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AN APPLICATION OF HÖLDER'S INEQUALITY FOR CONVOLUTIONS

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ABSTRACT. Let $\mathcal{A}_p(n)$ be the class of analytic and multivalent functions $f(z)$ in the open unit disk $\mathbb{U}$. Furthermore, let $\mathcal{S}_p(n,\alpha)$ and $\mathcal{T}_p(n,\alpha)$ be the subclasses of $\mathcal{A}_p(n)$ consisting of multivalent starlike functions $f(z)$ of order α and multivalent convex functions $f(z)$ of order α, respectively. Using the coefficient inequalities for $f(z)$ to be in $\mathcal{S}_p(n,\alpha)$ and $\mathcal{T}_p(n,\alpha)$, new subclasses $\mathcal{S}_p^*(n,\alpha)$ and $\mathcal{T}_p^*(n,\alpha)$ are introduced. Applying the Hölder inequality, some interesting properties of generalizations of convolutions (or Hadamard products) for functions $f(z)$ in the classes $\mathcal{S}_p^*(n,\alpha)$ and $\mathcal{T}_p^*(n,\alpha)$ are considered.

Key words and phrases: Analytic function, Multivalent starlike, Multivalent convex.

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1. INTRODUCTION

Let $\mathcal{A}_p(n)$ be the class of functions $f(z)$ of the form

$$f(z) = z^p + \sum_{k=p+n}^{\infty} a_k z^k$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} | |z| < 1\}$ for some natural numbers p and n. Let $\mathcal{S}_p(n,\alpha)$ be the subclass of $\mathcal{A}_p(n)$ consisting of functions $f(z)$ which satisfy

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \qquad (z \in \mathbb{U})$$

for some α $(0 \leqq \alpha < p)$. Also let $\mathcal{T}_p(n,\alpha)$ be the subclass of $\mathcal{A}_p(n)$ consisting of functions $f(z)$ satisfying $zf'(z)/p \in \mathcal{S}_p(n,\alpha)$, that is,

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \qquad (z \in \mathbb{U})$$

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for some α $(0 \leqq \alpha < p)$. These classes, $\mathcal{A}_p(n)$, $\mathcal{S}_p(n,\alpha)$ and $\mathcal{T}_p(n,\alpha)$, were studied by Owa [3]. It is easy to derive the following lemmas, which provide the sufficient conditions for functions $f(z) \in \mathcal{A}_p(n)$ to be in the classes $\mathcal{S}_p(n,\alpha)$ and $\mathcal{T}_p(n,\alpha)$, respectively.

Lemma 1.1. *If $f(z) \in \mathcal{A}_p(n)$ satisfies*

$$\sum_{k=p+n}^{\infty} (k-\alpha)|a_k| \leqq p-\alpha \tag{1.1}$$

for some α $(0 \leqq \alpha < p)$, then $f(z) \in \mathcal{S}_p(n,\alpha)$.

Lemma 1.2. *If $f(z) \in \mathcal{A}_p(n)$ satisfies*

$$\sum_{k=p+n}^{\infty} k(k-\alpha)|a_k| \leqq p(p-\alpha) \tag{1.2}$$

for some α $(0 \leqq \alpha < p)$, then $f(z) \in \mathcal{T}_p(n,\alpha)$.

Remark 1. We note that Silverman [4] has given Lemma 1.1 and Lemma 1.2 in the case of $p = 1$ and $n = 1$. Also, Srivastava, Owa and Chatterjea [5] have given the coefficient inequalities in the case of $p = 1$.

In view of Lemma 1.1 and Lemma 1.2, we introduce the subclass $\mathcal{S}_p^*(n,\alpha)$ consisting of functions $f(z)$ which satisfy the coefficient inequality (1.1), and the subclass $\mathcal{T}_p^*(n,\alpha)$ consisting of functions $f(z)$ which satisfy the coefficient inequality (1.2).

2. CONVOLUTION PROPERTIES FOR THE CLASSES $\mathcal{S}_p^*(n,\alpha)$ AND $\mathcal{T}_p^*(n,\alpha)$

For functions $f_j(z) \in \mathcal{A}_p(n)$ given by

$$f_j(z) = z^p + \sum_{k=p+n}^{\infty} a_{k,j} z^k \qquad (j = 1, 2, \ldots, m),$$

we define

$$G_m(z) = z^p + \sum_{k=p+n}^{\infty} \left(\prod_{j=1}^{m} a_{k,j} \right) z^k$$

and

$$H_m(z) = z^p + \sum_{k=p+n}^{\infty} \left(\prod_{j=1}^{m} a_{k,j}^{p_j} \right) z^k \qquad (p_j > 0).$$

Then $G_m(z)$ denotes the convolution of $f_j(z)$ $(j = 1, 2, \ldots, m)$. Therefore, $H_m(z)$ is the generalization of the convolutions. In the case of $p_j = 1$, we have $G_m(z) = H_m(z)$. The generalization of the convolution was considered by Choi, Kim and Owa [1].

In the present paper, we discuss an application of the Hölder inequality for $H_m(z)$ to be in the classes $\mathcal{S}_p^*(n,\alpha)$ and $\mathcal{T}_p^*(n,\alpha)$.

For $f_j(z) \in \mathcal{A}_p(n)$, the Hölder inequality is given by

$$\sum_{k=p+n}^{\infty} \left(\prod_{j=1}^{m} |a_{k,j}| \right) \leqq \prod_{j=1}^{m} \left(\sum_{k=p+n}^{\infty} |a_{k,j}|^{p_j} \right)^{\frac{1}{p_j}},$$

where $p_j > 1$ and $\sum_{j=1}^{m} \frac{1}{p_j} \geqq 1$.

Recently, Nishiwaki, Owa and Srivastava [2] have given some results of Hölder-type inequalities for a subclass of uniformly starlike functions.

Theorem 2.1. *If* $f_j(z) \in \mathcal{S}_p^*(n, \alpha_j)$ *for each* $j = 1, 2, \dots, m$*, then* $H_m(z) \in \mathcal{S}_p^*(n, \beta)$ *with*

$$\beta = \inf_{k \geqq p+n} \left\{ p - \frac{(k-p)\prod_{j=1}^m (p-\alpha_j)^{p_j}}{\prod_{j=1}^m (k-\alpha_j)^{p_j} - \prod_{j=1}^m (p-\alpha_j)^{p_j}} \right\},$$

where $p_j \geqq \frac{1}{q_j}$*,* $q_j > 1$ *and* $\sum_{j=1}^m \frac{1}{q_j} \geqq 1$*.*

Proof. For $f_j(z) \in \mathcal{S}_p^*(n, \alpha_j)$, Lemma 1.1 gives us that

$$\sum_{k=p+n}^{\infty} \left(\frac{k-\alpha_j}{p-\alpha_j}\right) |a_{k,j}| \leqq 1 \qquad (j = 1, 2, \dots, m),$$

which implies

$$\left\{ \sum_{k=p+n}^{\infty} \left(\frac{k-\alpha_j}{p-\alpha_j}\right) |a_{k,j}| \right\}^{\frac{1}{q_j}} \leqq 1$$

with $q_j > 1$ and $\sum_{j=1}^m \frac{1}{q_j} \geqq 1$. Applying the Hölder inequality, we have:

$$\sum_{k=p+n}^{\infty} \left\{ \prod_{j=1}^m \left(\frac{k-\alpha_j}{p-\alpha_j}\right)^{\frac{1}{q_j}} |a_{k,j}|^{\frac{1}{q_j}} \right\} \leqq 1.$$

Note that we have to find the largest β such that

$$\sum_{k=p+n}^{\infty} \left(\frac{k-\beta}{p-\beta}\right) \left(\prod_{j=1}^m |a_{k,j}|^{p_j}\right) \leqq 1,$$

that is,

$$\sum_{k=p+n}^{\infty} \left(\frac{k-\beta}{p-\beta}\right) \left(\prod_{j=1}^m |a_{k,j}|^{p_j}\right) \leqq \sum_{k=p+n}^{\infty} \left\{ \prod_{j=1}^m \left(\frac{k-\alpha_j}{p-\alpha_j}\right)^{\frac{1}{q_j}} |a_{k,j}|^{\frac{1}{q_j}} \right\}.$$

Therefore, we need to find the largest β such that

$$\left(\frac{k-\beta}{p-\beta}\right) \left(\prod_{j=1}^m |a_{k,j}|^{p_j}\right) \leqq \prod_{j=1}^m \left(\frac{k-\alpha_j}{p-\alpha_j}\right)^{\frac{1}{q_j}} |a_{k,j}|^{\frac{1}{q_j}},$$

which is equivalent to

$$\left(\frac{k-\beta}{p-\beta}\right) \left(\prod_{j=1}^m |a_{k,j}|^{p_j - \frac{1}{q_j}}\right) \leqq \prod_{j=1}^m \left(\frac{k-\alpha_j}{p-\alpha_j}\right)^{\frac{1}{q_j}}$$

for all $k \geqq p+n$. Since

$$\prod_{j=1}^m \left(\frac{k-\alpha_j}{p-\alpha_j}\right)^{p_j - \frac{1}{q_j}} |a_{k,j}|^{p_j - \frac{1}{q_j}} \leqq 1 \qquad \left(p_j - \frac{1}{q_j} \geqq 0\right),$$

we see that

$$\prod_{j=1}^m |a_{k,j}|^{p_j - \frac{1}{q_j}} \leqq \frac{1}{\prod_{j=1}^m \left(\frac{k-\alpha_j}{p-\alpha_j}\right)^{p_j - \frac{1}{q_j}}}.$$

This implies that

$$\frac{k-\beta}{p-\beta} \leqq \prod_{j=1}^m \left(\frac{k-\alpha_j}{p-\alpha_j}\right)^{p_j}$$

for all $k \geqq p+n$. Therefore, β should be

$$\beta \leqq p - \frac{(k-p)\prod_{j=1}^{m}(p-\alpha_j)^{p_j}}{\prod_{j=1}^{m}(k-\alpha_j)^{p_j} - \prod_{j=1}^{m}(p-\alpha_j)^{p_j}} \qquad (k \geqq p+n).$$

This completes the proof of the theorem. □

Taking $p_j = 1$ in Theorem 2.1, we obtain

Corollary 2.2. *If $f_j(z) \in \mathcal{S}_p^*(n,\alpha_j)$ for each $j = 1,2,\ldots,m$, then $G_m(z) \in \mathcal{S}_p^*(n,\beta)$ with*

$$\beta = p - \frac{n\prod_{j=1}^{m}(p-\alpha_j)}{\prod_{j=1}^{m}(p+n-\alpha_j) - \prod_{j=1}^{m}(p-\alpha_j)}.$$

Proof. In view of Theorem 2.1, we have

$$\beta \leqq \inf_{k \geqq p+n} \left\{ p - \frac{(k-p)\prod_{j=1}^{m}(p-\alpha_j)}{\prod_{j=1}^{m}(k-\alpha_j) - \prod_{j=1}^{m}(p-\alpha_j)} \right\}.$$

Let $F(k;m)$ be the right hand side of the above inequality. Further, let us define $G(k;m)$ by the numerator of $F'(k;m)$. When $m=2$,

$$\begin{aligned} G(k;2) &= -(p-\alpha_1)(p-\alpha_2)\{(k-\alpha_1)(k-\alpha_2) - (p-\alpha_1)(p-\alpha_2)\} \\ &\qquad + (k-p)(p-\alpha_1)(p-\alpha_2)\{(k-\alpha_1)+(k-\alpha_2)\} \\ &= (p-\alpha_1)(p-\alpha_2)(k-p)^2 > 0. \end{aligned}$$

Since $F(k;2)$ is an increasing function of k, we see that

$$\begin{aligned} F(k;2) &\geqq F(p+n;2) \\ &= p - \frac{n(p-\alpha_1)(p-\alpha_2)}{(p+n-\alpha_1)(p+n-\alpha_2) - (p-\alpha_1)(p-\alpha_2)}. \end{aligned} \tag{2.1}$$

Therefore, the corollary is true for $m=2$. Let us suppose that $G_{m-1}(z) \in \mathcal{S}_p^*(n,\beta^*)$ and $f_m(z) \in \mathcal{S}_p^*(n,\alpha_m)$, where

$$\beta^* = p - \frac{n\prod_{j=1}^{m-1}(p-\alpha_j)}{\prod_{j=1}^{m-1}(p+n-\alpha_j) - \prod_{j=1}^{m-1}(p-\alpha_j)}.$$

Then replacing α_1 by β^* and α_2 by α_m from (2.1), we see that

$$\begin{aligned} \beta &= p - \frac{n(p-\beta^*)(p-\alpha_m)}{(p+n-\beta^*)(p+n-\alpha_m) - (p-\beta^*)(p-\alpha_m)} \\ &= p - \frac{n\prod_{j=1}^{m}(p-\alpha_j)}{\prod_{j=1}^{m}(p+n-\alpha_j) - \prod_{j=1}^{m}(p-\alpha_j)}. \end{aligned}$$

Therefore, the corollary is true for the integer m. Using mathematical induction, we complete the proof of the corollary. □

Taking $\alpha_j = \alpha$ in Theorem 2.1, we have:

Corollary 2.3. *If $f_j(z) \in \mathcal{S}_p^*(n,\alpha)$ for all $j = 1,2,\ldots,m$, then $H_m(z) \in \mathcal{S}_p^*(n,\beta)$ with*

$$\beta = p - \frac{n(p-\alpha)^s}{(p+n-\alpha)^s - (p-\alpha)^s},$$

where

$$s=\sum_{j=1}^{m}p_j \geqq 1+\frac{p-\alpha}{n},\quad p_j \geqq \frac{1}{q_j},\quad q_j>1 \quad \textit{and} \quad \sum_{j=1}^{m}\frac{1}{q_j}\geqq 1.$$

Proof. By means of Theorem 2.1, we obtain that

$$\beta \leqq p-\frac{(k-p)(p-\alpha)^s}{(k-\alpha)^s-(p-\alpha)^s} \qquad (k \geqq p+n).$$

Let us define $F(k)$ by

$$F(k)=p-\frac{(k-p)(p-\alpha)^s}{(k-\alpha)^s-(p-\alpha)^s} \qquad (k \geqq p+n).$$

Then the numerator of $F'(k)$ can be written as

$$(p-\alpha)^s(k-\alpha)^s\left\{s(k-p)-(k-\alpha)\right\}+(p-\alpha)^s.$$

Since $s \geqq 1+\frac{p-\alpha}{n}$, we easily see that the numerator of $F'(k)$ is positive for $k \geqq p+n$. Therefore, $F(k)$ is increasing for $k \geqq p+n$. This gives the value of β in the corollary. □

We consider the example for Corollary 2.3.

Example 2.1. Let us define $f_j(z)$ by

$$f_j(z)=z^p+\frac{p-\alpha}{p+n-\alpha}\epsilon z^{p+n}+\frac{p-\alpha}{p+n+j-\alpha}\epsilon_j z^{p+n+j} \qquad (|\epsilon|+|\epsilon_j|\leqq 1)$$

for each $j=1,2,\dots,m$, which is equivalent to $f_j(z)\in\mathcal{S}_p^*(n,\alpha)$. Then $H_m(z)\in\mathcal{S}_p^*(n,\beta)$ with

$$\beta=p-\frac{n(p-\alpha)^s}{(p+n-\alpha)^s-(p-\alpha)^s}.$$

Because, for functions

$$f_j(z)=z^p+\frac{p-\alpha}{p+n-\alpha}\epsilon z^{p+n}+\frac{p-\alpha}{p+n+j-\alpha}\epsilon_j z^{p+n+j} \tag{2.2}$$

for each $j=1,2,\dots,m$, we have

$$\begin{aligned}\sum_{k=p+n}^{\infty}\frac{k-\alpha}{p-\alpha}|a_k| &= \frac{p+n-\alpha}{p-\alpha}|\epsilon||a_{p+n}|+\frac{p+n+j-\alpha}{p-\alpha}|\epsilon_j||a_{p+n+j}|\\ &=|\epsilon|+|\epsilon_j|\leqq 1\end{aligned}$$

from Lemma 1.1 which implies $f_j(z)\in\mathcal{S}_p^*(n,\alpha_j)$. From (2.2), we see that

$$H_m(z)=z^p+\left(\frac{p-\alpha}{p+n-\alpha}\epsilon\right)^s z^{p+n}.$$

Therefore $H_m(z)\in\mathcal{S}_p^*(n,\beta)$.

We also derive other results about $\mathcal{S}_p^*$ and $\mathcal{T}_p^*$.

Theorem 2.4. *If* $f_j(z)\in\mathcal{S}_p^*(n,\alpha_j)$ *for each* $j=1,2,\dots,m$, *then* $H_m(z)\in\mathcal{T}_p^*(n,\beta)$ *with*

$$\beta=\inf_{k\geqq p+n}\left\{p-\frac{k(k-p)\prod_{j=1}^{m}(p-\alpha_j)^{p_j}}{p\prod_{j=1}^{m}(k-\alpha_j)^{p_j}-k\prod_{j=1}^{m}(p-\alpha_j)^{p_j}}\right\},$$

where $p_j\geqq\frac{1}{q_j}$, $q_j>1$ *and* $\sum_{j=1}^{m}\frac{1}{q_j}\geqq 1$.

Proof. Using the same method as the proof in Theorem 2.1, we have to find the largest β such that

$$\left(\frac{k(k-\beta)}{p(p-\beta)}\right)\left(\prod_{j=1}^{m}|a_{k,j}|^{p_j}\right) \leqq \prod_{j=1}^{m}\left(\frac{k-\alpha_j}{p-\alpha_j}\right)^{\frac{1}{q_j}}|a_{k,j}|^{\frac{1}{q_j}},$$

which implies that

$$\beta \leqq p - \frac{k(k-p)\prod_{j=1}^{m}(p-\alpha_j)^{p_j}}{p\prod_{j=1}^{m}(k-\alpha_j)^{p_j} - k\prod_{j=1}^{m}(p-\alpha_j)^{p_j}}$$

for all $k \geqq p+n$. □

Corollary 2.5. *If $f_j(z) \in \mathcal{S}_p^*(n,\alpha_j)$ for each $j = 1,2,\dots,m$, then $G_m(z) \in \mathcal{T}_p^*(n,\beta)$ with*

$$\beta = p - \frac{n(p+n)\prod_{j=1}^{m}(p-\alpha_j)}{p\prod_{j=1}^{m}(p+n-\alpha_j) - (p+n)\prod_{j=1}^{m}(p-\alpha_j)}.$$

Theorem 2.6. *If $f_j(z) \in \mathcal{T}_p^*(n,\alpha_j)$ for each $j = 1,2,\dots,m$, then $H_m(z) \in \mathcal{T}_p^*(n,\beta)$ with*

$$\beta = \inf_{k \geqq p+n}\left\{p - \frac{k(k-p)\prod_{j=1}^{m}p^{p_j}(p-\alpha_j)^{p_j}}{p\prod_{j=1}^{m}k^{p_j}(k-\alpha_j)^{p_j} - k\prod_{j=1}^{m}p^{p_j}(p-\alpha_j)^{p_j}}\right\},$$

where $p_j \geqq \frac{1}{q_j}$, $q_j > 1$ and $\sum_{j=1}^{m}\frac{1}{q_j} \geqq 1$.

Proof. To prove the theorem, we have to find the largest β such that

$$\frac{k(k-\beta)}{p(p-\beta)}\left(\prod_{j=1}^{m}|a_{k,j}|^{p_j}\right) \leqq \prod_{j=1}^{m}\left(\frac{k(k-\alpha_j)}{p(p-\alpha_j)}\right)^{\frac{1}{q_j}}|a_{k,j}|^{\frac{1}{q_j}}$$

for all $k \geqq p+n$. □

Corollary 2.7. *If $f_j(z) \in \mathcal{T}_p^*(n,\alpha_j)$ for each $j = 1,2,\dots,m$, then $G_m(z) \in \mathcal{T}_p^*(n,\beta)$ with*

$$\beta = p - \frac{np^{m-1}\prod_{j=1}^{m}(p-\alpha_j)}{(p+n)^{m-1}\prod_{j=1}^{m}(p+n-\alpha_j) - p^{m-1}\prod_{j=1}^{m}(p-\alpha_j)}.$$

Theorem 2.8. *If $f_j(z) \in \mathcal{T}_p^*(n,\alpha_j)$ for each $j = 1,2,\dots,m$, then $H_m(z) \in \mathcal{S}_p^*(n,\beta)$ with*

$$\beta = \inf_{k \geqq p+n}\left\{p - \frac{(k-p)\prod_{j=1}^{m}p^{p_j}(p-\alpha_j)^{p_j}}{\prod_{j=1}^{m}k^{p_j}(k-\alpha_j)^{p_j} - \prod_{j=1}^{m}p^{p_j}(p-\alpha_j)^{p_j}}\right\},$$

where $p_j \geqq \frac{1}{q_j}$, $q_j > 1$ and $\sum_{j=1}^{m}\frac{1}{q_j} \geqq 1$.

Proof. We note that, we need to find the largest β such that

$$\frac{k-\beta}{p-\beta}\left(\prod_{j=1}^{m}|a_{k,j}|^{p_j}\right) \leqq \prod_{j=1}^{m}\left(\frac{k(k-\alpha_j)}{p(p-\alpha_j)}\right)^{\frac{1}{q_j}}|a_{k,j}|^{\frac{1}{q_j}}$$

for all $k \geqq p+n$. □

Corollary 2.9. *If $f_j(z) \in \mathcal{T}_p^*(n,\alpha_j)$ for each $j = 1,2,\dots,m$, then $G_m(z) \in \mathcal{S}_p^*(n,\beta)$ with*

$$\beta = p - \frac{np^{m}\prod_{j=1}^{m}(p-\alpha_j)}{(p+n)^{m}\prod_{j=1}^{m}(p+n-\alpha_j) - p^{m}\prod_{j=1}^{m}(p-\alpha_j)}.$$

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