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CHARACTERIZATION OF THE TRACE BY YOUNG'S INEQUALITY

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ABSTRACT. Let φ be a positive linear functional on the algebra of $n \times n$ complex matrices and p, q be positive numbers such that $\frac{1}{p} + \frac{1}{q} = 1$. We prove that if for any pair A, B of positive semi-definite $n \times n$ matrices the inequality

$$\varphi(|AB|) \leq \frac{\varphi(A^p)}{p} + \frac{\varphi(B^q)}{q}$$

holds, then φ is a positive scalar multiple of the trace.

Key words and phrases: Characterization of the trace, Matrix Young's inequality.

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In what follows, $\mathcal{M}_n$ stands for the *-algebra of $n \times n$ complex matrices, $\mathcal{M}_n^+$ stands for the cone of positive semi-definite matrices, p and q are positive numbers such that $\frac{1}{p} + \frac{1}{q} = 1$. For $A \in \mathcal{M}_n$, $|A|$ is understood as the modulus $|A| = (A^*A)^{1/2}$.

T. Ando proved in [1] that for any pair $A, B \in \mathcal{M}_n$ there is a unitary $U \in \mathcal{M}_n$ such that

$$U^*|AB^*|U \leq \frac{|A|^p}{p} + \frac{|B|^q}{q}.$$

It follows immediately that for any pair $A, B \in \mathcal{M}_n^+$ the following trace version of Young's inequality holds:

$$\mathrm{Tr}(|AB|) \leq \frac{\mathrm{Tr}(A^p)}{p} + \frac{\mathrm{Tr}(B^q)}{q}.$$

The aim of this note is to show that the latter inequality characterizes the trace among the positive linear functionals on $\mathcal{M}_n$.

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Theorem 1. *Let φ be a positive linear functional on $\mathcal{M}_n$ such that for any pair $A, B \in \mathcal{M}_n^+$ the inequality*

$$\varphi(|AB|) \le \frac{\varphi(A^p)}{p} + \frac{\varphi(B^q)}{q} \tag{1}$$

holds. Then $\varphi = k\,\mathrm{Tr}$ for some nonnegative number k.

Proof. As is well known, every positive linear functional φ on $\mathcal{M}_n$ can be represented in the form $\varphi(\cdot) = \mathrm{Tr}(S_\varphi \cdot)$ for some $S_\varphi \in \mathcal{M}_n^+$. It is easily seen that without loss of generality we can assume that $S_\varphi = \mathrm{diag}(\alpha_1, \alpha_2, \dots, \alpha_n)$, and we have to prove that $\alpha_i = \alpha_j$ for all $i, j = 1, \dots, n$. Clearly, it suffices to prove that $\alpha_1 = \alpha_2$. Inequality (1) must hold, in particular, for all matrices $A = [a_{ij}]_{i,j=1}^n$, $B = [b_{ij}]_{i,j=1}^n$ in $\mathcal{M}_n^+$ such that $0 = a_{ij} = b_{ij}$ if $3 \le i \le n$ or $3 \le j \le n$. Thus the proof of the theorem reduces to the following lemma.

Lemma 2. *Let $S = \mathrm{diag}\left(\frac{1}{2} + s, \frac{1}{2} - s\right)$, where $0 \le s \le \frac{1}{2}$. If for every pair $A, B \in \mathcal{M}_2^+$ the inequality*

$$\mathrm{Tr}(S\,|AB|) \le \frac{\mathrm{Tr}(SA^p)}{p} + \frac{\mathrm{Tr}(SB^q)}{q} \tag{2}$$

holds, then $s = 0$.

Proof of Lemma 2. Let $0 \le \varepsilon \le \frac{1}{2}$, $\delta = \frac{1}{4} - \varepsilon^2$. Let us consider two projections

$$P_1 = \begin{pmatrix} \frac{1}{2} - \varepsilon & \sqrt{\delta} \\ \sqrt{\delta} & \frac{1}{2} + \varepsilon \end{pmatrix}, \qquad P_2 = \begin{pmatrix} \frac{1}{2} + \varepsilon & \sqrt{\delta} \\ \sqrt{\delta} & \frac{1}{2} - \varepsilon \end{pmatrix}.$$

Calculate $|P_1P_2|$:

$$P_2P_1 = \begin{pmatrix} 2\delta & (1+2\varepsilon)\sqrt{\delta} \\ (1-2\varepsilon)\sqrt{\delta} & 2\delta \end{pmatrix}, \qquad P_2P_1P_2 = 4\delta P_2,$$

hence

$$|P_1P_2| = 2\sqrt{\delta}P_2 = \sqrt{1-4\varepsilon^2}P_2.$$

Substitute $A = \alpha P_1$, $B = \beta P_2$ with $\alpha, \beta > 0$ into (2) and perform the calculations. Then the left hand side in (2) becomes

$$\alpha\beta\sqrt{1-4\varepsilon^2}\left(\frac{1}{2} + 2\varepsilon s\right)$$

and the right hand one becomes

$$\frac{\alpha^p\left(\frac{1}{2} - 2\varepsilon s\right)}{p} + \frac{\beta^q\left(\frac{1}{2} + 2\varepsilon s\right)}{q}.$$

Now, take $\alpha = 1$, $\beta = \left(\frac{1-4\varepsilon s}{1+4\varepsilon s}\right)^{\frac{1}{q}}$. Then we obtain as an implication of (2):

$$\frac{1}{2}(1-4\varepsilon s)^{\frac{1}{q}}(1+4\varepsilon s)^{\frac{1}{p}}\sqrt{1-4\varepsilon^2} \le \frac{1}{2}(1-4\varepsilon s),$$

which implies

$$(1-4\varepsilon^2)^{\frac{p}{2}} \le \frac{1-4\varepsilon s}{1+4\varepsilon s}. \tag{3}$$

By the Taylor formulas,

$$(1-4\varepsilon^2)^{\frac{p}{2}} = 1 - 2p\varepsilon^2 + o(\varepsilon^2) = 1 + o(\varepsilon) \quad (\varepsilon \to 0),$$
$$\frac{1-4\varepsilon s}{1+4\varepsilon s} = 1 - 8\varepsilon s + o(\varepsilon) \quad (\varepsilon \to 0).$$

Since we have supposed that $0 \leq s$, the inequality (3) can hold for all $\varepsilon \in \left(0, \frac{1}{2}\right]$ only if $s = 0$. □

REFERENCES

[1] T. ANDO, Matrix Young inequalities, *Oper. Theory Adv. Appl.*, **75** (1995), 33–38.