Journal of Inequalities in Pure and Applied Mathematics

CHARACTERIZATION OF THE TRACE BY YOUNG'S INEQUALITY

A.M. BIKCHENTAEV AND O.E. TIKHONOV

Research Institute of Mathematics and Mechanics
Kazan State University
Nuzhina 17, Kazan, 420008, Russia.

EMail: Airat.Bikchentaev@ksu.ru

EMail: Oleg.Tikhonov@ksu.ru

volume 6, issue 2, article 49, 2005.

Received 25 March, 2005; accepted 11 April, 2005.

Communicated by: T. Ando

ISSN (electronic): 1443-5756
089-05

Abstract

Let φ be a positive linear functional on the algebra of $n \times n$ complex matrices and p, q be positive numbers such that $\frac{1}{p} + \frac{1}{q} = 1$. We prove that if for any pair A, B of positive semi-definite $n \times n$ matrices the inequality

$$\varphi(|AB|) \leq \frac{\varphi(A^p)}{p} + \frac{\varphi(B^q)}{q}$$

holds, then φ is a positive scalar multiple of the trace.

2000 Mathematics Subject Classification: 15A45.
Key words: Characterization of the trace, Matrix Young's inequality.

Supported by the Russian Foundation for Basic Research (grant no. 05-01-00799).

In what follows, $\mathcal{M}_n$ stands for the *-algebra of $n \times n$ complex matrices, $\mathcal{M}_n^+$ stands for the cone of positive semi-definite matrices, p and q are positive numbers such that $\frac{1}{p} + \frac{1}{q} = 1$. For $A \in \mathcal{M}_n$, $|A|$ is understood as the modulus $|A| = (A^*A)^{1/2}$.

T. Ando proved in [1] that for any pair $A, B \in \mathcal{M}_n$ there is a unitary $U \in \mathcal{M}_n$ such that

$$U^*|AB^*|U \leq \frac{|A|^p}{p} + \frac{|B|^q}{q}.$$

It follows immediately that for any pair $A, B \in \mathcal{M}_n^+$ the following trace version of Young's inequality holds:

$$\operatorname{Tr}(|AB|) \leq \frac{\operatorname{Tr}(A^p)}{p} + \frac{\operatorname{Tr}(B^q)}{q}.$$

The aim of this note is to show that the latter inequality characterizes the trace among the positive linear functionals on $\mathcal{M}_n$.

Theorem 1. *Let φ be a positive linear functional on $\mathcal{M}_n$ such that for any pair $A, B \in \mathcal{M}_n^+$ the inequality*

$$\varphi(|AB|) \leq \frac{\varphi(A^p)}{p} + \frac{\varphi(B^q)}{q} \tag{1}$$

holds. Then $\varphi = k \operatorname{Tr}$ for some nonnegative number k.

Proof. As is well known, every positive linear functional φ on $\mathcal{M}_n$ can be represented in the form $\varphi(\cdot) = \operatorname{Tr}(S_\varphi \cdot)$ for some $S_\varphi \in \mathcal{M}_n^+$. It is easily seen that without loss of generality we can assume that $S_\varphi = \operatorname{diag}(\alpha_1, \alpha_2, \dots, \alpha_n)$, and we have to prove that $\alpha_i = \alpha_j$ for all $i, j = 1, \dots, n$. Clearly, it suffices to prove that $\alpha_1 = \alpha_2$. Inequality (1) must hold, in particular, for all matrices $A = [a_{ij}]_{i,j=1}^n$, $B = [b_{ij}]_{i,j=1}^n$ in $\mathcal{M}_n^+$ such that $0 = a_{ij} = b_{ij}$ if $3 \leq i \leq n$ or $3 \leq j \leq n$. Thus the proof of the theorem reduces to the following lemma.

Lemma 2. *Let $S = \operatorname{diag}\left(\frac{1}{2} + s, \frac{1}{2} - s\right)$, where $0 \leq s \leq \frac{1}{2}$. If for every pair $A, B \in \mathcal{M}_2^+$ the inequality*

$$\operatorname{Tr}(S\,|AB|) \leq \frac{\operatorname{Tr}(SA^p)}{p} + \frac{\operatorname{Tr}(SB^q)}{q} \tag{2}$$

holds, then $s = 0$.

Proof of Lemma 2. Let $0 \le \varepsilon \le \frac{1}{2}$, $\delta = \frac{1}{4} - \varepsilon^2$. Let us consider two projections

$$P_1 = \begin{pmatrix} \frac{1}{2} - \varepsilon & \sqrt{\delta} \\ \sqrt{\delta} & \frac{1}{2} + \varepsilon \end{pmatrix}, \qquad P_2 = \begin{pmatrix} \frac{1}{2} + \varepsilon & \sqrt{\delta} \\ \sqrt{\delta} & \frac{1}{2} - \varepsilon \end{pmatrix}.$$

Calculate $|P_1 P_2|$:

$$P_2 P_1 = \begin{pmatrix} 2\delta & (1+2\varepsilon)\sqrt{\delta} \\ (1-2\varepsilon)\sqrt{\delta} & 2\delta \end{pmatrix}, \qquad P_2 P_1 P_2 = 4\delta P_2,$$

hence

$$|P_1 P_2| = 2\sqrt{\delta} P_2 = \sqrt{1 - 4\varepsilon^2} P_2.$$

Substitute $A = \alpha P_1$, $B = \beta P_2$ with $\alpha, \beta > 0$ into (2) and perform the calculations. Then the left hand side in (2) becomes

$$\alpha\beta\sqrt{1-4\varepsilon^2}\left(\frac{1}{2} + 2\varepsilon s\right)$$

and the right hand one becomes

$$\frac{\alpha^p\left(\frac{1}{2} - 2\varepsilon s\right)}{p} + \frac{\beta^q\left(\frac{1}{2} + 2\varepsilon s\right)}{q}.$$

Now, take $\alpha = 1$, $\beta = \left(\frac{1-4\varepsilon s}{1+4\varepsilon s}\right)^{\frac{1}{q}}$. Then we obtain as an implication of (2):

$$\frac{1}{2}(1-4\varepsilon s)^{\frac{1}{q}}(1+4\varepsilon s)^{\frac{1}{p}}\sqrt{1-4\varepsilon^2} \le \frac{1}{2}(1-4\varepsilon s),$$

which implies

$$(1-4\varepsilon^2)^{\frac{p}{2}} \leq \frac{1-4\varepsilon s}{1+4\varepsilon s}. \tag{3}$$

By the Taylor formulas,

$$(1-4\varepsilon^2)^{\frac{p}{2}} = 1 - 2p\varepsilon^2 + o(\varepsilon^2) = 1 + o(\varepsilon) \quad (\varepsilon \to 0),$$

$$\frac{1-4\varepsilon s}{1+4\varepsilon s} = 1 - 8\varepsilon s + o(\varepsilon) \quad (\varepsilon \to 0).$$

Since we have supposed that $0 \leq s$, the inequality (3) can hold for all $\varepsilon \in \left(0, \frac{1}{2}\right]$ only if $s = 0$. $\square$

Characterization of the Trace by Young's Inequality

A.M. Bikchentaev and O.E. Tikhonov

Title Page

Contents

Go Back

Close

Quit

References

[1] T. ANDO, Matrix Young inequalities, *Oper. Theory Adv. Appl.*, **75** (1995), 33–38.