PROPERTIES OF $q-$MEYER-KÖNIG-ZELLER DURRMEYER OPERATORS

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Abstract: We introduce a q analogue of the Meyer-König-Zeller Durrmeyer type operators and investigate their rate of convergence.

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Contents

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1. Introduction

Abel et al. [5] introduced the Meyer-Konig-Zeller Durrmeyer operators as

$$M_n(f;x) = \sum_{k=0}^{\infty} m_{n,k}(x) \int_0^1 b_{n,k}(t) f(t) dt, \qquad 0 \leq x < 1, \tag{1.1}$$

where

$$m_{n,k}(x) = \binom{n+k-1}{k} x^k (1-x)^n$$

and

$$b_{n,k}(t) = n \binom{n+k}{k} t^k (1-t)^{n-1}.$$

Very recently H. Wang [6], O. Dogru and V. Gupta [2], A. Altin, O. Dogru and M.A. Ozarslan [7] and T. Trif [3] studied the q-Meyer-Konig-Zeller operators. This motivated us to introduce the q analogue of the Meyer-Konig-Zeller Durrmeyer operators.

Before introducing the operators, we mention certain definitions based on $q-$integers; details can be found in [10] and [12].

For each non-negative integer k, the q-integer $[k]$ and the q-factorial $[k]!$ are respectively defined by

$$[k] := \begin{cases} (1-q^k)/(1-q), & q \neq 1 \\ k, & q = 1 \end{cases},$$

and

$$[k]! := \begin{cases} [k]\,[k-1]\cdots[1], & k \geq 1 \\ 1, & k = 0 \end{cases}.$$

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issn: 1443-5756

For the integers $n,\ k$ satisfying $n \geq k \geq 0$, the q-binomial coefficients are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix} := \frac{[n]!}{[k]![n-k]!} \ .$$

We use the following notations

$$(a+b)_q^n = \prod_{j=0}^{n-1}(a+q^j b) = (a+b)(a+qb)\cdots(a+q^{n-1}b)$$

and

$$(t;q)_0 = 1, \quad (t;q)_n = \prod_{j=0}^{n-1}(1-q^j t), \quad (t;q)_\infty = \prod_{j=0}^{\infty}(1-q^j t).$$

Also it can be seen that

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}.$$

The $q-$Beta function is defined as

$$B_q(m,n) = \int_0^1 t^{m-1}(1-qt)_q^{n-1} d_q t$$

for $m, n \in \mathbb{N}$ and we have

$$B_q(m,n) = \frac{[m-1]![n-1]!}{[m+n-1]!}. \tag{1.2}$$

It can be easily checked that

$$\prod_{j=0}^{n-1}(1-q^j x)\sum_{k=0}^{\infty}\begin{bmatrix} n+k-1 \\ k \end{bmatrix} x^k = 1. \tag{1.3}$$

$q-$Meyer-König-Zeller Durrmeyer Operators
Honey Sharma
vol. 10, iss. 4, art. 105, 2009

journal of inequalities in pure and applied mathematics
issn: 1443-5756

Now we introduce the q-Meyer-Konig-Zeller Durrmeyer operator as follows

$$M_{n,q}(f;x) = \sum_{k=0}^{\infty} m_{n,k,q}(x) \int_0^1 b_{n,k,q}(t) f(qt) d_q t, \quad 0 \leq x < 1 \tag{1.4}$$

$$:= \sum_{k=0}^{\infty} m_{n,k,q}(x) A_{n,k,q}(f), \tag{1.5}$$

where $0 < q < 1$ and

$$m_{n,k,q}(x) = P_{n-1}(x) \begin{bmatrix} n+k-1 \\ k \end{bmatrix} x^k, \tag{1.6}$$

$$b_{n,k,q}(t) = \frac{[n+k]!}{[k]![n-1]!} t^k (1-qt)_q^{n-1}. \tag{1.7}$$

Here

$$P_{n-1}(x) = \prod_{j=0}^{n-1} (1 - q^j x).$$

Remark 1. It can be seen that for $q \to 1^-$, the $q-$Meyer-Konig-Zeller Durrmeyer operator becomes the operator studied in [4] for $\alpha = 1$.

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issn: 1443-5756

2. Moments

Lemma 2.1. *For* $g_s(t) = t^s$, $s = 0, 1, 2, \ldots$, *we have*

$$\int_0^1 b_{n,k,q}(t) g_s(qt) d_q t = q^s \frac{[n+k]![k+s]!}{[k]![k+s+n]!}. \tag{2.1}$$

Proof. By using the $q-$Beta function (1.2), the above lemma can be proved easily. □

Here, we introduce two lemmas proved in [8], as follows:

Lemma 2.2. *For* $r = 0, 1, 2, \ldots$ *and* $n > r$, *we have*

$$P_{n-1}(x) \sum_{k=0}^{\infty} \begin{bmatrix} n+k-1 \\ k \end{bmatrix} \frac{x^k}{[n+k-1]_{\underline{r}}} = \frac{\prod_{j=1}^{r}(1-q^{n-j}x)}{[n-1]_{\underline{r}}}, \tag{2.2}$$

where $[n-1]_{\underline{r}} = [n-1][n-2]\cdots[n-r]$.

Lemma 2.3. *The identity*

$$\frac{1}{[n+k+r]} \leq \frac{1}{q^{r+1}[n+k-1]}, \qquad r \geq 0 \tag{2.3}$$

holds.

Theorem 2.4. *For all* $x \in [0,1]$, $n \in \mathbb{N}$ *and* $q \in (0,1)$, *we have*

$$M_{n,q}(e_0; x) = 1, \tag{2.4}$$

$$M_{n,q}(e_1; x) \leq x + \frac{(1-q^{n-1}x)}{q[n-1]}, \tag{2.5}$$

$$M_{n,q}(e_1; x) \geq \left(1 - \frac{(1+q^{n-2})}{[n+1]}\right) x + q^{n-2}(1-q)x^2, \tag{2.6}$$

journal of inequalities in pure and applied mathematics
issn: 1443-5756

$$M_{n,q}\left(e_2;x\right)\le x^2+\frac{(1+q)^2}{q^3}\frac{\left(1-q^{n-1}x\right)}{[n-1]}x \tag{2.7}$$
$$+\frac{(1+q)}{q^4}\frac{\left(1-q^{n-1}x\right)\left(1-q^{n-2}x\right)}{[n-1][n-2]}.$$

Proof. We have to estimate $M_{n,q}\left(e_s;x\right)$ for $s=0,1,2$. The result can be easily verified for $s=0$. Using the above lemmas and equation (1.3), we obtain relations (2.5) and (2.6) as follows

$$\begin{aligned} M_{n,q}(e_1,x) &= qP_{n-1}(x)\sum_{k=0}^{\infty}\begin{bmatrix} n+k-1 \\ k\end{bmatrix}\frac{[k+1]}{[n+k+1]}x^k \\ &\le qP_{n-1}(x)\sum_{k=0}^{\infty}\begin{bmatrix} n+k-1 \\ k\end{bmatrix}\frac{q[k]+1}{q^2[n+k-1]}x^k \\ &= xP_{n-1}(x)\sum_{k=0}^{\infty}\begin{bmatrix} n+k-1 \\ k\end{bmatrix}x^k \\ &\quad+\frac{P_{n-1}(x)}{q}\sum_{k=0}^{\infty}\begin{bmatrix} n+k-1 \\ k\end{bmatrix}\frac{x^k}{[n+k-1]} \\ &= x+\frac{\left(1-q^{n-1}x\right)}{q[n-1]}. \end{aligned}$$

Also,

$$\begin{aligned} M_{n,q}(e_1,x) &= qP_{n-1}(x)\sum_{k=1}^{\infty}\begin{bmatrix} n+k-2 \\ k-1\end{bmatrix}\frac{[k+1]}{[k]}\frac{[n+k-1]}{[n+k+1]}x^k \\ &\ge P_{n-1}(x)\sum_{k=0}^{\infty}\begin{bmatrix} n+k-1 \\ k\end{bmatrix}\left(\frac{[n+k+1]-1}{[n+k+2]}\right)x^{k+1} \end{aligned}$$

journal of inequalities in pure and applied mathematics
issn: 1443-5756

$$\geq P_{n-1}(x)\sum_{k=0}^{\infty}\begin{bmatrix} n+k-1 \\ k \end{bmatrix}\left(\frac{[n+k+1]}{[n+k+2]}-\frac{1}{[n+1]}\right)x^{k+1}$$

$$\geq P_{n-1}(x)\sum_{k=0}^{\infty}\begin{bmatrix} n+k-1 \\ k \end{bmatrix}\left(1-\frac{q^{n+k+1}}{[n+k+2]}\right)x^{k+1}-\frac{1}{[n+1]}x$$

$$\geq P_{n-1}(x)\sum_{k=0}^{\infty}\begin{bmatrix} n+k-1 \\ k \end{bmatrix}\left(1-\frac{q^{n-2}\left(1-(1-q)[k]\right)}{[n+k-1]}\right)x^{k+1}-\frac{1}{[n+1]}x$$

$$= x-\frac{q^{n-2}x}{[n+1]}+q^{n-2}(1-q)x^2P_{n-1}(x)\sum_{k=0}^{\infty}\begin{bmatrix} n+k-1 \\ k \end{bmatrix}x^k-\frac{1}{[n+1]}x$$

$$= \left(1-\frac{(1+q^{n-2})}{[n+1]}\right)x+q^{n-2}(1-q)x^2.$$

Similar calculations reveal the relation (2.7) as follows

$$M_{n,q}(e_2,x)=q^2P_{n-1}(x)\sum_{k=0}^{\infty}\begin{bmatrix} n+k-1 \\ k \end{bmatrix}\frac{[k+1][k+2]}{[n+k+1][n+k+2]}x^k$$

$$\leq \frac{1}{q^4}P_{n-1}(x)\sum_{k=0}^{\infty}\begin{bmatrix} n+k-1 \\ k \end{bmatrix}\frac{q^3[k]^2+(2q+1)q[k]+(q+1)}{[n+k-1][n+k-2]}x^k$$

$$= \frac{P_{n-1}(x)}{q}\sum_{k=0}^{\infty}\frac{[n+k-2]!}{[k]![n-1]!}(q[k]+1)x^{k+1}$$

$$+\frac{P_{n-1}(x)(2q+1)x}{q^3}\sum_{k=0}^{\infty}\begin{bmatrix} n+k-1 \\ k \end{bmatrix}\frac{x^k}{[n+k-1]}$$

journal of inequalities in pure and applied mathematics
issn: 1443-5756

$$
\begin{aligned}
&+\frac{P_{n-1}(x)(1+q)}{q^4}\sum_{k=0}^{\infty}\begin{bmatrix} n+k-1\\ k\end{bmatrix}\frac{x^k}{[n+k-1]^{\underline{2}}}\\
&=x^2P_{n-1}(x)\sum_{k=0}^{\infty}\begin{bmatrix} n+k-1\\ k\end{bmatrix}x^k\\
&\quad+x\frac{P_{n-1}(x)}{q}\sum_{k=0}^{\infty}\begin{bmatrix} n+k-1\\ k\end{bmatrix}\frac{x^k}{[n+k-1]}+x\frac{(2q+1)}{q^3}\frac{(1-q^{n-1}x)}{[n-1]}\\
&\quad+\frac{(1+q)}{q^4}\frac{(1-q^{n-1}x)(1-q^{n-2}x)}{[n-1][n-2]}\\
&=x^2+\frac{(1+q)^2}{q^3}\frac{(1-q^{n-1}x)}{[n-1]}x+\frac{(1+q)}{q^4}\frac{(1-q^{n-1}x)(1-q^{n-2}x)}{[n-1][n-2]}.
\end{aligned}
$$

□

Remark 2. From Lemma 2.3, it is observed that for $q \to 1^-$, we obtain

$$
\begin{aligned}
&M_n(e_0;x)=1,\\
&M_n(e_1;x)\le x+\frac{(1-x)}{(n-1)},\\
&M_n(e_1;x)\ge\left(1-\frac{2}{(n+1)}\right)x,\\
&M_n(e_2;x)\le x^2+\frac{4x(1-x)}{(n-1)}+\frac{2(1-x)^2}{(n-1)(n-2)},
\end{aligned}
$$

which are moments for a new generalization of the Meyer-Konig-Zeller operators for $\alpha = 1$ in [4].

journal of inequalities in pure and applied mathematics

issn: 1443-5756

Corollary 2.5. *The central moments of $M_{n,q}$ are*

$$M_{n,q}(\psi_0;x) = 1,$$
$$M_{n,q}(\psi_1;x) \leq \frac{(1-q^{n-1}x)}{q[n-1]},$$
$$M_{n,q}(\psi_2;x) \leq \frac{(1+q)^2}{q^3}\frac{(1-q^{n-1}x)}{[n-1]}x + \frac{(1+q)}{q^4}\frac{(1-q^{n-1}x)(1-q^{n-2}x)}{[n-1][n-2]} + 2\frac{(1+q^{n-2})}{[n+1]}x^2,$$

where $\psi_i(x) = (t-x)^i$ *for* $i = 0, 1, 2$.

Proof. By the linearity of $M_{n,q}$ and Theorem 2.4, we directly get the first two central moments. Using simple computations, the third moment can be easily verified as follows

$$\begin{aligned} M_{n,q}(\psi_2;x) &= M_{n,q}(e_2;x) + x^2 M_{n,q}(e_0;x) - 2xM_{n,q}(e_1;x) \\ &\leq \frac{(1+q)^2}{q^3}\frac{(1-q^{n-1}x)}{[n-1]}x + \frac{(1+q)}{q^4}\frac{(1-q^{n-1}x)(1-q^{n-2}x)}{[n-1][n-2]} \\ &\quad + \left(1 - \frac{(1+q^{n-2})}{[n+1]}\right)x - q^{n-2}(1-q)x^2 \\ &\leq \frac{(1+q)^2}{q^3}\frac{(1-q^{n-1}x)}{[n-1]}x + \frac{(1+q)}{q^4}\frac{(1-q^{n-1}x)(1-q^{n-2}x)}{[n-1][n-2]} \\ &\quad + 2\frac{(1+q^{n-2})}{[n+1]}x^2. \end{aligned}$$

□

journal of inequalities in pure and applied mathematics
issn: 1443-5756

Remark 3. For $q \to 1^-$, we get

$$M_n(\psi_2; x) \leq \frac{4x}{n-1} + \frac{2(1-x)^2}{(n-1)(n-2)}$$

which is similar to the result in [4].

Theorem 2.6. *The sequence $M_{n,q_n}(f)$ converges to f uniformly on $C[0,1]$ for each $f \in C[0,1]$ iff $q_n \to 1$ as $n \to \infty$.*

Proof. By the Korovkin theorem (see [1]), $M_{n,q_n}(f;x)$ converges to f uniformly on $[0,1]$ as $n \to \infty$ for $f \in C[0,1]$ iff $M_{n,q_n}(t^i;x) \to x^i$ for $i = 1, 2$ uniformly on $[0,1]$ as $n \to \infty$.

From the definition of $M_{n,q}$ and Theorem 2.4, M_{n,q_n} is a linear operator and reproduces constant functions.

Moreover, as $q_n \to 1$, then $[n]_{q_n} \to \infty$, therefore by Theorem 2.4, we get

$$M_{n,q_n}(t^i;x) \to x^i$$

for $i = 0, 1, 2$.

Hence, $M_{n,q_n}(f)$ converges to f uniformly on $C[0,1]$.

Conversely, suppose that $M_{n,q_n}(f)$ converges to f uniformly on $C[0,1]$ and q_n does not tend to 1 as $n \to \infty$. Then there exists a subsequence (q_{n_k}) of (q_n) s.t. $q_{n_k} \to q_0$ $(q_0 \neq 1)$ as $k \to \infty$. Thus

$$\frac{1}{[n]_{q_{n_k}}} = \frac{1 - q_{n_k}}{1 - {q_{n_k}}^n} \to (1 - q_0).$$

Taking $n = n_k$ and $q = q_{n_k}$ in $M_{n,q}(e_2, x)$, we have

$$M_{n,q_{n_k}}(e_2; x) \leq x + \frac{(1 - q_{n_k}^{n-1}x)(1 - q_0)}{q_{n_k}} \neq x$$

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issn: 1443-5756

which is a contradiction. Hence $q_n \to 1$. This completes the proof. □

Remark 4. Similar results are proved for the $q-$Bernstein-Durrmeyer operator in [11].

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issn: 1443-5756

3. Weighted Statistical Approximation Properties

In this section, we present the statistical approximation properties of the operator $M_{n,q}$ by using a Bohman-Korovkin type theorem [9].

Firstly, we recall the concepts of A-statistical convergence, weight functions and weighted spaces as considered in [9].

Let $A = (a_{jn})_{j,n}$ be a non-negative regular summability matrix. A sequence $(x_n)_n$ is said to be A-statistically convergent to a number L if, for every $\varepsilon > 0$, $\lim_j \sum_{n:|x_n-L|\geq\varepsilon} a_{jn} = 0$. It is denoted by $st_A - \lim_n x_n = L$. For $A := C_1$, the Cesàro matrix of order one is defined as

$$c_{jn} := \begin{cases} \frac{1}{j} & 1 \leq n \leq j \\ 0 & n > j. \end{cases}$$

A-statistical convergence coincides with statistical convergence.

A weight function is a real continuous function ρ on $\mathbb{R}$ s.t. $\lim_{|x|\to\infty} \rho(x) = \infty$, $\rho(x) \geq 1$ for all $x \in \mathbb{R}$.

The weighted space of real-valued functions f (denoted as $B_\rho(\mathbb{R})$) is defined on $\mathbb{R}$ with the property $|f(x)| \leq M_f \rho(x)$ for all $x \in \mathbb{R}$, where M_f is a constant depending on the function f. We also consider the weighted subspace $C_\rho(\mathbb{R})$ of $B_\rho(\mathbb{R})$ given by

$$C_\rho(\mathbb{R}) := \{f \in B_\rho(\mathbb{R}) : f \text{ continuous on } \mathbb{R}\}.$$

$B_\rho(\mathbb{R})$ and $C_\rho(\mathbb{R})$ are Banach spaces with the norm $\|\cdot\|_\rho$, where $\|f\|_\rho := \sup_{x\in\mathbb{R}} \frac{|f(x)|}{\rho(x)}$.

We next present a Bohman-Korovkin type theorem ([9, Theorem 3]) as follows.

Theorem 3.1. *Let $A = (a_{jn})_{j,n}$ be a non-negative regular summability matrix and let $(L_n)_n$ be a sequence of positive linear operators from $C_{\rho_1}(\mathbb{R})$ into $B_{\rho_2}(\mathbb{R})$, where*

journal of inequalities in pure and applied mathematics

issn: 1443-5756

ρ_1 and ρ_2 satisfy

$$\lim_{|x|\to\infty} \frac{\rho_1(x)}{\rho_2(x)} = 0.$$

Then

$$st_A - \lim_n \|L_n f - f\|_{\rho_2} = 0 \quad \textit{for all} \quad f \in C_{\rho_1}(\mathbb{R})$$

if and only if

$$st_A - \lim_n \|L_n F_v - F_v\|_{\rho_1} = 0, \qquad v = 0, 1, 2,$$

where $F_v(x) = \frac{x^v \rho_1(x)}{1+x^2}$, $v = 0, 1, 2.$

We next consider a sequence $(q_n)_n$, $q_n \in (0, 1)$, such that

$$st - \lim_n q_n = 1. \tag{3.1}$$

Theorem 3.2. *Let $(q_n)_n$ be a sequence satisfying (3.1). Then for all $f \in C_{\rho_0}(\mathbb{R}_+)$, we have*

$$st - \lim_n \|M_{n,q}(f; \cdot) - f\|_{\rho_\alpha} = 0, \quad \alpha > 0.$$

Proof. It is clear that

$$st - \lim_n \|M_{n,q_n}(e_0; \cdot) - e_0\|_{\rho_0} = 0. \tag{3.2}$$

Based on equation (2.5), we have

$$\begin{aligned} \frac{|M_{n,q_n}(e_1, x) - e_1(x)|}{1 + x^2} &\leq \| e_0 \| \frac{1}{q_n^2 [n-1]_{q_n}} \\ &\leq \frac{1}{[n-1]_{q_n}}. \end{aligned}$$

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issn: 1443-5756

Since $st - \lim_n q_n = 1$, we get $st - \lim_n \frac{1}{[n-1]_{q_n}} = 0$ and thus

$$st - \lim_n \|M_{n,q_n}(e_1;\cdot) - e_1\|_{\rho_0} = 0. \tag{3.3}$$

By using (2.7), we have

$$\begin{aligned} \frac{|M_{n,q_n}(e_2,x) - e_2(x)|}{1+x^2} &\leq \| e_0 \| \left(\frac{1}{[n-1]_{q_n}} + \frac{1}{[n-1]_{q_n}[n-2]_{q_n}} \right) \\ &\leq \frac{1}{[n-1]_{q_n}} + \frac{1}{[n-2]_{q_n}^2}. \end{aligned}$$

Consequently,

$$st - \lim_n \|K_{n,q_n}(e_2;\cdot) - e_2\|_{\rho_0} = 0. \tag{3.4}$$

Finally, using (3.2), (3.3) and (3.4), the proof follows from Theorem 3.1 by choosing $A = C_1$, the Cesàro matrix of order one and $\rho_1(x) = 1 + x^2$, $\rho_2(x) = 1 + x^{2+\alpha}$, $x \in \mathbb{R}_+$, $\alpha > 0$. □

journal of inequalities in pure and applied mathematics
issn: 1443-5756

4. Order of Approximation

We now recall the concept of modulus of continuity. The modulus of continuity of $f(x) \in C[0,a]$, denoted by $\omega(f,\delta)$, is defined by

$$\omega(f,\delta) = \sup_{|x-y|\leq\delta;\, x,y\in[0,a]} |f(x)-f(y)|. \tag{4.1}$$

The modulus of continuity possesses the following properties (see [9]):

$$\omega(f,\lambda\delta) \leq (1+\lambda)\omega(f,\delta) \tag{4.2}$$

and

$$\omega(f,n\delta) \leq n\omega(f,\delta), \quad n\in\mathbb{N}.$$

Theorem 4.1. *Let $(q_n)_n$ be a sequence satisfying (3.1). Then*

$$|M_{n,q}(f;x)-f| \leq 2\omega(f,\sqrt{\delta_n}) \tag{4.3}$$

for all $f\in C[0,1]$, where

$$\delta_n = M_{n,q}\left((qt-x)^2;x\right). \tag{4.4}$$

Proof. By the linearity and monotonicity of $M_{n,q}$, we get

$$\begin{aligned}|M_{n,q}(f;x)-f| &\leq M_{n,q}(|f(t)-f(x)|;x)\\ &= \sum_{k=0}^{\infty} m_{n,k,q}(x)\int_0^1 b_{n,k,q}(t)|f(qt)-f(x)|d_qt.\end{aligned}$$

Also

$$|f(qt)-f(x)| \leq \left(1+\frac{(qt-x)^2}{\delta^2}\right)\omega(f,\delta). \tag{4.5}$$

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issn: 1443-5756

By using (4.5), we obtain

$$|M_{n,q}(f;x) - f| \leq \sum_{k=0}^{\infty} m_{n,k,q}(x) \int_0^1 b_{n,k,q}(t) \left(1 + \frac{(qt-x)^2}{\delta^2}\right) \omega(f,\delta) d_q t$$
$$= \left(M_{n,q}(e_0;x) + \frac{1}{\delta^2} M_{n,q}\left((qt-x)^2;x\right)\right) \omega(f,\delta)$$

and

$$M_{n,q}\left((qt-x)^2;x\right) = q^2 M_{n,q}(e_2;x) + x^2 M_{n,q}(e_0;x) - 2qx M_{n,q}(e_1;x)$$
$$\leq (1-q)^2 x^2 + \frac{(1+q)^2}{q} \frac{(1-q^{n-1}x)}{[n-1]} x$$
$$+ \frac{(1+q)}{q^2} \frac{(1-q^{n-1}x)(1-q^{n-2}x)}{[n-1][n-2]}$$
$$+ 2xq^2 \left(\frac{(1+q^{n-2})}{[n+1]}\right) - 2q^{n-1}(1-q)x^3.$$

By (3.1) and the above equation, we get

$$\lim_{n\to\infty, q_n\to 1} M_{n,q}\left((qt-x)^2;x\right) = 0. \tag{4.6}$$

So, letting $\delta_n = M_{n,q}\left((qt-x)^2;x\right)$ and taking $\delta = \sqrt{\delta_n}$, we finally obtain

$$|M_{n,q}(f;x) - f| \leq 2\omega(f, \sqrt{\delta_n}).$$

□

As usual, a function $f \in Lip_M(\alpha)$, ($M > 0$ and $0 < \alpha \leq 1$), if the inequality

$$|f(t) - f(x)| \leq M|t-x|^{\alpha} \tag{4.7}$$

for all $t, x \in [0,1]$.

vol. 10, iss. 4, art. 105, 2009

journal of inequalities in pure and applied mathematics
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Theorem 4.2. *For all $f \in Lip_M(\alpha)$ and $x \in [0,1]$, we have*

$$|M_{n,q}(f;x) - f| \leq M\delta_n^{\alpha/2}, \tag{4.8}$$

where $\delta_n = M_{n,q}(\psi_2;x)$.

Proof. Using inequality (4.7) and Hölder's inequality with $p = \frac{2}{\alpha}$, $q = \frac{2}{2-\alpha}$, we get

$$\begin{aligned} |M_{n,q}(f;x) - f| &\leq M_{n,q}(|f(t) - f(x)|;x) \\ &\leq M M_{n,q}(|t-x|^{\alpha};x) \\ &\leq M M_{n,q}(|t-x|^2;x)^{\alpha/2}. \end{aligned}$$

Taking $\delta_n = M_{n,q}(\psi_2;x)$, we get

$$|M_{n,q}(f;x) - f| \leq M\delta_n^{\alpha/2}.$$

□

Theorem 4.3. *For all $f \in C[0,1]$ and $f(1) = 0$, we have*

$$|A_{n,k,q}(f)| \leq A_{n,k,q}(|f|) \leq \omega(f, q^n)(1 + q^{-n}), \quad (0 \leq k \leq n). \tag{4.9}$$

Proof. Clearly

$$\begin{aligned} |f(qt)| &= |f(qt) - f(1)| \\ &\leq \omega(f, q^n(1-qt)) \\ &\leq \omega(f, q^n)\left(1 + \frac{(1-qt)}{q^n}\right). \end{aligned}$$

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Thus by using Lemma 2.1, we get

$$\begin{aligned}|A_{n,k,q}(f)| &\leq A_{n,k,q}(|f|)\\ &= \int_0^1 b_{n,k,q}(t)|f(qt)|d_qt\\ &\leq \omega(f,q^n)\int_0^1 b_{n,k,q}(t)\left(1+\frac{(1-qt)}{q^n}\right)d_qt\\ &= \omega(f,q^n)\left(\left(1+\frac{1}{q^n}\right)\int_0^1 b_{n,k,q}(t)d_qt-\frac{1}{q^n}\int_0^1 b_{n,k,q}(t)(qt)d_qt\right)\\ &= \omega(f,q^n)\left(\left(1+\frac{1}{q^n}\right)-\frac{1}{q^{n-1}}\frac{[k+1]}{[k+n+1]}\right)\\ &\leq \omega(f,q^n)(1+q^{-n}).\end{aligned}$$

$\square$

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