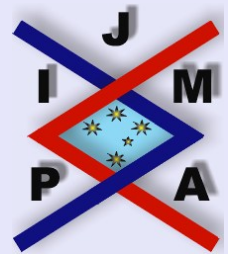


NEW PERTURBED ITERATIONS FOR A GENERALIZED CLASS OF STRONGLY NONLINEAR OPERATOR INCLUSION PROBLEMS IN BANACH SPACES

HENG-YOU LAN, HUANG-LIN ZENG AND ZUO-AN LI

Department of Mathematics
Sichuan University of Science and Engineering
Zigong, Sichuan 643000, P.R. China.

*E*Mail: hengyoulan@163.com



volume 7, issue 3, article 91,
2006.

Received 05 March, 2006;
accepted 19 March, 2006.

Communicated by: R.U. Verma

Abstract

Contents



Home Page

Go Back

Close

Quit

Abstract

The purpose of this paper is to introduce and study a new kind of generalized strongly nonlinear operator inclusion problems involving generalized m -accretive mapping in Banach spaces. By using the resolvent operator technique for generalized m -accretive mapping due to Huang and Fang, we also prove the existence theorem of the solution for this kind of operator inclusion problems and construct a new class of perturbed iterative algorithm with mixed errors for solving this kind of generalized strongly nonlinear operator inclusion problems in Banach spaces. Further, we discuss the convergence and stability of the iterative sequence generated by the perturbed algorithm. Our results improve and generalize the corresponding results of [3, 6, 11, 12].

2000 Mathematics Subject Classification: 68Q25, 49J40, 47H19, 47H12.

Key words: Generalized m -accretive mapping; Generalized strongly nonlinear operator inclusion problems; Perturbed iterative algorithm with errors; Existence; Convergence and stability.

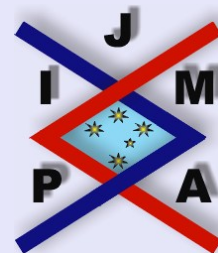
This work was supported by the Educational Science Foundation of Sichuan, Sichuan of China No. 2004C018, 2005A140.

The authors thank Prof. Ram U. Verma and Y. J. Cho for their valuable suggestions.

Contents

1	Introduction	3
2	Generalized m -Accretive Mapping	7
3	Existence Theorems	11
4	Perturbed Algorithm and Stability	15

References



New Perturbed Iterations for a Generalized Class of Strongly Nonlinear Operator Inclusion Problems in Banach Spaces

Heng-you Lan, Huang-Lin Zeng
and Zuo-An Li

Title Page

Contents



Go Back

Close

Quit

Page 2 of 22

1. Introduction

Let X be a real Banach space and $T : X \rightarrow 2^X$ is a multi-valued operator, where 2^X denotes the family of all the nonempty subsets of X . The following operator inclusion problem of finding $x \in X$ such that

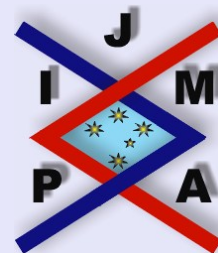
$$(1.1) \quad 0 \in T(u)$$

has been studied extensively because of its role in the modelization of unilateral problems, nonlinear dissipative systems, convex optimizations, equilibrium problems, knowledge engineering, etc. For details, we can refer to [1] – [6], [8] – [15] and the references therein. Concerning the development of iterative algorithms for the problem (1.1) in the literature, a very common assumption is that T is a maximal monotone operator or m -accretive operator. When T is maximal monotone or m -accretive, many iterative algorithms have been constructed to approximate the solutions of the problem (1.1).

In many practical cases, T is split in the form $T = F + M$, where $F, M : X \rightarrow 2^X$ are two multi-valued operators. So the problem (1.1) reduces to the following: Find $x \in X$ such that

$$(1.2) \quad 0 \in F(x) + M(x),$$

which is called the variational inclusion problem. When both F and M are maximal monotone or M is m -accretive, some approximate solutions for the problem (1.2) have been developed (see [10, 13] and the references therein). If $M = \partial\varphi$, where $\partial\varphi$ is the subdifferential of a proper convex lower semi-continuous functional $\varphi : X \rightarrow \mathbb{R} \cup \{+\infty\}$, then the problem (1.2) reduces to the variational inequality problem:



New Perturbed Iterations for a Generalized Class of Strongly Nonlinear Operator Inclusion Problems in Banach Spaces

Heng-you Lan, Huang-Lin Zeng
and Zuo-An Li

Title Page

Contents



Go Back

Close

Quit

Page 3 of 22

Find $x \in X$ and $u \in F(x)$ such that

$$(1.3) \quad \langle u, y - x \rangle + \varphi(y) - \varphi(x) \geq 0, \quad y \in X.$$

Many iterative algorithms have been established to approximate the solution of the problem (1.3) when F is strongly monotone. Recently, the problem (1.2) was studied by several authors when F and M need not to be maximal monotone or m -accretive. Further, Ding [3], Huang [6], and Lan et al. [11] developed some iterative algorithms to solve the following quasi-variational inequality problem of finding $x \in X$ and $u \in F(x)$, $v \in V(x)$ such that

$$(1.4) \quad \langle u, y - x \rangle + \varphi(y, v) - \varphi(x, v) \geq 0, \quad \forall y \in X$$

by introducing the concept of subdifferential $\partial\varphi(\cdot, t)$ of a proper functional $\varphi(\cdot, \cdot)$ for $t \in X$, which is defined by

$$\partial\varphi(\cdot, t) = \{f \in X : \varphi(y, t) - \varphi(x, t) \geq \langle f, y - x \rangle, \quad y \in X\},$$

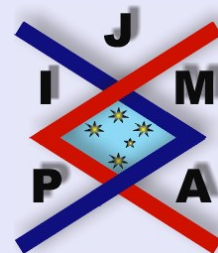
where $\varphi(\cdot, t) : X \rightarrow \mathbb{R} \cup \{+\infty\}$ is a proper convex lower semi-continuous functional for all $t \in X$.

It is easy to see that the problem (1.4) is equivalent to the following:

Find $x \in X$ such that

$$(1.5) \quad 0 \in F(x) + \partial\varphi(x, V(x)).$$

Recently, Huang and Fang [7] first introduced the concept of a generalized m -accretive mapping, which is a generalization of an m -accretive mapping, and gave the definition and properties of the resolvent operator for the generalized



New Perturbed Iterations for a Generalized Class of Strongly Nonlinear Operator Inclusion Problems in Banach Spaces

Heng-you Lan, Huang-Lin Zeng
and Zuo-An Li

Title Page

Contents



Go Back

Close

Quit

Page 4 of 22

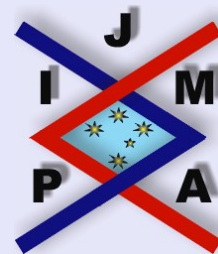
m -accretive mapping in a Banach space. Later, by using the resolvent operator technique, which is a very important method for finding solutions of variational inequality and variational inclusion problems, a number of nonlinear variational inclusions and many systems of variational inequalities, variational inclusions, complementarity problems and equilibrium problems. Bi, Huang, Jin and other authors introduced and studied some new classes of nonlinear variational inclusions involving generalized m -accretive mappings in Banach spaces, they also obtained some new corresponding existence and convergence results (see, [2, 5, 8] and the references therein). On the other hand, Huang, Lan, Zeng, Wang et al. discussed the stability of the iterative sequence generated by the algorithm for solving what they studied (see [6, 11, 15, 19]).

Motivated and inspired by the above works, in this paper, we introduce and study the following new class of generalized strongly nonlinear operator inclusion problems involving generalized m -accretive mappings:

Find $x \in X$ such that $(p(x), g(x)) \in \text{Dom } M$ and

$$(1.6) \quad f \in N(S(x), T(x), U(x)) + M(p(x), g(x)),$$

where f is an any given element on X , a real Banach space, $S, T, U, p, g : X \rightarrow X$ and $N : X \times X \times X \rightarrow X$ are single-valued mappings and $M : X \times X \rightarrow 2^X$ is a generalized m -accretive mapping with respect to the first argument, 2^X denotes the family of all the nonempty subsets of X . By using the resolvent operator technique for generalized m -accretive mappings due to Huang and Fang [7, 8], we prove the existence theorems of the solution for these types of operator inclusion problems in Banach spaces, and discuss the convergence and stability of a new perturbed iterative algorithm for solving this



New Perturbed Iterations for a Generalized Class of Strongly Nonlinear Operator Inclusion Problems in Banach Spaces

Heng-you Lan, Huang-Lin Zeng
and Zuo-An Li

Title Page

Contents



Go Back

Close

Quit

Page 5 of 22

class of nonlinear operator inclusion problems in Banach spaces. Our results improve and generalize the corresponding results of [3, 6, 11, 12].

We remark that for a suitable choice of f , the mappings $N, \eta, S, T, U, M, p, g$ and the space X , a number of known or new classes of variational inequalities, variational inclusions and corresponding optimization problems can be obtained as special cases of the nonlinear quasi-variational inclusion problem (1.6). Moreover, these classes of variational inclusions provide us with a general and unified framework for studying a wide range of interesting and important problems arising in mechanics, optimization and control, equilibrium theory of transportation and economics, management sciences, and other branches of mathematical and engineering sciences, etc. See for more details [1, 3, 4, 6, 9, 11, 15, 17, 18] and the references therein.



New Perturbed Iterations for a Generalized Class of Strongly Nonlinear Operator Inclusion Problems in Banach Spaces

Heng-you Lan, Huang-Lin Zeng
and Zuo-An Li

Title Page

Contents



Go Back

Close

Quit

Page 6 of 22

2. Generalized m -Accretive Mapping

Throughout this paper, let X be a real Banach space with dual space X^* , $\langle \cdot, \cdot \rangle$ the dual pair between X and X^* , and 2^X denote the family of all the nonempty subsets of X . The generalized duality mapping $J_q : X \rightarrow 2^{X^*}$ is defined by

$$J_q(x) = \{x^* \in X^* : \langle x, x^* \rangle = \|x\|^q, \|x^*\| = \|x\|^{q-1}\}, \quad \forall x \in X,$$

where $q > 1$ is a constant. In particular, J_2 is the usual normalized duality mapping. It is well known that, in general, $J_q(x) = \|x\|^{q-2}J_2(x)$ for all $x \neq 0$ and J_q is single-valued if X^* is strictly convex (see, for example, [16]). If $X = H$ is a Hilbert space, then J_2 becomes the identity mapping of H . In what follows we shall denote the single-valued generalized duality mapping by j_q .

Definition 2.1. *The mapping $g : X \rightarrow X$ is said to be*

1. α -strongly accretive, if for any $x, y \in X$, there exists $j_q(x - y) \in J_q(x - y)$ such that

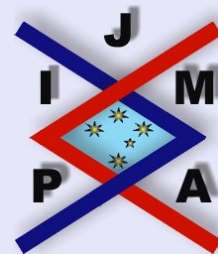
$$\langle g(x) - g(y), j_q(x - y) \rangle \geq \alpha \|x - y\|^q,$$

where $\alpha > 0$ is a constant;

2. β -Lipschitz continuous, if there exists a constant $\beta > 0$ such that

$$\|g(x) - g(y)\| \leq \beta \|x - y\|, \quad \forall x, y \in X.$$

Definition 2.2. *Let $h, g : X \rightarrow X$ be two single-valued mappings. The mapping $N : X \times X \times X \rightarrow X$ is said to be*



New Perturbed Iterations for a Generalized Class of Strongly Nonlinear Operator Inclusion Problems in Banach Spaces

Heng-you Lan, Huang-Lin Zeng
and Zuo-An Li

Title Page

Contents



Go Back

Close

Quit

Page 7 of 22

1. σ -strongly accretive with respect to h in the first argument, if for any $x, y \in X$, there exists $j_q(x - y) \in J_q(x - y)$ such that

$$\langle N(h(x), \cdot, \cdot) - N(h(y), \cdot, \cdot), j_q(x - y) \rangle \geq \sigma \|x - y\|^q,$$

where $\sigma > 0$ is a constant;

2. ς -relaxed accretive with respect to g in the second argument, if for any $x, y \in X$, there exists $j_q(x - y) \in J_q(x - y)$ such that

$$\langle N(\cdot, g(x), \cdot) - N(\cdot, g(y), \cdot), j_q(x - y) \rangle \geq -\varsigma \|x - y\|^q$$

where $\varsigma > 0$ is a constant;

3. ϵ -Lipschitz continuous with respect to the first argument, if there exists a constant $\epsilon > 0$ such that

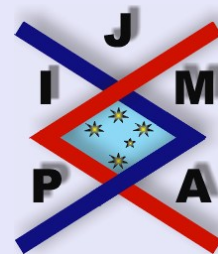
$$\|N(x, \cdot, \cdot) - N(y, \cdot, \cdot)\| \leq \epsilon \|x - y\|, \quad \forall x, y \in X.$$

Similarly, we can define the ξ , γ -Lipschitz continuity in the second and third argument of $N(\cdot, \cdot, \cdot)$, respectively.

Definition 2.3 ([7]). Let $\eta : X \times X \rightarrow X^*$ be a single-valued mapping and $A : X \rightarrow 2^X$ be a multi-valued mapping. Then A is said to be

1. η -accretive if

$$\langle u - v, \eta(x, y) \rangle \geq 0, \quad \forall x, y \in X, u \in A(x), v \in A(y);$$



New Perturbed Iterations for a Generalized Class of Strongly Nonlinear Operator Inclusion Problems in Banach Spaces

Heng-you Lan, Huang-Lin Zeng
and Zuo-An Li

Title Page

Contents



Go Back

Close

Quit

Page 8 of 22

2. *generalized m -accretive if A is η -accretive and $(I + \lambda A)(X) = X$ for all (equivalently, for some) $\lambda > 0$.*

Remark 1. *Huang and Fang gave one example of the generalized m -accretive mapping in [7]. If $X = X^* = H$ is a Hilbert space, then (1), (2) of Definition 2.3 reduce to the definition of η -monotonicity and maximal η -monotonicity respectively; if X is uniformly smooth and $\eta(x, y) = J_2(x - y)$, then (1) and (2) of Definition 2.3 reduce to the definitions of accretivity and m -accretivity in uniformly smooth Banach spaces, respectively (see [7, 8]).*

Definition 2.4. *The mapping $\eta : X \times X \rightarrow X^*$ is said to be*

1. *δ -strongly monotone, if there exists a constant $\delta > 0$ such that*

$$\langle x - y, \eta(x, y) \rangle \geq \delta \|x - y\|^2, \quad \forall x, y \in X;$$

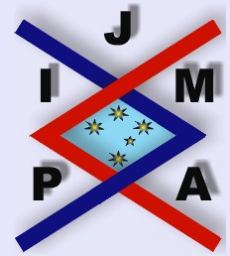
2. *τ -Lipschitz continuous, if there exists a constant $\tau > 0$ such that*

$$\|\eta(x, y)\| \leq \tau \|x - y\|, \quad \forall x, y \in X.$$

The modulus of smoothness of X is the function $\rho_X : [0, \infty) \rightarrow [0, \infty)$ defined by

$$\rho_X(t) = \sup \left\{ \frac{1}{2} \|x + y\| + \|x - y\| - 1 : \|x\| \leq 1, \|y\| \leq t \right\}.$$

A Banach space X is called uniformly smooth if $\lim_{t \rightarrow 0} \frac{\rho_X(t)}{t} = 0$ and X is called q -uniformly smooth if there exists a constant $c > 0$ such that $\rho_X \leq ct^q$, where $q > 1$ is a real number.



New Perturbed Iterations for a Generalized Class of Strongly Nonlinear Operator Inclusion Problems in Banach Spaces

Heng-you Lan, Huang-Lin Zeng and Zuo-An Li

Title Page

Contents



Go Back

Close

Quit

Page 9 of 22

It is well known that Hilbert spaces, L_p (or l_p) spaces, $1 < p < \infty$, and the Sobolev spaces $W^{m,p}$, $1 < p < \infty$, are all q -uniformly smooth. In the study of characteristic inequalities in q -uniformly smooth Banach spaces, Xu [16] proved the following result:

Lemma 2.1. *Let $q > 1$ be a given real number and X be a real uniformly smooth Banach space. Then X is q -uniformly smooth if and only if there exists a constant $c_q > 0$ such that for all $x, y \in X$, $j_q(x) \in J_q(x)$, there holds the following inequality*

$$\|x + y\|^q \leq \|x\|^q + q\langle y, j_q(x) \rangle + c_q\|y\|^q.$$

In [7], Huang and Fang show that for any $\rho > 0$, inverse mapping $(I + \rho A)^{-1}$ is single-valued, if $\eta : X \times X \rightarrow X^*$ is strict monotone and $A : X \rightarrow 2^X$ is a generalized m -accretive mapping, where I is the identity mapping. Based on this fact, Huang and Fang [7] gave the following definition:

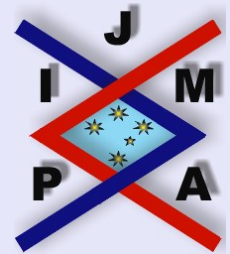
Definition 2.5. *Let $A : X \rightarrow 2^X$ be a generalized m -accretive mapping. Then the resolvent operator J_A^ρ for A is defined as follows:*

$$J_A^\rho(z) = (I + \rho A)^{-1}(z), \quad \forall z \in X,$$

where $\rho > 0$ is a constant and $\eta : X \times X \rightarrow X^*$ is a strictly monotone mapping.

Lemma 2.2 ([7, 8]). *Let $\eta : X \times X \rightarrow X^*$ be τ -Lipschitz continuous and δ -strongly monotone. Let $A : X \rightarrow 2^X$ be a generalized m -accretive mapping. Then the resolvent operator J_A^ρ for A is Lipschitz continuous with constant $\frac{\tau}{\delta}$, i.e.,*

$$\|J_A^\rho(x) - J_A^\rho(y)\| \leq \frac{\tau}{\delta}\|x - y\|, \quad \forall x, y \in X.$$



New Perturbed Iterations for a Generalized Class of Strongly Nonlinear Operator Inclusion Problems in Banach Spaces

Heng-you Lan, Huang-Lin Zeng
and Zuo-An Li

Title Page

Contents



Go Back

Close

Quit

Page 10 of 22

3. Existence Theorems

In this section, we shall give the existence theorem of problem (1.6). Firstly, from the definition of the resolvent operator for a generalized m -accretive mapping, we have the following lemma:

Lemma 3.1. x is the solution of problem (1.6) if and only if

$$(3.1) \quad p(x) = J_{M(\cdot, g(x))}^\rho [p(x) - \rho(N(S(x), T(x), U(x)) - f)],$$

where $J_{M(\cdot, g(x))}^\rho = (I + \rho M(\cdot, g(x)))^{-1}$ and $\rho > 0$ is a constant.

Theorem 3.2. Let X be a q -uniformly smooth Banach space, $\eta : X \times X \rightarrow X^*$ be τ -Lipschitz continuous and δ -strongly monotone, $M : X \times X \rightarrow 2^X$ be a generalized m -accretive mapping with respect to the first argument, and mappings $S, T, U : X \rightarrow X$ be κ, μ, ν -Lipschitz continuous, respectively. Let $p : X \rightarrow X$ be α -strongly accretive and β -Lipschitz continuous, $g : X \rightarrow X$ be ι -Lipschitz continuous, $N : X \times X \times X \rightarrow X$ be σ -strongly accretive with respect to S in the first argument and ς -relaxed accretive with respect to T in the second argument, and ϵ, ξ, γ -Lipschitz continuous in the first, second and third argument, respectively. Suppose that there exist constants $\rho > 0$ and $\zeta > 0$ such that for each $x, y, z \in X$,

$$(3.2) \quad \left\| J_{M(\cdot, x)}^\rho(z) - J_{M(\cdot, x)}^\rho(z) \right\| \leq \zeta \|x - y\|$$

and

$$(3.3) \quad \begin{cases} h = \zeta \iota + \left(1 + \frac{\tau}{\delta}\right) (1 - q\alpha + c_q \beta^q)^{\frac{1}{q}} < 1, \\ \tau \left[(1 - q\rho(\sigma - \varsigma) + c_q \rho^q (\epsilon\kappa + \xi\mu)^q)^{\frac{1}{q}} + \rho\gamma\nu \right] < \delta(1 - h), \end{cases}$$



New Perturbed Iterations for a Generalized Class of Strongly Nonlinear Operator Inclusion Problems in Banach Spaces

Heng-you Lan, Huang-Lin Zeng and Zuo-An Li

Title Page

Contents



Go Back

Close

Quit

Page 11 of 22

where c_q is the same as in Lemma 2.1, then problem (1.6) has a unique solution x^* .

Proof. From Lemma 3.1, problem (1.6) is equivalent to the fixed problem (3.1), equation (3.1) can be rewritten as follows:

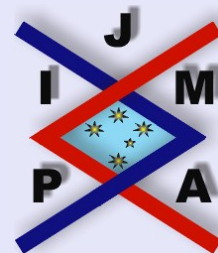
$$x = x - p(x) - J_{M(\cdot, g(x))}^\rho [p(x) - \rho(N(S(x), T(x), U(x)) - f)].$$

For every $x \in X$, take

$$(3.4) \quad Q(x) = x - p(x) - J_{M(\cdot, g(x))}^\rho [p(x) - \rho(N(S(x), T(x), U(x)) - f)].$$

Then x^* is the unique solution of problem (1.6) if and only if x^* is the unique fixed point of Q . In fact, it follows from (3.2), (3.4) and Lemma 2.2 that

$$\begin{aligned} & \|Q(x) - Q(y)\| \\ & \leq \|x - y - (p(x) - p(y))\| \\ & \quad + \left\| J_{M(\cdot, g(x))}^\rho [p(x) - \rho(N(S(x), T(x), U(x)) - f)] \right. \\ & \quad \left. - J_{M(\cdot, g(y))}^\rho [p(y) - \rho(N(S(y), T(y), U(y)) - f)] \right\| \\ & \leq \|x - y - (p(x) - p(y))\| \\ & \quad + \left\| J_{M(\cdot, g(x))}^\rho [p(x) - \rho(N(S(x), T(x), U(x)) - f)] \right. \\ & \quad \left. - J_{M(\cdot, g(x))}^\rho [p(y) - \rho(N(S(y), T(y), U(y)) - f)] \right\| \\ & \quad + \left\| J_{M(\cdot, g(x))}^\rho [p(y) - \rho(N(S(y), T(y), U(y)) - f)] \right. \\ & \quad \left. - J_{M(\cdot, g(y))}^\rho [p(y) - \rho(N(S(y), T(y), U(y)) - f)] \right\| \end{aligned}$$



New Perturbed Iterations for a Generalized Class of Strongly Nonlinear Operator Inclusion Problems in Banach Spaces

Heng-you Lan, Huang-Lin Zeng
and Zuo-An Li

Title Page

Contents

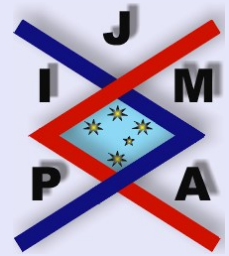


Go Back

Close

Quit

Page 12 of 22



$$\begin{aligned}
 &\leq \left(1 + \frac{\tau}{\delta}\right) \|x - y - (p(x) - p(y))\| \\
 &\quad + \frac{\tau}{\delta} \{ \|x - y - \rho[(N(S(x), T(x), U(x)) - N(S(y), T(x), U(x))) \\
 &\quad + (N(S(y), T(x), U(x)) - N(S(y), T(y), U(x)))]\| \\
 &\quad + \rho \|N(S(y), T(y), U(x)) - N(S(y), T(y), U(y))\| \} \\
 (3.5) \quad &+ \zeta \|g(x) - g(y)\|.
 \end{aligned}$$

By the hypothesis of g, p, S, T, U, N and Lemma 2.1, now we know there exists $c_q > 0$ such that

$$(3.6) \quad \|g(x) - g(y)\| \leq \iota \|x - y\|,$$

$$(3.7) \quad \|x - y - (p(x) - p(y))\|^q \leq (1 - q\alpha + c_q\beta^q) \|x - y\|^q,$$

$$(3.8) \quad \|N(S(y), T(y), U(x)) - N(S(y), T(y), U(y))\| \leq \gamma\nu \|x - y\|,$$

$$\begin{aligned}
 &\|x - y - \rho[(N(S(x), T(x), U(x)) - N(S(y), T(x), U(x))) \\
 &\quad + (N(S(y), T(x), U(x)) - N(S(y), T(y), U(x)))]\|^q \\
 &\leq \|x - y\|^q - q\rho \langle (N(S(x), T(x), U(x)) - N(S(y), T(x), U(x))) \\
 &\quad + (N(S(y), T(x), U(x)) - N(S(y), T(y), U(x))), j_q(x - y) \rangle \\
 &\quad + c_q\rho^q \| (N(S(x), T(x), U(x)) - N(S(y), T(x), U(x))) \\
 &\quad + (N(S(y), T(x), U(x)) - N(S(y), T(y), U(x))) \|^q \\
 &\leq \|x - y\|^q \\
 &\quad - q\rho \langle N(S(x), T(x), U(x)) - N(S(y), T(x), U(x)), j_q(x - y) \rangle \\
 &\quad + \langle N(S(y), T(x), U(x)) - N(S(y), T(y), U(x)), j_q(x - y) \rangle
 \end{aligned}$$

New Perturbed Iterations for a Generalized Class of Strongly Nonlinear Operator Inclusion Problems in Banach Spaces

Heng-you Lan, Huang-Lin Zeng and Zuo-An Li

Title Page

Contents

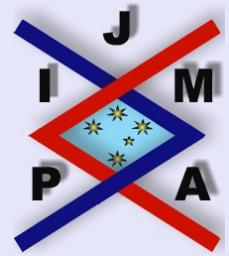


Go Back

Close

Quit

Page 13 of 22



Title Page

Contents



Go Back

Close

Quit

Page 14 of 22

$$\begin{aligned}
 & + c_q \rho^q \| \|N(S(x), T(x), U(x)) - N(S(y), T(x), U(x))\| \\
 & + \|N(S(y), T(x), U(x)) - N(S(y), T(y), U(x))\| \|^q \\
 (3.9) \quad & \leq [1 - q\rho(\sigma - \varsigma) + c_q \rho^q (\epsilon\kappa + \xi\mu)^q] \|x - y\|^q.
 \end{aligned}$$

Combining (3.5) – (3.9), we get

$$(3.10) \quad \|Q(x) - Q(y)\| \leq \theta \|x - y\|,$$

where

$$\begin{aligned}
 (3.11) \quad \theta & = h + \frac{\tau}{\delta} \left[(1 - q\rho(\sigma - \varsigma) + c_q \rho^q (\epsilon\kappa + \xi\mu)^q)^{\frac{1}{q}} + \rho\gamma\nu \right], \\
 h & = \zeta\iota + \left(1 + \frac{\tau}{\delta}\right) (1 - q\alpha + c_q \beta^q)^{\frac{1}{q}}.
 \end{aligned}$$

It follows from (3.3) that $0 < \theta < 1$ and so $Q : X \rightarrow X$ is a contractive mapping, i.e., Q has a unique fixed point in X . This completes the proof. \square

Remark 2. If X is a 2-uniformly smooth Banach space and there exists $\rho > 0$ such that

$$\left\{ \begin{array}{l}
 h = \zeta\iota + \left(1 + \frac{\tau}{\delta}\right) \sqrt{1 - 2\alpha + c_2\beta^2} < 1, \\
 0 < \rho < \frac{\delta(1-h)}{\tau\gamma\nu}, \quad \gamma\nu < \sqrt{c_2}(\epsilon\kappa + \xi\mu), \\
 \tau(\sigma - \varsigma) > \delta\gamma\nu(1-h) + \sqrt{[c_2(\epsilon\kappa + \xi\mu)^2 - \gamma^2\nu^2][\tau^2 - \delta^2(1-h)^2]}, \\
 \left| \rho - \frac{\tau(\sigma - \varsigma) + \delta\gamma\nu(h-1)}{\tau[c_2(\epsilon\kappa + \xi\mu)^2 - \gamma^2\nu^2]} \right| < \frac{[\tau(\sigma - \varsigma) - \delta\gamma\nu(1-h)]^2 - [c_2(\epsilon\kappa + \xi\mu)^2 - \gamma^2\nu^2][\tau^2 - \delta^2(1-h)^2]}{\tau[c_2(\epsilon\kappa + \xi\mu)^2 - \gamma^2\nu^2]},
 \end{array} \right.$$

then (3.3) holds. We note that the Hilbert space and L_p (or l_p) ($2 \leq p < \infty$) spaces are 2-uniformly Banach spaces.

4. Perturbed Algorithm and Stability

In this section, by using the following definition and lemma, we construct a new perturbed iterative algorithm with mixed errors for solving problem (1.6) and prove the convergence and stability of the iterative sequence generated by the algorithm.

Definition 4.1. Let S be a selfmap of X , $x_0 \in X$, and let $x_{n+1} = h(S, x_n)$ define an iteration procedure which yields a sequence of points $\{x_n\}_{n=0}^{\infty}$ in X . Suppose that $\{x \in X : Sx = x\} \neq \emptyset$ and $\{x_n\}_{n=0}^{\infty}$ converges to a fixed point x^* of S . Let $\{u_n\} \subset X$ and let $\epsilon_n = \|u_{n+1} - h(S, u_n)\|$. If $\lim \epsilon_n = 0$ implies that $u_n \rightarrow x^*$, then the iteration procedure defined by $x_{n+1} = h(S, x_n)$ is said to be S -stable or stable with respect to S .

Lemma 4.1 ([12]). Let $\{a_n\}, \{b_n\}, \{c_n\}$ be three nonnegative real sequences satisfying the following condition:

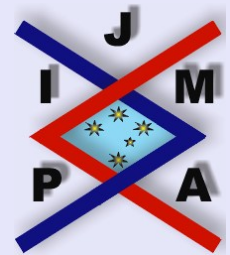
there exists a natural number n_0 such that

$$a_{n+1} \leq (1 - t_n)a_n + b_n t_n + c_n, \quad \forall n \geq n_0,$$

where $t_n \in [0, 1]$, $\sum_{n=0}^{\infty} t_n = \infty$, $\lim_{n \rightarrow \infty} b_n = 0$, $\sum_{n=0}^{\infty} c_n < \infty$. Then $a_n \rightarrow 0$ ($n \rightarrow \infty$).

The relation (3.1) allows us to construct the following perturbed iterative algorithm with mixed errors.

Algorithm 1. Step 1. Choose $x_0 \in X$.



New Perturbed Iterations for a Generalized Class of Strongly Nonlinear Operator Inclusion Problems in Banach Spaces

Heng-you Lan, Huang-Lin Zeng
and Zuo-An Li

Title Page

Contents

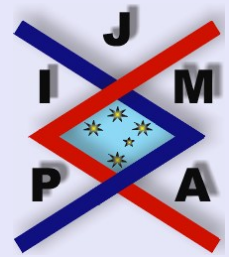


Go Back

Close

Quit

Page 15 of 22



Title Page

Contents



Go Back

Close

Quit

Page 16 of 22

Step 2. Let

$$(4.1) \begin{cases} x_{n+1} = (1 - \alpha_n)x_n + \alpha_n[y_n - p(y_n) \\ \quad + J_{M(\cdot, g(y_n))}^\rho(p(y_n) - \rho(N(S(y_n), T(y_n), U(y_n)) - f))] \\ \quad + \alpha_n u_n + \omega_n, \\ y_n = (1 - \beta_n)x_n + \beta_n[x_n - p(x_n) \\ \quad + J_{M(\cdot, g(x_n))}^\rho(p(x_n) - \rho(N(S(x_n), T(x_n), U(x_n)) - f))] + v_n, \end{cases}$$

Step 3. Choose sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{u_n\}$, $\{v_n\}$ and $\{\omega_n\}$ such that for $n \geq 0$, $\{\alpha_n\}$, $\{\beta_n\}$ are two sequences in $[0, 1]$, $\{u_n\}$, $\{v_n\}$, $\{\omega_n\}$ are sequences in X satisfying the following conditions:

- (i) $u_n = u'_n + u''_n$;
- (ii) $\lim_{n \rightarrow \infty} \|u'_n\| = \lim_{n \rightarrow \infty} \|v_n\| = 0$;
- (iii) $\sum_{n=0}^{\infty} \|u''_n\| < \infty$, $\sum_{n=0}^{\infty} \|\omega_n\| < \infty$,

Step 4. If x_{n+1} , y_n , α_n , β_n , u_n , v_n and ω_n satisfy (4.1) to sufficient accuracy, go to Step 5; otherwise, set $n := n + 1$ and return to Step 2.

Step 5. Let $\{z_n\}$ be any sequence in X and define $\{\varepsilon_n\}$ by

$$(4.2) \begin{cases} \varepsilon_n = \|z_{n+1} - \{(1 - \alpha_n)z_n + \alpha_n[t_n - p(t_n) \\ \quad + J_{M(\cdot, g(t_n))}^\rho(p(t_n) - \rho(N(S(t_n), T(t_n), U(t_n)) - f))] \\ \quad + \alpha_n u_n + \omega_n]\|, \\ t_n = (1 - \beta_n)z_n + \beta_n[z_n - p(z_n) \\ \quad + J_{M(\cdot, g(z_n))}^\rho(p(z_n) - \rho(N(S(z_n), T(z_n), U(z_n)) - f))] + v_n. \end{cases}$$

Step 6. If $\varepsilon_n, z_{n+1}, t_n, \alpha_n, \beta_n, u_n, v_n$ and ω_n satisfy (4.2) to sufficient accuracy, stop; otherwise, set $n := n + 1$ and return to Step 3.

Theorem 4.2. Suppose that $X, S, T, U, p, g, N, \eta$ and M are the same as in Theorem 3.2, θ is defined by (3.11). If $\sum_{n=0}^{\infty} \alpha_n = \infty$ and conditions (3.2), (3.3) hold, then the perturbed iterative sequence $\{x_n\}$ defined by (4.1) converges strongly to the unique solution of problem (1.6). Moreover, if there exists $a \in (0, \alpha_n]$ for all $n \geq 0$, then $\lim_{n \rightarrow \infty} z_n = x^*$ if and only if $\lim_{n \rightarrow \infty} \varepsilon_n = 0$, where ε_n is defined by (4.2).

Proof. From Theorem 3.2, we know that problem (1.6) has a unique solution $x^* \in X$. It follows from (4.1), (3.11) and the proof of (3.10) in Theorem 3.2 that

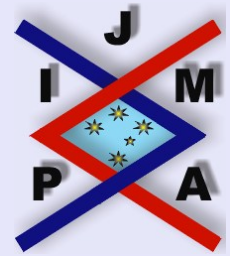
$$\begin{aligned}
 & \|x_{n+1} - x^*\| \\
 & \leq (1 - \alpha_n)\|x_n - x^*\| + \alpha_n\theta\|y_n - x^*\| + \alpha_n(\|u'_n\| + \|u''_n\|) + \|\omega_n\| \\
 (4.3) \quad & \leq (1 - \alpha_n)\|x_n - x^*\| + \alpha_n\theta\|y_n - x^*\| + \alpha_n\|u'_n\| + (\|u''_n\| + \|\omega_n\|).
 \end{aligned}$$

Similarly, we have

$$(4.4) \quad \|y_n - x^*\| \leq (1 - \beta_n + \beta_n\theta)\|x_n - x^*\| + \|v_n\|.$$

Combining (4.3) – (4.4), we obtain

$$\begin{aligned}
 (4.5) \quad \|x_{n+1} - x^*\| & \leq [1 - \alpha_n(1 - \theta(1 - \beta_n + \beta_n\theta))]\|x_n - x^*\| \\
 & \quad + \alpha_n(\|u'_n\| + \theta\|v_n\|) + (\|u''_n\| + \|\omega_n\|).
 \end{aligned}$$



New Perturbed Iterations for a Generalized Class of Strongly Nonlinear Operator Inclusion Problems in Banach Spaces

Heng-you Lan, Huang-Lin Zeng
and Zuo-An Li

Title Page

Contents



Go Back

Close

Quit

Page 17 of 22

Since $\theta < 1$, $0 < \beta_n \leq 1$ ($n \geq 0$), we have $1 - \beta_n + \beta_n\theta < 1$ and $1 - \theta(1 - \beta_n + \beta_n\theta) > 1 - \theta > 0$. Therefore, (4.5) implies

$$(4.6) \quad \|x_{n+1} - x^*\| \leq [1 - \alpha_n(1 - \theta)]\|x_n - x^*\| + \alpha_n(1 - \theta) \cdot \frac{1}{1 - \theta} (\|u'_n\| + \theta\|v_n\|) + (\|u''_n\| + \|\omega_n\|).$$

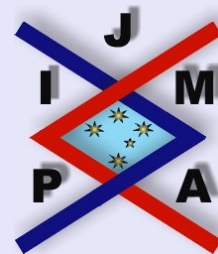
Since $\sum_{n=0}^{\infty} \alpha_n = \infty$, it follows from Lemma 4.1 and (4.6) that $\|x_n - x^*\| \rightarrow 0$ ($n \rightarrow \infty$), i.e., $\{x_n\}$ converges strongly to the unique solution x^* of the problem (1.6).

Now we prove the second conclusion. By (4.2), we know

$$(4.7) \quad \|z_{n+1} - x^*\| \leq \|(1 - \alpha_n)z_n + \alpha_n[t_n - p(t_n) + J_{M(\cdot, g(t_n))}^p(p(t_n) - \rho(N(S(t_n), T(t_n), U(t_n)) - f)) + \alpha_n u_n + \omega_n - x^*\| + \varepsilon_n.$$

As the proof of inequality (4.6), we have

$$(4.8) \quad \|(1 - \alpha_n)z_n + \alpha_n[t_n - p(t_n) + J_{M(\cdot, g(t_n))}^p(p(t_n) - \rho(N(S(t_n), T(t_n), U(t_n)) - f)) + \alpha_n u_n + \omega_n - x^*\| \leq [1 - \alpha_n(1 - \theta)]\|z_n - x^*\| + \alpha_n(1 - \theta) \cdot \frac{1}{1 - \theta} (\|u'_n\| + \theta\|v_n\|) + (\|u''_n\| + \|\omega_n\|).$$



New Perturbed Iterations for a Generalized Class of Strongly Nonlinear Operator Inclusion Problems in Banach Spaces

Heng-you Lan, Huang-Lin Zeng
and Zuo-An Li

Title Page

Contents



Go Back

Close

Quit

Page 18 of 22

Since $0 < a \leq \alpha_n$, it follows from (4.7) and (4.8) that

$$\begin{aligned} \|z_{n+1} - x^*\| &\leq [1 - \alpha_n(1 - \theta)]\|z_n - x^*\| \\ &\quad + \alpha_n(1 - \theta) \cdot \frac{1}{1 - \theta} (\|u'_n\| + \theta\|v_n\|) + (\|u''_n\| + \|\omega_n\|) + \varepsilon_n \\ &\leq [1 - \alpha_n(1 - \theta)]\|z_n - x^*\| \\ &\quad + \alpha_n(1 - \theta) \cdot \frac{1}{1 - \theta} \left(\|u'_n\| + \theta\|v_n\| + \frac{\varepsilon_n}{a} \right) + (\|u''_n\| + \|\omega_n\|). \end{aligned}$$

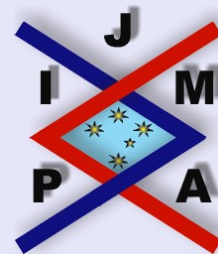
Suppose that $\lim \varepsilon_n = 0$. Then from $\sum_{n=0}^{\infty} \alpha_n = \infty$ and Lemma 4.1, we have $\lim z_n = x^*$.

Conversely, if $\lim z_n = x^*$, then we get

$$\begin{aligned} \varepsilon_n &= \|z_{n+1} - \{(1 - \alpha_n)z_n + \alpha_n[t_n - p(t_n) \\ &\quad + J_{M(\cdot, g(t_n))}^{\rho}(p(t_n) - \rho(N(S(t_n), T(t_n), U(t_n)) - f))] + \alpha_n u_n + \omega_n\}\| \\ &\leq \|z_{n+1} - x^*\| + \|(1 - \alpha_n)z_n + \alpha_n[t_n - p(t_n) \\ &\quad + J_{M(\cdot, g(t_n))}^{\rho}(p(t_n) - \rho(N(S(t_n), T(t_n), U(t_n)) - f))] + \alpha_n u_n + \omega_n - x^*\| \\ &\leq \|z_{n+1} - x^*\| + [1 - \alpha_n(1 - \theta)]\|z_n - x^*\| \\ &\quad + \alpha_n(\|u'_n\| + \theta\|v_n\|) + (\|u''_n\| + \|\omega_n\|) \rightarrow 0 \quad (n \rightarrow \infty). \end{aligned}$$

This completes the proof. \square

Remark 3. If $u_n = v_n = \omega_n = 0$ ($n \geq 0$) in Algorithm 1, then the conclusions of Theorem 4.2 also hold. The results of Theorems 3.2 and 4.2 improve and generalize the corresponding results of [3, 6, 11, 12].



New Perturbed Iterations for a Generalized Class of Strongly Nonlinear Operator Inclusion Problems in Banach Spaces

Heng-you Lan, Huang-Lin Zeng
and Zuo-An Li

Title Page

Contents



Go Back

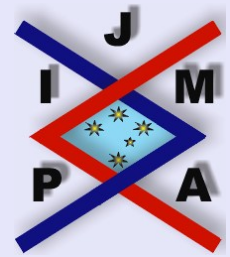
Close

Quit

Page 19 of 22

References

- [1] C. BAIOCCHI AND A. CAOPELO, *Variational and Quasivariational Inequalities, Application to Free Boundary Problems*, Wiley, New York, 1984.
- [2] Z.S. BI, Z. HAN AND Y.P. FANG, Sensitivity analysis for nonlinear variational inclusions involving generalized m -accretive mappings, *J. Sichuan Univ.*, **40**(2) (2003), 240–243.
- [3] X.P. DING, Existence and algorithm of solutions for generalized mixed implicit quasi-variational inequalities, *Appl. Math. Comput.* **113** (2000), 67–80.
- [4] F. GIANNESI AND A. MAUGERI, *Variational Inequalities and Network Equilibrium Problems*, Plenum, New York, 1995.
- [5] N.J. HUANG, Nonlinear implicit quasi-variational inclusions involving generalized m -accretive mappings, *Arch. Inequal. Appl.*, **2**(4) (2004), 413–425.
- [6] N.J. HUANG, M.R. BAI, Y.J. CHO AND S.M. KANG, Generalized nonlinear mixed quasi-variational inequalities, *Comput. Math. Appl.*, **40** (2000), 205–216.
- [7] N.J. HUANG, Y.P. FANG, Generalized m -accretive mappings in Banach spaces, *J. Sichuan Univ.*, **38**(4) (2001), 591–592.
- [8] N.J. HUANG, Y.P. FANG AND C.X. DENG, Nonlinear variational inclusions involving generalized m -accretive mappings, *Proceedings of the*



New Perturbed Iterations for a Generalized Class of Strongly Nonlinear Operator Inclusion Problems in Banach Spaces

Heng-you Lan, Huang-Lin Zeng
and Zuo-An Li

Title Page

Contents



Go Back

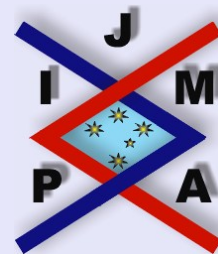
Close

Quit

Page 20 of 22

Bellman Continuum: International Workshop on Uncertain Systems and Soft Computing, Beijing, China, July 24-27, 2002, pp. 323–327.

- [9] M.M. JIN, Sensitivity analysis for strongly nonlinear quasi-variational inclusions involving generalized m -accretive mappings, *Nonlinear Anal. Forum*, **8**(1) (2003), 93–99.
- [10] J.S. JUNG AND C.H. MORALES, The Mann process for perturbed m -accretive operators in Banach spaces, *Nonlinear Anal.*, **46**(2) (2001), 231–243.
- [11] H.Y. LAN, J.K. KIM AND N.J. HUANG, On the generalized nonlinear quasi-variational inclusions involving non-monotone set-valued mappings, *Nonlinear Funct. Anal. & Appl.*, **9**(3) (2004), 451–465.
- [12] L.S. LIU, Ishikawa and Mann iterative process with errors for nonlinear strongly accretive mappings in Banach spaces, *J. Math. Anal. Appl.*, **194** (1995), 114–135.
- [13] A. MOUDAFI, On the regularization of the sum of two maximal monotone operators, *Nonlin. Anal. TMA*, **42** (2000), 1203–1208.
- [14] W.L. WANG, Z.Q. LIU, C. FENG AND S.H. KANG, Three-step iterative algorithm with errors for generalized strongly nonlinear quasi-variational inequalities, *Adv. Nonlinear Var. Inequal.*, **7**(2) (2004), 27–34.
- [15] M.V. SOLODOV AND B.F. SVAITER, A hybrid projection-proximal point algorithm, *J. Convex Anal.*, **6**(1) (1999), 59–70.



New Perturbed Iterations for a Generalized Class of Strongly Nonlinear Operator Inclusion Problems in Banach Spaces

Heng-you Lan, Huang-Lin Zeng
and Zuo-An Li

Title Page

Contents



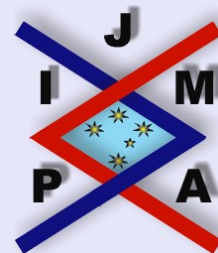
Go Back

Close

Quit

Page 21 of 22

- [16] H.K. XU, Inequalities in Banach spaces with applications, *Nonlinear Anal.*, **16**(12) (1991), 1127–1138.
- [17] G.X.Z. YUAN, *KKM Theory and Applications*, Marcel Dekker, 1999.
- [18] H.L. ZENG, A fast learning algorithm for solving systems of linear equations and related problems, *J. Adv. Modelling Anal.*, **29** (1995), 67–71.
- [19] H.L. ZENG, Stability analysis on continuous neural networks with slowly synapse-varying structures, *Proceedings of IEEE: International Symposium on Circuits and Systems*, United States of America, 1992, pp. 443–447.



New Perturbed Iterations for a Generalized Class of Strongly Nonlinear Operator Inclusion Problems in Banach Spaces

Heng-you Lan, Huang-Lin Zeng
and Zuo-An Li

Title Page

Contents



Go Back

Close

Quit

Page 22 of 22