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A NEW UPPER BOUND OF THE LOGARITHMIC MEAN

GAO JIA AND JINDE CAO

Department of Applied Mathematics,
Hunan City University,
Yiyang 413000, China.

Department of Mathematics,
Southeast University,
Nanjing 210096, China.
EMail: jdcao@seu.edu.cn

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Abstract

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Abstract

Let a and b be positive numbers with $a \neq b$. The inequalities about the logarithmic-mean

$$L(a,b) < H_p(a,b) < M_q(a,b)$$

are obtained, where $p \geq \frac{1}{2}$ and $q \geq \frac{2}{3}p$. We would point out that $p = \frac{1}{2}$ and $q = \frac{1}{3}$ are the best constants such that above inequalities hold.

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1. Introduction and Main Results

The aim of this paper is to establish a new upper bound for the logarithmic mean.

Let a and b be positive numbers with $a \neq b$, $p > 0, q > 0$. The logarithmic mean is defined as

$$L(a,b) = \frac{b-a}{\log b - \log a},$$

The power mean is defined by

$$M_q(a,b) = \left(\frac{a^q + b^q}{2}\right)^{\frac{1}{q}},$$

and the Heron mean is defined as

$$H_p(a,b) = \left(\frac{a^p + (ab)^{p/2} + b^p}{3}\right)^{\frac{1}{p}}.$$

There are many important results concerning $L(a,b)$, $M_p(a,b)$ and $H_q(a,b)$. The well known Lin Tong-Po inequality (see [1]) is stated as

$$L(a,b) < M_{\frac{1}{3}}(a,b). \tag{1.1}$$

In [2], Yang Z.H. obtained the inequalities

$$L(a,b) < M_{\frac{1}{2}}(a,b) < H_1(a,b). \tag{1.2}$$

In [1], Kuang J. C. summarized and stated the interpolation inequalities

$$L(a,b) < M_{\frac{1}{3}}(a,b) < M_{\frac{1}{2}}(a,b) < H_1(a,b) < M_{\frac{2}{3}}(a,b). \tag{1.3}$$

In this paper, we further improve the upper bound of the logarithmic mean and obtain the following theorem:

Theorem 1.1. *Let* $p \geq \frac{1}{2}$, $q \geq \frac{2}{3}p$, *and* a, b *be positive numbers with* $a \neq b$. *We then have*

$$L(a,b) < H_p(a,b) < M_q(a,b). \tag{1.4}$$

Furthermore, $p = \frac{1}{2}$, $q = \frac{2}{3}$ *are the best constants for (1.4).*

2. Proof of Theorem 1.1

In this section, there are two goals: the first is to state and prove some fundamental lemmas. The second is to prove our main result by virtue of these lemmas.

Lemma 2.1. *([3], [4]). Suppose a and b are fixed positive numbers with $a \neq b$. For $p > 0$, then $H_p(a,b)$ and $M_p(a,b)$ are strictly monotone increasing functions with respect to p.*

Lemma 2.2. *Let $x > 1$. Then*

$$\frac{x-1}{\log x} < \left(\frac{x^{\frac{1}{2}} + x^{\frac{1}{4}} + 1}{3} \right)^2. \tag{2.1}$$

Proof. Taking $t = x^{\frac{1}{4}}$, where $x > 1$, it is easy to see that inequality (2.1) is equivalent to

$$\frac{t^4 - 1}{4 \log t} < \frac{1}{9}(t^2 + t + 1)^2. \tag{2.2}$$

Define the function

$$f(t) = \frac{4}{9} \log t - \frac{t^4 - 1}{(t^2 + t + 1)^2}. \tag{2.3}$$

Calculating the derivative for $f(t)$, we get

$$\begin{aligned} f'(t) &= \frac{4}{9t} - \frac{4t^3(t^2+t+1) - 2(t^4-1)(2t+1)}{(t^2+t+1)^3} \\ &= \frac{2(t-1)^4(2t^2+5t+2)}{9t(t^2+t+1)^3}. \end{aligned}$$

Since $t = x^{\frac{1}{4}} > 1$, we find that $f'(t) > 0$. Obviously, $f'(1) = 0$. So $f(t) > 0$ for $t > 1$. i.e. (2.1) holds. $\square$

Lemma 2.3. *Let $x > 1$, then the following inequality holds*

$$\left(\frac{x^{\frac{1}{2}} + x^{\frac{1}{4}} + 1}{3}\right)^2 < \left(\frac{x^{\frac{1}{3}} + 1}{2}\right)^3. \tag{2.4}$$

Proof. Taking $t = x^{\frac{1}{12}}$, where $x > 1$, it is easy to see that inequality (2.4) is equivalent to

$$9(t^4 + 1)^3 > 8(t^6 + t^3 + 1)^2. \tag{2.5}$$

Define a function $g(t)$ as

$$g(t) = 9(t^4 + 1)^3 - 8(t^6 + t^3 + 1)^2.$$

Factorizing $g(t)$, we obtain

$$\begin{aligned} g(t) &= (t-1)^4(1 + 4t + 10t^2 + 4t^3 - 2t^4 + 4t^5 + 10t^6 + 4t^7 + t^8) \\ &= (t-1)^4((t^4 - 1)^2 + 4t + 10t^2 + 4t^3 + 4t^5 + 10t^6 + 4t^7). \end{aligned}$$

The proof is completed. $\square$

Proof of Theorem 1.1. We first prove, for $p = \frac{1}{2}, q = \frac{1}{3}$, that (1.4) is true. In fact, since $a > 0, b > 0$ and $a \neq b$, there is no harm in supposing $b > a$. If we take $x = \frac{b}{a}$, using Lemma 2.2 and Lemma 2.3, we have

$$L(a,b) < H_{\frac{1}{2}}(a,b) < M_{\frac{1}{3}}(a,b). \tag{2.6}$$

For $q \geq \frac{2}{3}p$, there is the known result ([1])

$$H_p(a,b) < M_q(a,b), \quad (a \neq b). \tag{2.7}$$

Using Lemma 2.1, combining (2.6) and (2.7), we can conclude that

$$L(a,b) < H_{\frac{1}{2}}(a,b) < H_p(a,b) < M_q(a,b), \quad \left(p \geq \frac{1}{2},\ q \geq \frac{2}{3}p\right).$$

Next, we prove that $p = \frac{1}{2}$ and $q = \frac{1}{3}$ are the best constants for (1.4). Suppose we know that the following inequalities

$$L(x,1) < H_p(x,1) < M_q(x,1), \tag{2.8}$$

hold for any $x > 1$. There is no harm in supposing $1 < x \leq 2$. (In fact, if $n < x \leq n+1$, we can take $t = x - n$, where n is a positive integer.) Taking $t = x - 1$, applying Taylor's Theorem to the functions $L(x,1)$, $H_p(x,1)$ and $M_q(x,1)$, we have

$$L(x,1) = L(t+1,1) = 1 + \frac{1}{2}t - \frac{1}{12}t^2 + \cdots, \tag{2.9}$$

$$H_p(x,1) = H_p(t+1,1) = 1 + \frac{1}{2}t + \frac{2p-3}{24}t^2 + \cdots, \tag{2.10}$$

$$M_q(x,1) = M_q(t+1,1) = 1 + \frac{1}{2}t + \frac{q-1}{8}t^2 + \cdots, \tag{2.11}$$

With simple manipulations (2.9), (2.10) and (2.11), together with (2.8), yield

$$-\frac{1}{12} \le \frac{2p-3}{24} \le \frac{q-1}{8}. \tag{2.12}$$

From (2.12), it immediately follows that

$$p \ge \frac{1}{2}, \quad \text{and} \quad q \ge \frac{2}{3}p.$$

We then have, by virtue of Lemma 2.1, that $p = \frac{1}{2}$ and $q = \frac{1}{3}$ are the best constants for (1.4). $\square$

Remark 2.1. *It is easy to see that the best lower bound of the logarithmic mean is $H_0(a,b) = \sqrt{ab}$, namely $H_0 = G$, the geometric mean. In addition, using Lemma 2.1, combining (1.4), (2.7), (2.8) and the related results in [1], we derive the following graceful inequalities*

$$\sqrt{ab} < L(a,b) < H_{\frac{1}{2}}(a,b) < M_{\frac{1}{3}}(a,b) < M_\alpha(a,b) < H_\beta(a,b) < M_\gamma(a,b),$$

where $\frac{1}{3} < \alpha < \frac{\log 2}{\log 3}\beta$, $\gamma \ge \frac{2}{3}\beta$, $\beta > \frac{\log 3}{3\log 2}$.

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