

## DIFFERENTIAL SUBORDINATIONS AND SUPERORDINATIONS FOR ANALYTIC FUNCTIONS DEFINED BY THE DZIOK-SRIVASTAVA LINEAR OPERATOR

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## Abstract

In the present investigation, we obtain some subordination and superordination results involving Dziok-Srivastava linear operator  $H_m^l[\alpha_1]$  for certain normalized analytic functions in the open unit disk. Our results extend corresponding previously known results.

*2000 Mathematics Subject Classification:* Primary 30C45; Secondary 30C80.

*Key words:* Univalent functions, Starlike functions, Convex functions, Differential subordination, Convolution, Dziok-Srivastava linear operator.

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# 1. Introduction

Let  $\mathcal{H}$  be the class of functions analytic in  $\Delta := \{z : |z| < 1\}$  and  $\mathcal{H}(a, n)$  be the subclass of  $\mathcal{H}$  consisting of functions of the form  $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots$ . Let  $\mathcal{A}$  be the subclass of  $\mathcal{H}$  consisting of functions of the form  $f(z) = z + a_2 z^2 + \dots$ . Let  $p, h \in \mathcal{H}$  and let  $\phi(r, s, t; z) : \mathbb{C}^3 \times \Delta \rightarrow \mathbb{C}$ . If  $p$  and  $\phi(p(z), zp'(z), z^2 p''(z); z)$  are univalent and if  $p$  satisfies the second order superordination

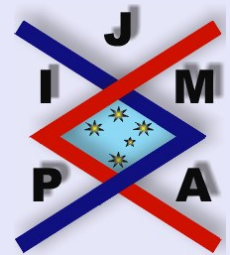
$$(1.1) \quad h(z) \prec \phi(p(z), zp'(z), z^2 p''(z); z),$$

then  $p$  is a solution of the differential superordination (1.1). (If  $f$  is subordinate to  $F$ , then  $F$  is superordinate to  $f$ .) An analytic function  $q$  is called a *subordinant* if  $q \prec p$  for all  $p$  satisfying (1.1). A univalent subordinant  $\tilde{q}$  that satisfies  $q \prec \tilde{q}$  for all subordinants  $q$  of (1.1) is said to be the best subordinant. Recently Miller and Mocanu [14] obtained conditions on  $h, q$  and  $\phi$  for which the following implication holds:

$$h(z) \prec \phi(p(z), zp'(z), z^2 p''(z); z) \Rightarrow q(z) \prec p(z).$$

Using the results of Miller and Mocanu [14], Bulboacă [5] considered certain classes of first order differential subordinations as well as superordination-preserving integral operators [4]. Ali et al. [1] have used the results of Bulboacă [5] and obtained sufficient conditions for certain normalized analytic functions  $f(z)$  to satisfy

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z),$$



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where  $q_1$  and  $q_2$  are given univalent functions in  $\Delta$  with  $q_1(0) = 1$  and  $q_2(0) = 1$ . Shanmugam et al. [19] obtained sufficient conditions for a normalized analytic function  $f(z)$  to satisfy

$$q_1(z) \prec \frac{f(z)}{zf'(z)} \prec q_2(z) \quad \text{and} \quad q_1(z) \prec \frac{z^2 f'(z)}{\{f(z)\}^2} \prec q_2(z)$$

where  $q_1$  and  $q_2$  are given univalent functions in  $\Delta$  with  $q_1(0) = 1$  and  $q_2(0) = 1$ .

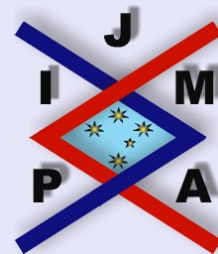
In [2], for functions  $f \in \mathcal{A}$  such that  $\delta > 0$ ,

$$\Re \left\{ \frac{zf'(z)}{f(z)} \left( \frac{f(z)}{z} \right)^\delta \right\} > 0, \quad z \in \Delta,$$

a class of Bazilevic type functions was considered and certain properties were studied. In this paper motivated by Liu [11], we define a class

$$B(\lambda, \delta, A, B) := \left\{ f \in \mathcal{A} : (1 - \lambda) \left( \frac{f(z)}{z} \right)^\delta + \lambda \frac{zf'(z)}{f(z)} \left( \frac{f(z)}{z} \right)^\delta \prec \frac{1 + Az}{1 + Bz} \right\},$$

where  $\delta > 0$ ,  $\lambda \geq 0$ ,  $-1 \leq B < A \leq 1$  and studied certain interesting properties based on subordination. Further we obtained a sandwich result for functions in the class  $B(\lambda, \delta, A, B)$ .



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## 2. Preliminaries

For our present investigation, we shall need the following definition and results.

**Definition 2.1** ([14, Definition 2, p. 817]). Denote by  $Q$ , the set of all functions  $f(z)$  that are analytic and injective on  $\overline{\Delta} - E(f)$ , where

$$E(f) = \left\{ \zeta \in \partial\Delta : \lim_{z \rightarrow \zeta} f(z) = \infty \right\}$$

and are such that  $f'(\zeta) \neq 0$  for  $\zeta \in \partial\Delta - E(f)$ .

**Lemma 2.1** ([13, Theorem 3.4h, p. 132]). Let  $q(z)$  be univalent in the unit disk  $\Delta$  and  $\theta$  and  $\phi$  be analytic in a domain  $D$  containing  $q(\Delta)$  with  $\phi(w) \neq 0$  when  $w \in q(\Delta)$ . Set  $Q(z) = zq'(z)\phi(q(z))$ ,  $h(z) = \theta(q(z)) + Q(z)$ . Suppose that

1.  $Q(z)$  is starlike univalent in  $\Delta$ , and
2.  $\Re \left\{ \frac{zh'(z)}{Q(z)} \right\} > 0$  for  $z \in \Delta$ .

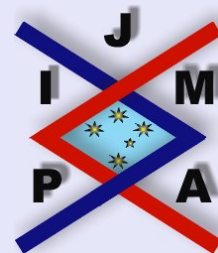
If

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)),$$

then  $p(z) \prec q(z)$  and  $q(z)$  is the best dominant.

**Lemma 2.2** ([19]). Let  $q$  be a convex univalent function in  $\Delta$  and  $\psi, \gamma \in \mathbb{C}$  with

$$\Re \left\{ 1 + \frac{zq''(z)}{q'(z)} + \frac{\psi}{\gamma} \right\} > 0.$$



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If  $p(z)$  is analytic in  $\Delta$  and

$$\psi p(z) + \gamma z p'(z) \prec \psi q(z) + \gamma z q'(z)$$

then  $p(z) \prec q(z)$  and  $q(z)$  is the best dominant.

**Lemma 2.3 ([5]).** Let  $q(z)$  be convex univalent in the unit disk  $\Delta$  and  $\vartheta$  and  $\varphi$  be analytic in a domain  $D$  containing  $q(\Delta)$ . Suppose that

1.  $\Re [\vartheta'(q(z))/\varphi(q(z))] > 0$  for  $z \in \Delta$ ,
2.  $zq'(z)\varphi(q(z))$  is starlike univalent in  $\Delta$ .

If  $p(z) \in \mathcal{H}[q(0), 1] \cap \mathcal{Q}$ , with  $p(\Delta) \subseteq D$ , and  $\vartheta(p(z)) + zp'(z)\varphi(p(z))$  is univalent in  $\Delta$ , and

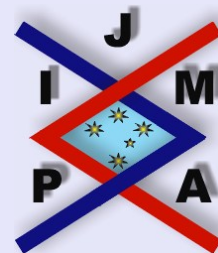
$$(2.1) \quad \vartheta(q(z)) + zq'(z)\varphi(q(z)) \prec \vartheta(p(z)) + zp'(z)\varphi(p(z)),$$

then  $q(z) \prec p(z)$  and  $q(z)$  is the best subordinator.

**Lemma 2.4 ([14, Theorem 8, p. 822]).** Let  $q$  be convex univalent in  $\Delta$  and  $\gamma \in \mathbb{C}$ . Further assume that  $\Re [\bar{\gamma}] > 0$ . If  $p(z) \in \mathcal{H}[q(0), 1] \cap \mathcal{Q}$ ,  $p(z) + \gamma zp'(z)$  is univalent in  $\Delta$ , then

$$q(z) + \gamma zq'(z) \prec p(z) + \gamma zp'(z)$$

implies  $q(z) \prec p(z)$  and  $q(z)$  is the best subordinator.



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For two functions  $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$  and  $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ , the Hadamard product (or convolution) of  $f$  and  $g$  is defined by

$$(f * g)(z) := z + \sum_{n=2}^{\infty} a_n b_n z^n =: (g * f)(z).$$

For  $\alpha_j \in \mathbb{C}$  ( $j = 1, 2, \dots, l$ ) and  $\beta_j \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$  ( $j = 1, 2, \dots, m$ ), the *generalized hypergeometric function*  ${}_l F_m(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m; z)$  is defined by the infinite series

$${}_l F_m(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m; z) := \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_l)_n}{(\beta_1)_n \cdots (\beta_m)_n} \frac{z^n}{n!}$$

$$(l \leq m + 1; l, m \in \mathbb{N}_0 := \{0, 1, 2, \dots\})$$

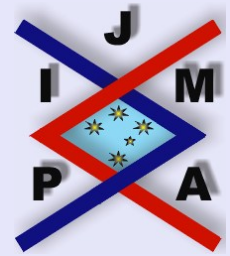
where  $(a)_n$  is the Pochhammer symbol defined by

$$(a)_n := \frac{\Gamma(a+n)}{\Gamma(a)}$$

$$= \begin{cases} 1, & (n = 0); \\ a(a+1)(a+2) \cdots (a+n-1), & (n \in \mathbb{N} := \{1, 2, 3, \dots\}). \end{cases}$$

Corresponding to the function

$$h(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m; z) := z {}_l F_m(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m; z),$$



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the Dziok-Srivastava operator [7] (see also [8, 20])  $H_m^l(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m)$  is defined by the Hadamard product

$$\begin{aligned} (2.2) \quad H_m^l(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m)f(z) & \\ & := h(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m; z) * f(z) \\ & = z + \sum_{n=2}^{\infty} \frac{(\alpha_1)_{n-1} \cdots (\alpha_l)_{n-1}}{(\beta_1)_{n-1} \cdots (\beta_m)_{n-1}} \frac{a_n z^n}{(n-1)!}. \end{aligned}$$

For brevity, we write

$$H_m^l[\alpha_1]f(z) := H_m^l(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m)f(z).$$

It is easy to verify from (2.2) that

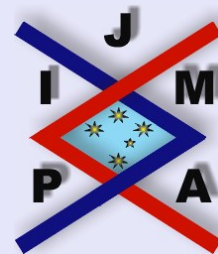
$$(2.3) \quad z(H_m^l[\alpha_1]f(z))' = \alpha_1 H_m^l[\alpha_1 + 1]f(z) - (\alpha_1 - 1)H_m^l[\alpha_1]f(z).$$

Special cases of the Dziok-Srivastava linear operator include the Hohlov linear operator [9], the Carlson-Shaffer linear operator  $L(a, c)$  [6], the Ruscheweyh derivative operator  $D^n$  [18], the generalized Bernardi-Libera-Livingston linear integral operator (cf. [3], [10], [12]) and the Srivastava-Owa fractional derivative operators (cf. [16], [17]).

The main object of the present paper is to find sufficient conditions for certain normalized analytic functions  $f(z)$  to satisfy

$$q_1(z) \prec \left( \frac{H_m^l[\alpha_1]f(z)}{z} \right)^\delta \prec q_2(z)$$

where  $q_1$  and  $q_2$  are given univalent functions in  $\Delta$ . Also, we obtain the number of known results as special cases.



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### 3. Main Results

We begin with the following:

**Theorem 3.1.** Let  $q(z)$  be univalent in  $\Delta$ ,  $\lambda \in \mathbb{C}$  and  $\alpha_1 > 0$ ,  $\delta > 0$ . Suppose  $q(z)$  satisfies

$$(3.1) \quad \Re \left\{ 1 + \frac{zq''(z)}{q'(z)} + \frac{\lambda}{\delta} \right\} > 0.$$

If  $f \in \mathcal{A}$  satisfies the subordination,

$$(3.2) \quad (1 - \lambda\alpha_1) \left( \frac{H_m^l[\alpha_1]f(z)}{z} \right)^\delta + \lambda\alpha_1 \left( \frac{H_m^l[\alpha_1]f(z)}{z} \right)^\delta \prec \left( \frac{H_m^l[\alpha_1 + 1]f(z)}{H_m^l[\alpha_1]f(z)} \right) \prec q(z) + \frac{\lambda}{\delta} zq'(z),$$

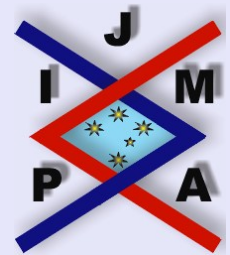
then

$$\left( \frac{H_m^l[\alpha_1]f(z)}{z} \right)^\delta \prec q(z)$$

and  $q(z)$  is the best dominant.

*Proof.* Define the function  $p(z)$  by

$$(3.3) \quad p(z) := \left( \frac{H_m^l[\alpha_1]f(z)}{z} \right)^\delta.$$



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Then

$$\frac{zp'(z)}{\delta} := \alpha_1 \left( \frac{H_m^l[\alpha_1]f(z)}{z} \right)^\delta \left( \frac{H_m^l[\alpha_1 + 1]f(z)}{H_m^l[\alpha_1]f(z)} - 1 \right),$$

hence the hypothesis (3.2) of Theorem 3.1 yields the subordination:

$$p(z) + \frac{\lambda zp'(z)}{\delta} \prec q(z) + \frac{\lambda zq'(z)}{\delta}.$$

Now Theorem 3.1 follows by applying Lemma 2.2 with  $\psi = 1$  and  $\gamma = \frac{\lambda}{\delta}$ .  $\square$

When  $l = 2$ ,  $m = 1$ ,  $\alpha_1 = a$ ,  $\alpha_2 = 1$ , and  $\beta_1 = c$  in Theorem 3.1, we have the following corollary.

**Corollary 3.2.** *Let  $q(z)$  be univalent in  $\Delta$ ,  $\lambda \in C$  and  $\alpha_1 > 0$ ,  $\delta > 0$ . Suppose  $q(z)$  satisfies (3.1). If  $f \in \mathcal{A}$  and satisfies the subordination,*

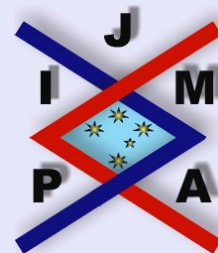
$$(3.4) \quad (1 - \lambda a) \left( \frac{L(a, c)f(z)}{z} \right)^\delta + \lambda a \left( \frac{L(a, c)f(z)}{z} \right)^\delta \left( \frac{L(a + 1, c)f(z)}{L(a, c)f(z)} \right) \prec q(z) + \frac{\lambda}{\delta} zq'(z),$$

then

$$\left( \frac{L(a, c)f(z)}{z} \right)^\delta \prec q(z)$$

and  $q(z)$  is the best dominant.

By taking  $l = 1$ ,  $m = 0$  and  $\alpha_1 = 1$  in Theorem 3.1, we get the following corollary.



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**Corollary 3.3.** Let  $q(z)$  be univalent in  $\Delta$ ,  $\lambda \in \mathbb{C}$  and  $\alpha_1 > 0$ ,  $\delta > 0$ . Suppose  $q(z)$  satisfies (3.1). If  $f \in \mathcal{A}$  and satisfies the subordination,

$$(3.5) \quad (1 - \lambda) \left( \frac{f(z)}{z} \right)^\delta + \lambda \left( \frac{f(z)}{z} \right)^\delta \left( \frac{zf'(z)}{f(z)} \right) \prec q(z) + \frac{\lambda}{\delta} zq'(z),$$

then

$$\left( \frac{f(z)}{z} \right)^\delta \prec q(z)$$

and  $q(z)$  is the best dominant.

**Corollary 3.4.** Let  $-1 \leq B < A \leq 1$  and (3.1) hold. If  $f \in \mathcal{A}$  and

$$(1 - \lambda\alpha_1) \left( \frac{H_m^l[\alpha_1]f(z)}{z} \right)^\delta + \lambda\alpha_1 \left( \frac{H_m^l[\alpha_1]f(z)}{z} \right)^\delta \left( \frac{H_m^l[\alpha_1 + 1]f(z)}{H_m^l[\alpha_1]f(z)} \right) \prec \frac{\lambda(A - B)z}{\delta(1 + Bz)^2} + \frac{1 + Az}{1 + Bz},$$

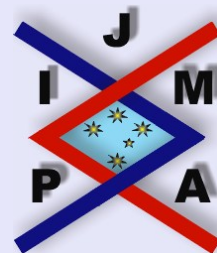
then

$$\left( \frac{H_m^l[\alpha_1]f(z)}{z} \right)^\delta \prec \frac{1 + Az}{1 + Bz}$$

and  $\frac{1+Az}{1+Bz}$  is the best dominant.

**Theorem 3.5.** Let  $q(z)$  be univalent in  $\Delta$ ,  $\lambda, \delta \in \mathbb{C}$ . Suppose  $q(z)$  satisfies

$$(3.6) \quad \Re \left\{ 1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} \right\} > 0.$$



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If  $f \in \mathcal{A}$  satisfies the subordination:

$$(3.7) \quad 1 + \gamma \delta \alpha_1 \left( \frac{H_m^l[\alpha_1 + 1]f(z)}{H_m^l[\alpha_1]f(z)} - 1 \right) \prec 1 + \gamma \frac{zq'(z)}{q(z)},$$

then

$$\left( \frac{H_m^l[\alpha_1]f(z)}{z} \right)^\delta \prec q(z)$$

and  $q(z)$  is the best dominant.

*Proof.* Define the function  $p(z)$  by

$$p(z) = \left( \frac{H_m^l[\alpha_1]f(z)}{z} \right)^\delta.$$

It is clear that  $p(0) = 1$  and  $p(z)$  is analytic in  $\Delta$ . By using the identity (2.3), from (3.3) we get,

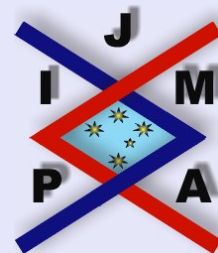
$$(3.8) \quad \frac{zp'(z)}{p(z)} = \alpha_1 \delta \left( \frac{H_m^l[\alpha_1 + 1]f(z)}{H_m^l[\alpha_1]f(z)} - 1 \right).$$

Using (3.8) in (3.7), we see that the subordination becomes

$$1 + \gamma \frac{zp'(z)}{p(z)} \prec 1 + \gamma \frac{zq'(z)}{q(z)}.$$

By setting

$$\theta(w) = 1 \quad \text{and} \quad \varphi(w) = \frac{\gamma}{w},$$



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we observe that  $\varphi$  and  $\theta$  are analytic in  $\mathbb{C} \setminus \{0\}$ . Also we see that

$$Q(z) := zq'(z)\varphi(q(z)) = \frac{\gamma zq'(z)}{q(z)},$$

and

$$h(z) := \vartheta(q(z)) + Q(z) = 1 + \gamma \frac{zq'(z)}{q(z)}.$$

It is clear that  $Q(z)$  is starlike univalent in  $\Delta$  and

$$\Re \frac{zh'(z)}{Q(z)} = \Re \left[ 1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} \right] \geq 0.$$

By the hypothesis of Theorem 3.5, the result now follows by an application of Lemma 2.1.  $\square$

Specializing the values of  $l = 1$ ,  $m = 0$ ,  $\alpha_1 = 1$  and  $q(z) = \frac{1}{(1-z)^{2b}}$  ( $b \in \mathbb{C} - \{0\}$ ),  $\gamma = \frac{1}{b}$  and  $\delta = 1$  in Theorem 3.5 above, we have the following corollary as stated in [21].

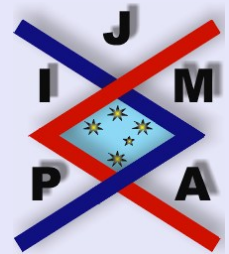
**Corollary 3.6.** *Let  $b$  be a non zero complex number. If  $f \in \mathcal{A}$  and*

$$1 + \frac{1}{b} \left[ \frac{zf'(z)}{f(z)} - 1 \right] \prec \frac{1+z}{1-z},$$

then

$$\frac{f(z)}{z} \prec \frac{1}{(1-z)^{2b}}$$

and  $\frac{1}{(1-z)^{2b}}$  is the best dominant.



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Choosing the values of  $l = 1$ ,  $m = 0$ ,  $\alpha_1 = 1$  and  $q(z) = \frac{1}{(1-z)^{2ab}}$  ( $b \in C - \{0\}$ ),  $\gamma = \frac{1}{b}$  and  $\delta = a \neq 0$  in Theorem 3.5 above, we have the following corollary as stated in [15].

**Corollary 3.7.** *Let  $b$  be a non zero complex number. If  $f \in \mathcal{A}$  and*

$$1 + \frac{1}{b} \left[ \frac{zf'(z)}{f(z)} - 1 \right] \prec \frac{1+z}{1-z},$$

then

$$\left( \frac{f(z)}{z} \right)^a \prec \frac{1}{(1-z)^{2ab}}$$

where  $a \neq 0$  is a complex number and  $\frac{1}{(1-z)^{2ab}}$  is the best dominant.

Similarly for  $l = 2$ ,  $m = 1$ ,  $\alpha_1 = 1$ ,  $\alpha_2 = 1$ ,  $\beta_1 = 1$  and  $q(z) = \frac{1}{(1-z)^{2b}}$  ( $b \in C - \{0\}$ ),  $\gamma = \frac{1}{b}$  and  $\delta = 1$  in Theorem 3.5 above, we get the following result as stated in [21].

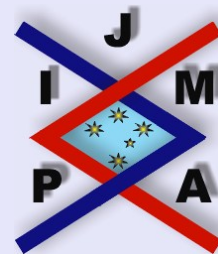
**Corollary 3.8.** *Let  $b$  be a non zero complex number. If  $f \in \mathcal{A}$  and*

$$1 + \frac{1}{b} \left[ \frac{zf''(z)}{f'(z)} - 1 \right] \prec \frac{1+z}{1-z},$$

then

$$f'(z) \prec \frac{1}{(1-z)^{2b}}$$

and  $\frac{1}{(1-z)^{2b}}$  is the best dominant.



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Next, applying Lemma 2.3, we have the following theorem.

**Theorem 3.9.** Let  $q(z)$  be convex univalent in  $\Delta$ ,  $\lambda \in \mathcal{C}$  and  $0 < \delta < 1$ . Suppose  $f \in \mathcal{A}$  satisfies

$$(3.9) \quad \operatorname{Re} \left\{ \frac{\delta}{\lambda} \right\} > 0$$

and  $\left( \frac{H_m^l[\alpha_1]f(z)}{z} \right)^\delta \in H[q(0), 1] \cap Q$ . Let

$$(1 - \lambda\alpha_1) \left( \frac{H_m^l[\alpha_1]f(z)}{z} \right)^\delta + \lambda\alpha_1 \left( \frac{H_m^l[\alpha_1]f(z)}{z} \right)^\delta \left( \frac{H_m^l[\alpha_1 + 1]f(z)}{H_m^l[\alpha_1]f(z)} \right)$$

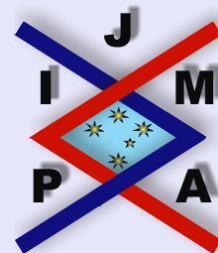
be univalent in  $\Delta$ . If  $f \in \mathcal{A}$  satisfies the superordination,

$$(3.10) \quad q(z) + \frac{\lambda}{\delta} z q'(z) \prec (1 - \lambda\alpha_1) \left( \frac{H_m^l[\alpha_1]f(z)}{z} \right)^\delta + \lambda\alpha_1 \left( \frac{H_m^l[\alpha_1]f(z)}{z} \right)^\delta \left( \frac{H_m^l[\alpha_1 + 1]f(z)}{H_m^l[\alpha_1]f(z)} \right)$$

then

$$q(z) \prec \left( \frac{H_m^l[\alpha_1]f(z)}{z} \right)^\delta$$

and  $q(z)$  is the best subordinant.



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*Proof.* Define the function  $p(z)$  by

$$(3.11) \quad p(z) := \left( \frac{H_m^l[\alpha_1]f(z)}{z} \right)^\delta.$$

Using (3.11), simple computation produces

$$\frac{zp'(z)}{\delta} := \alpha_1 \left( \frac{H_m^l[\alpha_1]f(z)}{z} \right)^\delta \left( \frac{H_m^l[\alpha_1 + 1]f(z)}{H_m^l[\alpha_1]f(z)} - 1 \right),$$

then

$$q(z) + \frac{\lambda}{\delta} zq'(z) \prec p(z) + \frac{\lambda}{\delta} zp'(z).$$

By setting  $\vartheta(w) = w$  and  $\phi(w) = \frac{\lambda}{\delta}$ , it is easily observed that  $\vartheta(w)$  is analytic in  $C$ . Also,  $\phi(w)$  is analytic in  $C \setminus \{0\}$  and  $\phi(w) \neq 0$ , ( $w \in C \setminus \{0\}$ ).

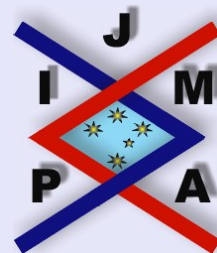
Since  $q(z)$  is a convex univalent function, it follows that

$$\Re \left\{ \frac{\vartheta'(q(z))}{\phi(q(z))} \right\} = \Re \left\{ \frac{\delta}{\lambda} \right\} > 0, \quad z \in \Delta, \quad \delta, \lambda \in C, \delta, \lambda \neq 0.$$

Now Theorem 3.9 follows by applying Lemma 2.3.  $\square$

Concluding the results of differential subordination and superordination, we state the following sandwich result.

**Theorem 3.10.** *Let  $q_1$  and  $q_2$  be convex univalent in  $\Delta$ ,  $\lambda \in \mathbb{C}$  and  $0 < \delta < 1$ . Suppose  $q_2$  satisfies (3.1) and  $q_1$  satisfies (3.9). If  $\left( \frac{H_m^l[\alpha_1]f(z)}{z} \right)^\delta \in \mathcal{H}[q(0), 1] \cap$*



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$Q$ ,

$$(1 - \lambda\alpha_1) \left( \frac{H_m^l[\alpha_1]f(z)}{z} \right)^\delta + \lambda\alpha_1 \left( \frac{H_m^l[\alpha_1]f(z)}{z} \right)^\delta \left( \frac{H_m^l[\alpha_1 + 1]f(z)}{H_m^l[\alpha_1]f(z)} \right)$$

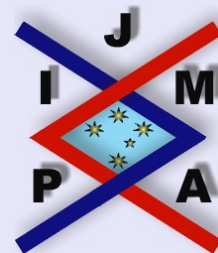
is univalent in  $\Delta$ . If  $f \in \mathcal{A}$  satisfies

$$(3.12) \quad q_1(z) + \frac{\lambda}{\delta} z q_1'(z) \prec (1 - \lambda\alpha_1) \left( \frac{H_m^l[\alpha_1]f(z)}{z} \right)^\delta + \lambda\alpha_1 \left( \frac{H_m^l[\alpha_1]f(z)}{z} \right)^\delta \left( \frac{H_m^l[\alpha_1 + 1]f(z)}{H_m^l[\alpha_1]f(z)} \right) \prec q_2(z) + \frac{\lambda}{\delta} z q_2'(z),$$

then

$$q_1(z) \prec \left( \frac{H_m^l[\alpha_1]f(z)}{z} \right)^\delta \prec q_2(z)$$

and  $q_1, q_2$  are respectively the best subordinant and best dominant.



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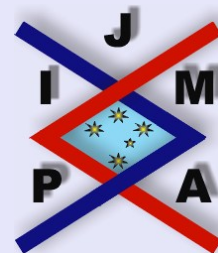
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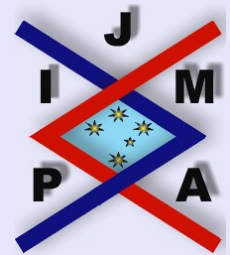
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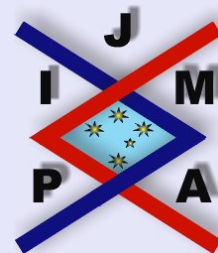
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