k \to \infty$ such that $\lim_{p_k \to \infty} p_k \|h_{p_k} - h^*\| = 0$. In this case $\lim_{p_k \to \infty} p_k \, |h_{p_k,w}(j) - 1| = 0$ for all $j \in I_n$. This implies that $h_{p_k,w}^{p_k}(j) \to 1$ as $p_k \to \infty$ and

$$a_j^* = \lim_{p_k \to \infty} w_j \left(\frac{h_{p_k,w}(j)}{\|h_{p_k,w}\|_{p_k,w}} \right)^{p_k} = \frac{w_j}{N}, \qquad j = 1, 2, \ldots, n,$$

which means (see the proof of Lemma 2.2) that the hyperplane (2.8), with $h^*(j) = \rho = 1$ for all $j \in I_n$, is a strongly separating hyperplane at h^*. From Theorem 2.4 (c), $h_{p,w} = h^*$ for all $p \geq 1$, which contradicts the hypothesis of the theorem.

(2) There exists a sequence $p_k \to p_0$, $1 \leq p_0 < \infty$, such that $\lim_{p_k \to p_0} p_k \, \|h_{p_k,w} - h^*\| = 0$. Since $h_{p_k,w} \to h_{p_0,w}$, we deduce that $\|h_{p_0,w} - h^*\| = \lim_{p_k \to p_0} \|h_{p_k,w} - h^*\| = 0$ and so $h_{p_0,w} = h^*$. Now, using the statement (b) of Theorem 2.4, we conclude that $h_{p,w} = h^*$, for all $p \geq 1$. A contradiction.

□

2.2. A Numerical Example in Isotonic Approximation.
Let $f = (\underbrace{a+1, \dots, a+1}_{r}, \underbrace{a-1, \dots, a-1}_{n-r}) \in \mathbb{R}^n$, and let K be the convex set of the nondecreasing vectors in $\mathbb{R}^n$, i.e.

$$K = \{h \in \mathbb{R}^n \, : \, h(i) \le h(j) \; \forall \, i, j \in I_n, \, i < j\} \, .$$

In this case, the (unique) best uniform approximation to f from K is the element $h^* = (a, a, \dots, a)$. Thus $h_{p,w} \to h^*$ as $p \to \infty$. Furthermore, it is easy to see that

$$h_{p,w} = (x_{p,w}, x_{p,w}, \dots, x_{p,w}) \in \mathbb{R}^n, \quad 1 < p < \infty,$$

for some $x_{p,w}$ satisfying $a - 1 \le x_{p,w} \le a + 1$.

In order to translate h^* to a vertex of the ℓ^n_∞-ball, we consider the closed, convex set

$$\widetilde{K} = \{\widetilde{h} \in \mathbb{R}^n \, : \, \widetilde{h}(j) = h(j) - f(j) \, , \, j \in I_n, \; h \in K\}.$$

In this way we obtain

- $\widetilde{f} = (0, 0, \dots, 0)$;
- $\widetilde{h}^* = h^* - f = (\underbrace{-1, \dots, -1}_{r}, \underbrace{1, \dots, 1}_{n-r})$.

To simplify the notation, we will write $\sigma_j = \operatorname{sgn}(\widetilde{h}^*(j))$, $j \in I_n$. Now, we are interested in obtaining a strongly separating hyperplane at $\widetilde{h}^*$, i.e., a hyperplane

$$\pi := \left\{ (x(1), x(2), \dots, x(n)) : \sum_{j=1}^{n} a_j \, \sigma_j \, x(j) = 1 \right\}$$

such that

(p1) $0 < a_j < 1$, all $j \in I_n$, and $\sum_1^n a_j = 1$;
(p2) $\sum_{j=1}^n \sigma_j \, a_j \, \widetilde{h}(j) \ge 1 \; \forall \, \widetilde{h} \in \widetilde{K}$.

Proposition 2.6. *Let $S := \sum_{j=1}^{r} w_j$. Then the above hyperplane π, with*

$$a_j = \frac{w_j}{2\,S} \;\; \textit{if} \;\; 1 \le j \le r, \quad a_j = \frac{w_j}{2(N - S)} \;\; \textit{if} \;\; r + 1 \le j \le n,$$

satisfies (p1) *and* (p2)*, and therefore it is a strongly separating hyperplane at $\widetilde{h}^*$.*

Proof. By definition, $0 < a_j < 1$ for all $j \in I_n$. Furthermore,

$$\sum_{j=1}^{n} \sigma_j \, a_j \, \widetilde{h}^*(j) = \sum_{j=1}^{n} a_j = \sum_{j=1}^{r} \frac{w_j}{2\,S} + \sum_{j=r+1}^{n} \frac{w_j}{2(N - S)} = \frac{1}{2} + \frac{1}{2} = 1.$$

Then (p1) holds.

Since

$$\sum_{j=1}^{n} \sigma_j \, a_j \, f(j) = -(a+1) \sum_{j=1}^{r} \frac{w_j}{2\,S} + (a-1) \sum_{j=r+1}^{n} \frac{w_j}{2(N - S)} = -1,$$

(p2) is equivalent to

$$\sum_{j=1}^{n} \sigma_j \, a_j \, h(j) \ge 0 \quad \forall \, h \in K. \tag{2.10}$$

But if h is a nondecreasing vector, then (2.10) is immediate because

$$\sum_{j=1}^{r} w_j h(j) \leq h(r) \sum_{j=1}^{r} w_j = S\, h(r) \text{ and } \sum_{j=r+1}^{n} h(j) \geq h(r) \sum_{j=r+1}^{n} w_j = (N-S)h(r),$$

and therefore

$$\begin{aligned}\sum_{j=1}^{n} \sigma_j\, a_j\, h(j) &= -\frac{1}{2\,S} \sum_{j=1}^{r} w_j h(j) + \frac{1}{2(N-S)} \sum_{j=r+1}^{n} w_j h(j) \\ &\geq -\frac{1}{2S}\, S\, h(r) + \frac{1}{2(N-S)}\,(N-S)h(r) = 0.\end{aligned}$$

This concludes the proof. □

From Proposition 2.6 we deduce that if $S = N/2$, then

$$\left\{(x(1), x(2), \ldots, x(n)) : \sum_{j=1}^{n} \sigma_j \frac{w_j}{N}\, x(j) = 1\right\} \tag{2.11}$$

is a strongly separating hyperplane at $\widetilde{h}^*$, and from Theorem 2.4 this is equivalent to $\widetilde{h}_{p,w} = \widetilde{h}^*$ for all $1 \leq p < \infty$. In the case that $S \neq N/2$, we claim that $\widetilde{h}_{p,w} \to \widetilde{h}^*$ as $p \to \infty$ exactly at a rate $O\left(\frac{1}{p}\right)$. From Proposition 2.6 and Theorems 2.4 and 2.5 we only need to show that (2.11) is not a strongly separating hyperplane at $\widetilde{h}^*$. This last assertion is true since (2.10), with $a_j = w_j/N$, all $j \in I_n$, implies

$$\sum_{j=r+1}^{n} w_j h(j) \geq \sum_{j=1}^{r} w_j h(j) \quad \forall\, h \in K. \tag{2.12}$$

On the other hand, if $S < N/2$ then $h = (-1, -1, \ldots, -1) \in K$ does not satisfy (2.12), and an analogous conclusion is valid for $h = (1, 1, \ldots, 1) \in K$ if $S > N/2$. This proves the claim.

In what follows we obtain these same results calculating directly the best $\ell^n_{p,w}-$approximations to f from K, namely, $h_{p,w} = (x_{p,w}, x_{p,w}, \ldots, x_{p,w})$. It is easy to check that

$$x_{p,w} = \frac{a - 1 + a\left(\frac{S}{N-S}\right)^{\frac{1}{p}} + \left(\frac{S}{N-S}\right)^{\frac{1}{p}}}{1 + \left(\frac{S}{N-S}\right)^{\frac{1}{p}}}, \quad 1 < p < \infty.$$

Then we immediately conclude that if $S = N/2$, then $h_{p,w} = h^*$ for $p > 1$, and if $S \neq N/2$, then $h_{p,w} \to h^*$ as $p \to \infty$. Moreover, we can calculate the rate of convergence. Indeed,

$$\lim_{p\to\infty} \frac{h_{p,w}(j) - h^*(j)}{1/p} = \lim_{p\to\infty} \frac{\frac{\left(\frac{S}{N-S}\right)^{1/p} - 1}{1 + \left(\frac{S}{N-S}\right)^{1/p}}}{1/p} = \frac{1}{2} \lim_{p\to\infty} \frac{\left(\frac{S}{N-S}\right)^{\frac{1}{p}} - 1}{1/p} = \frac{1}{2} \ln\left(\frac{S}{N-S}\right).$$

The rate of convergence is exactly $O\left(\frac{1}{p}\right)$.

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