

PROPERTIES OF q -MEYER-KÖNIG-ZELLER DURRMEYER OPERATORS

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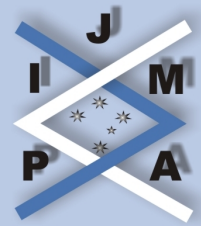
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Key words: q -integers, q -Meyer-König-Zeller Durrmeyer type operators, A-Statistical convergence, Weighted space, Weighted modulus of smoothness, Lipschitz class.

Abstract: We introduce a q analogue of the Meyer-König-Zeller Durrmeyer type operators and investigate their rate of convergence.

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q -Meyer-König-Zeller
Durrmeyer Operators

Honey Sharma

vol. 10, iss. 4, art. 105, 2009

[Title Page](#)

[Contents](#)



Page 1 of 21

[Go Back](#)

[Full Screen](#)

[Close](#)

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Contents

1	Introduction	3
2	Moments	6
3	Weighted Statistical Approximation Properties	13
4	Order of Approximation	16



q-Meyer-König-Zeller
Durrmeyer Operators

Honey Sharma

vol. 10, iss. 4, art. 105, 2009

Title Page

Contents



Page 2 of 21

Go Back

Full Screen

Close

journal of **inequalities**
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Title Page

Contents



Page 3 of 21

Go Back

Full Screen

Close

1. Introduction

Abel et al. [5] introduced the Meyer-König-Zeller Durrmeyer operators as

$$(1.1) \quad M_n(f; x) = \sum_{k=0}^{\infty} m_{n,k}(x) \int_0^1 b_{n,k}(t) f(t) dt, \quad 0 \leq x < 1,$$

where

$$m_{n,k}(x) = \binom{n+k-1}{k} x^k (1-x)^n$$

and

$$b_{n,k}(t) = n \binom{n+k}{k} t^k (1-t)^{n-1}.$$

Very recently H. Wang [6], O. Dogru and V. Gupta [2], A. Altin, O. Dogru and M.A. Ozarslan [7] and T. Trif [3] studied the q -Meyer-König-Zeller operators. This motivated us to introduce the q analogue of the Meyer-König-Zeller Durrmeyer operators.

Before introducing the operators, we mention certain definitions based on q -integers; details can be found in [10] and [12].

For each non-negative integer k , the q -integer $[k]$ and the q -factorial $[k]!$ are respectively defined by

$$[k] := \begin{cases} (1 - q^k)/(1 - q), & q \neq 1 \\ k, & q = 1 \end{cases},$$

and

$$[k]! := \begin{cases} [k][k-1] \cdots [1], & k \geq 1 \\ 1, & k = 0 \end{cases}.$$



Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 4 of 21

Go Back

Full Screen

Close

For the integers n, k satisfying $n \geq k \geq 0$, the q -binomial coefficients are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix} := \frac{[n]!}{[k]![n-k]!}.$$

We use the following notations

$$(a+b)_q^n = \prod_{j=0}^{n-1} (a+q^j b) = (a+b)(a+qb) \cdots (a+q^{n-1}b)$$

and

$$(t; q)_0 = 1, \quad (t; q)_n = \prod_{j=0}^{n-1} (1 - q^j t), \quad (t; q)_\infty = \prod_{j=0}^{\infty} (1 - q^j t).$$

Also it can be seen that

$$(a; q)_n = \frac{(a; q)_\infty}{(aq^n; q)_\infty}.$$

The q -Beta function is defined as

$$B_q(m, n) = \int_0^1 t^{m-1} (1-qt)_q^{n-1} d_q t$$

for $m, n \in \mathbb{N}$ and we have

$$(1.2) \quad B_q(m, n) = \frac{[m-1]![n-1]!}{[m+n-1]}.$$

It can be easily checked that

$$(1.3) \quad \prod_{j=0}^{n-1} (1 - q^j x) \sum_{k=0}^{\infty} \begin{bmatrix} n+k-1 \\ k \end{bmatrix} x^k = 1.$$



Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 5 of 21

Go Back

Full Screen

Close

Now we introduce the *q*-Meyer-König-Zeller Durrmeyer operator as follows

$$(1.4) \quad M_{n,q}(f; x) = \sum_{k=0}^{\infty} m_{n,k,q}(x) \int_0^1 b_{n,k,q}(t) f(qt) d_q t, \quad 0 \leq x < 1$$

$$(1.5) \quad := \sum_{k=0}^{\infty} m_{n,k,q}(x) A_{n,k,q}(f),$$

where $0 < q < 1$ and

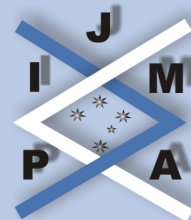
$$(1.6) \quad m_{n,k,q}(x) = P_{n-1}(x) \begin{bmatrix} n+k-1 \\ k \end{bmatrix} x^k,$$

$$(1.7) \quad b_{n,k,q}(t) = \frac{[n+k]!}{[k]![n-1]!} t^k (1-qt)_q^{n-1}.$$

Here

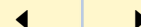
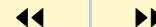
$$P_{n-1}(x) = \prod_{j=0}^{n-1} (1 - q^j x).$$

Remark 1. It can be seen that for $q \rightarrow 1^-$, the *q*-Meyer-König-Zeller Durrmeyer operator becomes the operator studied in [4] for $\alpha = 1$.



[Title Page](#)

[Contents](#)



Page 6 of 21

[Go Back](#)

[Full Screen](#)

[Close](#)

2. Moments

Lemma 2.1. For $g_s(t) = t^s$, $s = 0, 1, 2, \dots$, we have

$$(2.1) \quad \int_0^1 b_{n,k,q}(t)g_s(qt)d_qt = q^s \frac{[n+k]![k+s]!}{[k]![k+s+n]!}.$$

Proof. By using the q -Beta function (1.2), the above lemma can be proved easily. \square

Here, we introduce two lemmas proved in [8], as follows:

Lemma 2.2. For $r = 0, 1, 2, \dots$ and $n > r$, we have

$$(2.2) \quad P_{n-1}(x) \sum_{k=0}^{\infty} \begin{bmatrix} n+k-1 \\ k \end{bmatrix} \frac{x^k}{[n+k-1]^r} = \frac{\prod_{j=1}^r (1 - q^{n-j}x)}{[n-1]^r},$$

where $[n-1]^r = [n-1][n-2] \cdots [n-r]$.

Lemma 2.3. The identity

$$(2.3) \quad \frac{1}{[n+k+r]} \leq \frac{1}{q^{r+1}[n+k-1]}, \quad r \geq 0$$

holds.

Theorem 2.4. For all $x \in [0, 1]$, $n \in \mathbb{N}$ and $q \in (0, 1)$, we have

$$(2.4) \quad M_{n,q}(e_0; x) = 1,$$

$$(2.5) \quad M_{n,q}(e_1; x) \leq x + \frac{(1 - q^{n-1}x)}{q[n-1]},$$

$$(2.6) \quad M_{n,q}(e_1; x) \geq \left(1 - \frac{(1 + q^{n-2})}{[n+1]}\right) x + q^{n-2}(1 - q)x^2,$$



Title Page

Contents



Page 7 of 21

Go Back

Full Screen

Close

$$(2.7) \quad M_{n,q}(e_2; x) \leq x^2 + \frac{(1+q)^2(1-q^{n-1}x)}{q^3[n-1]}x + \frac{(1+q)(1-q^{n-1}x)(1-q^{n-2}x)}{q^4[n-1][n-2]}.$$

Proof. We have to estimate $M_{n,q}(e_s; x)$ for $s = 0, 1, 2$. The result can be easily verified for $s = 0$. Using the above lemmas and equation (1.3), we obtain relations (2.5) and (2.6) as follows

$$\begin{aligned} M_{n,q}(e_1, x) &= qP_{n-1}(x) \sum_{k=0}^{\infty} \begin{bmatrix} n+k-1 \\ k \end{bmatrix} \frac{[k+1]}{[n+k+1]} x^k \\ &\leq qP_{n-1}(x) \sum_{k=0}^{\infty} \begin{bmatrix} n+k-1 \\ k \end{bmatrix} \frac{q[k]+1}{q^2[n+k-1]} x^k \\ &= xP_{n-1}(x) \sum_{k=0}^{\infty} \begin{bmatrix} n+k-1 \\ k \end{bmatrix} x^k \\ &\quad + \frac{P_{n-1}(x)}{q} \sum_{k=0}^{\infty} \begin{bmatrix} n+k-1 \\ k \end{bmatrix} \frac{x^k}{[n+k-1]} \\ &= x + \frac{(1-q^{n-1}x)}{q[n-1]}. \end{aligned}$$

Also,

$$\begin{aligned} M_{n,q}(e_1, x) &= qP_{n-1}(x) \sum_{k=1}^{\infty} \begin{bmatrix} n+k-2 \\ k-1 \end{bmatrix} \frac{[k+1]}{[k]} \frac{[n+k-1]}{[n+k+1]} x^k \\ &\geq P_{n-1}(x) \sum_{k=0}^{\infty} \begin{bmatrix} n+k-1 \\ k \end{bmatrix} \left(\frac{[n+k+1]-1}{[n+k+2]} \right) x^{k+1} \end{aligned}$$



$$\begin{aligned}
 &\geq P_{n-1}(x) \sum_{k=0}^{\infty} \begin{bmatrix} n+k-1 \\ k \end{bmatrix} \left(\frac{[n+k+1]}{[n+k+2]} - \frac{1}{[n+1]} \right) x^{k+1} \\
 &\geq P_{n-1}(x) \sum_{k=0}^{\infty} \begin{bmatrix} n+k-1 \\ k \end{bmatrix} \left(1 - \frac{q^{n+k+1}}{[n+k+2]} \right) x^{k+1} - \frac{1}{[n+1]} x \\
 &\geq P_{n-1}(x) \sum_{k=0}^{\infty} \begin{bmatrix} n+k-1 \\ k \end{bmatrix} \left(1 - \frac{q^{n-2}(1-(1-q)[k])}{[n+k-1]} \right) x^{k+1} - \frac{1}{[n+1]} x \\
 &= x - \frac{q^{n-2}x}{[n+1]} + q^{n-2}(1-q)x^2 P_{n-1}(x) \sum_{k=0}^{\infty} \begin{bmatrix} n+k-1 \\ k \end{bmatrix} x^k - \frac{1}{[n+1]} x \\
 &= \left(1 - \frac{(1+q^{n-2})}{[n+1]} \right) x + q^{n-2}(1-q)x^2.
 \end{aligned}$$

Similar calculations reveal the relation (2.7) as follows

$$\begin{aligned}
 M_{n,q}(e_2, x) &= q^2 P_{n-1}(x) \sum_{k=0}^{\infty} \begin{bmatrix} n+k-1 \\ k \end{bmatrix} \frac{[k+1][k+2]}{[n+k+1][n+k+2]} x^k \\
 &\leq \frac{1}{q^4} P_{n-1}(x) \sum_{k=0}^{\infty} \begin{bmatrix} n+k-1 \\ k \end{bmatrix} \frac{q^3[k]^2 + (2q+1)q[k] + (q+1)}{[n+k-1][n+k-2]} x^k \\
 &= \frac{P_{n-1}(x)}{q} \sum_{k=0}^{\infty} \frac{[n+k-2]!}{[k]![n-1]!} (q[k]+1)x^{k+1} \\
 &\quad + \frac{P_{n-1}(x)(2q+1)x}{q^3} \sum_{k=0}^{\infty} \begin{bmatrix} n+k-1 \\ k \end{bmatrix} \frac{x^k}{[n+k-1]}
 \end{aligned}$$

Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 8 of 21

Go Back

Full Screen

Close

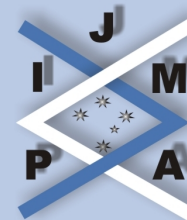
$$\begin{aligned}
& + \frac{P_{n-1}(x)(1+q)}{q^4} \sum_{k=0}^{\infty} \begin{bmatrix} n+k-1 \\ k \end{bmatrix} \frac{x^k}{[n+k-1]^2} \\
= & x^2 P_{n-1}(x) \sum_{k=0}^{\infty} \begin{bmatrix} n+k-1 \\ k \end{bmatrix} x^k \\
& + x \frac{P_{n-1}(x)}{q} \sum_{k=0}^{\infty} \begin{bmatrix} n+k-1 \\ k \end{bmatrix} \frac{x^k}{[n+k-1]} + x \frac{(2q+1)(1-q^{n-1}x)}{q^3 [n-1]} \\
& + \frac{(1+q)(1-q^{n-1}x)(1-q^{n-2}x)}{q^4 [n-1][n-2]} \\
= & x^2 + \frac{(1+q)^2(1-q^{n-1}x)}{q^3 [n-1]} x + \frac{(1+q)(1-q^{n-1}x)(1-q^{n-2}x)}{q^4 [n-1][n-2]}.
\end{aligned}$$

□

Remark 2. From Lemma 2.3, it is observed that for $q \rightarrow 1^-$, we obtain

$$\begin{aligned}
M_n(e_0; x) &= 1, \\
M_n(e_1; x) &\leq x + \frac{(1-x)}{(n-1)}, \\
M_n(e_1; x) &\geq \left(1 - \frac{2}{(n+1)}\right) x, \\
M_n(e_2; x) &\leq x^2 + \frac{4x(1-x)}{(n-1)} + \frac{2(1-x)^2}{(n-1)(n-2)},
\end{aligned}$$

which are moments for a new generalization of the Meyer-Konig-Zeller operators for $\alpha = 1$ in [4].



[Title Page](#)

[Contents](#)

◀◀ ▶▶

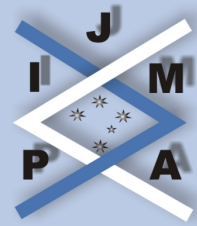
◀ ▶

Page 9 of 21

[Go Back](#)

[Full Screen](#)

[Close](#)



Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 10 of 21

Go Back

Full Screen

Close

Corollary 2.5. *The central moments of $M_{n,q}$ are*

$$M_{n,q}(\psi_0; x) = 1,$$

$$M_{n,q}(\psi_1; x) \leq \frac{(1 - q^{n-1}x)}{q[n-1]},$$

$$M_{n,q}(\psi_2; x) \leq \frac{(1+q)^2}{q^3} \frac{(1 - q^{n-1}x)}{[n-1]} x + \frac{(1+q)}{q^4} \frac{(1 - q^{n-1}x)(1 - q^{n-2}x)}{[n-1][n-2]} \\ + 2 \frac{(1 + q^{n-2})}{[n+1]} x^2,$$

where $\psi_i(x) = (t - x)^i$ for $i = 0, 1, 2$.

Proof. By the linearity of $M_{n,q}$ and Theorem 2.4, we directly get the first two central moments. Using simple computations, the third moment can be easily verified as follows

$$M_{n,q}(\psi_2; x) = M_{n,q}(e_2; x) + x^2 M_{n,q}(e_0; x) - 2x M_{n,q}(e_1; x) \\ \leq \frac{(1+q)^2}{q^3} \frac{(1 - q^{n-1}x)}{[n-1]} x + \frac{(1+q)}{q^4} \frac{(1 - q^{n-1}x)(1 - q^{n-2}x)}{[n-1][n-2]} \\ + \left(1 - \frac{(1 + q^{n-2})}{[n+1]}\right) x - q^{n-2}(1 - q)x^2 \\ \leq \frac{(1+q)^2}{q^3} \frac{(1 - q^{n-1}x)}{[n-1]} x + \frac{(1+q)}{q^4} \frac{(1 - q^{n-1}x)(1 - q^{n-2}x)}{[n-1][n-2]} \\ + 2 \frac{(1 + q^{n-2})}{[n+1]} x^2.$$

□



Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 11 of 21

Go Back

Full Screen

Close

Remark 3. For $q \rightarrow 1^-$, we get

$$M_n(\psi_2; x) \leq \frac{4x}{n-1} + \frac{2(1-x)^2}{(n-1)(n-2)}$$

which is similar to the result in [4].

Theorem 2.6. *The sequence $M_{n,q_n}(f)$ converges to f uniformly on $C[0, 1]$ for each $f \in C[0, 1]$ iff $q_n \rightarrow 1$ as $n \rightarrow \infty$.*

Proof. By the Korovkin theorem (see [1]), $M_{n,q_n}(f; x)$ converges to f uniformly on $[0, 1]$ as $n \rightarrow \infty$ for $f \in C[0, 1]$ iff $M_{n,q_n}(t^i; x) \rightarrow x^i$ for $i = 1, 2$ uniformly on $[0, 1]$ as $n \rightarrow \infty$.

From the definition of $M_{n,q}$ and Theorem 2.4, M_{n,q_n} is a linear operator and reproduces constant functions.

Moreover, as $q_n \rightarrow 1$, then $[n]_{q_n} \rightarrow \infty$, therefore by Theorem 2.4, we get

$$M_{n,q_n}(t^i; x) \rightarrow x^i$$

for $i = 0, 1, 2$.

Hence, $M_{n,q_n}(f)$ converges to f uniformly on $C[0, 1]$.

Conversely, suppose that $M_{n,q_n}(f)$ converges to f uniformly on $C[0, 1]$ and q_n does not tend to 1 as $n \rightarrow \infty$. Then there exists a subsequence (q_{n_k}) of (q_n) s.t. $q_{n_k} \rightarrow q_0$ ($q_0 \neq 1$) as $k \rightarrow \infty$. Thus

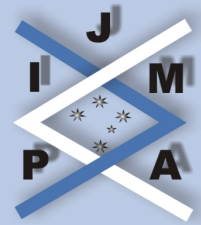
$$\frac{1}{[n]_{q_{n_k}}} = \frac{1 - q_{n_k}}{1 - q_{n_k}^n} \rightarrow (1 - q_0).$$

Taking $n = n_k$ and $q = q_{n_k}$ in $M_{n,q}(e_2, x)$, we have

$$M_{n,q_{n_k}}(e_2; x) \leq x + \frac{(1 - q_{n_k}^{n-1}x)(1 - q_0)}{q_{n_k}} \neq x$$

which is a contradiction. Hence $q_n \rightarrow 1$. This completes the proof. \square

Remark 4. Similar results are proved for the q -Bernstein-Durrmeyer operator in [11].



q-Meyer-König-Zeller
Durrmeyer Operators

Honey Sharma

vol. 10, iss. 4, art. 105, 2009

Title Page

Contents



Page 12 of 21

Go Back

Full Screen

Close

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[Title Page](#)

[Contents](#)

◀◀ ▶▶

◀ ▶

Page 13 of 21

[Go Back](#)

[Full Screen](#)

[Close](#)

3. Weighted Statistical Approximation Properties

In this section, we present the statistical approximation properties of the operator $M_{n,q}$ by using a Bohman-Korovkin type theorem [9].

Firstly, we recall the concepts of A -statistical convergence, weight functions and weighted spaces as considered in [9].

Let $A = (a_{jn})_{j,n}$ be a non-negative regular summability matrix. A sequence $(x_n)_n$ is said to be A -statistically convergent to a number L if, for every $\varepsilon > 0$, $\lim_j \sum_{n:|x_n-L|\geq\varepsilon} a_{jn} = 0$. It is denoted by $st_A - \lim_n x_n = L$. For $A := C_1$, the Cesàro matrix of order one is defined as

$$c_{jn} := \begin{cases} \frac{1}{j} & 1 \leq n \leq j \\ 0 & n > j. \end{cases}$$

A -statistical convergence coincides with statistical convergence.

A weight function is a real continuous function ρ on \mathbb{R} s.t. $\lim_{|x| \rightarrow \infty} \rho(x) = \infty$, $\rho(x) \geq 1$ for all $x \in \mathbb{R}$.

The weighted space of real-valued functions f (denoted as $B_\rho(\mathbb{R})$) is defined on \mathbb{R} with the property $|f(x)| \leq M_f \rho(x)$ for all $x \in \mathbb{R}$, where M_f is a constant depending on the function f . We also consider the weighted subspace $C_\rho(\mathbb{R})$ of $B_\rho(\mathbb{R})$ given by

$$C_\rho(\mathbb{R}) := \{f \in B_\rho(\mathbb{R}) : f \text{ continuous on } \mathbb{R}\}.$$

$B_\rho(\mathbb{R})$ and $C_\rho(\mathbb{R})$ are Banach spaces with the norm $\|\cdot\|_\rho$, where $\|f\|_\rho := \sup_{x \in \mathbb{R}} \frac{|f(x)|}{\rho(x)}$.

We next present a Bohman-Korovkin type theorem ([9, Theorem 3]) as follows.

Theorem 3.1. *Let $A = (a_{jn})_{j,n}$ be a non-negative regular summability matrix and let $(L_n)_n$ be a sequence of positive linear operators from $C_{\rho_1}(\mathbb{R})$ into $B_{\rho_2}(\mathbb{R})$, where*



Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 14 of 21

Go Back

Full Screen

Close

ρ_1 and ρ_2 satisfy

$$\lim_{|x| \rightarrow \infty} \frac{\rho_1(x)}{\rho_2(x)} = 0.$$

Then

$$st_A - \lim_n \|L_n f - f\|_{\rho_2} = 0 \quad \text{for all } f \in C_{\rho_1}(\mathbb{R})$$

if and only if

$$st_A - \lim_n \|L_n F_v - F_v\|_{\rho_1} = 0, \quad v = 0, 1, 2,$$

where $F_v(x) = \frac{x^v \rho_1(x)}{1+x^2}$, $v = 0, 1, 2$.

We next consider a sequence $(q_n)_n$, $q_n \in (0, 1)$, such that

$$(3.1) \quad st - \lim_n q_n = 1.$$

Theorem 3.2. *Let $(q_n)_n$ be a sequence satisfying (3.1). Then for all $f \in C_{\rho_0}(\mathbb{R}_+)$, we have*

$$st - \lim_n \|M_{n,q}(f; \cdot) - f\|_{\rho_\alpha} = 0, \quad \alpha > 0.$$

Proof. It is clear that

$$(3.2) \quad st - \lim_n \|M_{n,q_n}(e_0; \cdot) - e_0\|_{\rho_0} = 0.$$

Based on equation (2.5), we have

$$\begin{aligned} \frac{|M_{n,q_n}(e_1, x) - e_1(x)|}{1+x^2} &\leq \|e_0\| \frac{1}{q_n^2 [n-1]_{q_n}} \\ &\leq \frac{1}{[n-1]_{q_n}}. \end{aligned}$$



Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 15 of 21

Go Back

Full Screen

Close

Since $st - \lim_n q_n = 1$, we get $st - \lim_n \frac{1}{[n-1]_{q_n}} = 0$ and thus

$$(3.3) \quad st - \lim_n \|M_{n,q_n}(e_1; \cdot) - e_1\|_{\rho_0} = 0.$$

By using (2.7), we have

$$\begin{aligned} \frac{|M_{n,q_n}(e_2, x) - e_2(x)|}{1+x^2} &\leq \|e_0\| \left(\frac{1}{[n-1]_{q_n}} + \frac{1}{[n-1]_{q_n}[n-2]_{q_n}} \right) \\ &\leq \frac{1}{[n-1]_{q_n}} + \frac{1}{[n-2]_{q_n}^2}. \end{aligned}$$

Consequently,

$$(3.4) \quad st - \lim_n \|K_{n,q_n}(e_2; \cdot) - e_2\|_{\rho_0} = 0.$$

Finally, using (3.2), (3.3) and (3.4), the proof follows from Theorem 3.1 by choosing $A = C_1$, the Cesàro matrix of order one and $\rho_1(x) = 1 + x^2$, $\rho_2(x) = 1 + x^{2+\alpha}$, $x \in \mathbb{R}_+$, $\alpha > 0$. \square



Title Page

Contents



Page 16 of 21

Go Back

Full Screen

Close

4. Order of Approximation

We now recall the concept of modulus of continuity. The modulus of continuity of $f(x) \in C[0, a]$, denoted by $\omega(f, \delta)$, is defined by

$$(4.1) \quad \omega(f, \delta) = \sup_{|x-y| \leq \delta; x, y \in [0, a]} |f(x) - f(y)|.$$

The modulus of continuity possesses the following properties (see [9]):

$$(4.2) \quad \omega(f, \lambda\delta) \leq (1 + \lambda)\omega(f, \delta)$$

and

$$\omega(f, n\delta) \leq n\omega(f, \delta), \quad n \in \mathbb{N}.$$

Theorem 4.1. Let $(q_n)_n$ be a sequence satisfying (3.1). Then

$$(4.3) \quad |M_{n,q}(f; x) - f| \leq 2\omega(f, \sqrt{\delta_n})$$

for all $f \in C[0, 1]$, where

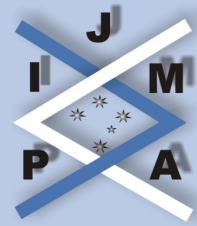
$$(4.4) \quad \delta_n = M_{n,q}((qt - x)^2; x).$$

Proof. By the linearity and monotonicity of $M_{n,q}$, we get

$$\begin{aligned} |M_{n,q}(f; x) - f| &\leq M_{n,q}(|f(t) - f(x)|; x) \\ &= \sum_{k=0}^{\infty} m_{n,k,q}(x) \int_0^1 b_{n,k,q}(t) |f(qt) - f(x)| d_q t. \end{aligned}$$

Also

$$(4.5) \quad |f(qt) - f(x)| \leq \left(1 + \frac{(qt - x)^2}{\delta^2}\right) \omega(f, \delta).$$



[Title Page](#)

[Contents](#)

◀◀ ▶▶

◀ ▶

Page 17 of 21

[Go Back](#)

[Full Screen](#)

[Close](#)

By using (4.5), we obtain

$$\begin{aligned} |M_{n,q}(f; x) - f| &\leq \sum_{k=0}^{\infty} m_{n,k,q}(x) \int_0^1 b_{n,k,q}(t) \left(1 + \frac{(qt-x)^2}{\delta^2}\right) \omega(f, \delta) d_q t \\ &= \left(M_{n,q}(e_0; x) + \frac{1}{\delta^2} M_{n,q}((qt-x)^2; x)\right) \omega(f, \delta) \end{aligned}$$

and

$$\begin{aligned} M_{n,q}((qt-x)^2; x) &= q^2 M_{n,q}(e_2; x) + x^2 M_{n,q}(e_0; x) - 2qx M_{n,q}(e_1; x) \\ &\leq (1-q)^2 x^2 + \frac{(1+q)^2 (1-q^{n-1}x)}{q [n-1]} x \\ &\quad + \frac{(1+q)(1-q^{n-1}x)(1-q^{n-2}x)}{q^2 [n-1][n-2]} \\ &\quad + 2xq^2 \left(\frac{(1+q^{n-2})}{[n+1]}\right) - 2q^{n-1}(1-q)x^3. \end{aligned}$$

By (3.1) and the above equation, we get

$$(4.6) \quad \lim_{n \rightarrow \infty, q_n \rightarrow 1} M_{n,q}((qt-x)^2; x) = 0.$$

So, letting $\delta_n = M_{n,q}((qt-x)^2; x)$ and taking $\delta = \sqrt{\delta_n}$, we finally obtain

$$|M_{n,q}(f; x) - f| \leq 2\omega(f, \sqrt{\delta_n}).$$

□

As usual, a function $f \in Lip_M(\alpha)$, ($M > 0$ and $0 < \alpha \leq 1$), if the inequality

$$(4.7) \quad |f(t) - f(x)| \leq M|t-x|^\alpha$$

for all $t, x \in [0, 1]$.



Title Page

Contents



Page 18 of 21

Go Back

Full Screen

Close

Theorem 4.2. For all $f \in Lip_M(\alpha)$ and $x \in [0, 1]$, we have

$$(4.8) \quad |M_{n,q}(f; x) - f| \leq M\delta_n^{\alpha/2},$$

where $\delta_n = M_{n,q}(\psi_2; x)$.

Proof. Using inequality (4.7) and Hölder's inequality with $p = \frac{2}{\alpha}$, $q = \frac{2}{2-\alpha}$, we get

$$\begin{aligned} |M_{n,q}(f; x) - f| &\leq M_{n,q}(|f(t) - f(x)|; x) \\ &\leq MM_{n,q}(|t - x|^\alpha; x) \\ &\leq MM_{n,q}(|t - x|^2; x)^{\alpha/2}. \end{aligned}$$

Taking $\delta_n = M_{n,q}(\psi_2; x)$, we get

$$|M_{n,q}(f; x) - f| \leq M\delta_n^{\alpha/2}.$$

□

Theorem 4.3. For all $f \in C[0, 1]$ and $f(1) = 0$, we have

$$(4.9) \quad |A_{n,k,q}(f)| \leq A_{n,k,q}(|f|) \leq \omega(f, q^n)(1 + q^{-n}), \quad (0 \leq k \leq n).$$

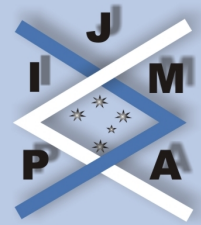
Proof. Clearly

$$\begin{aligned} |f(qt)| &= |f(qt) - f(1)| \\ &\leq \omega(f, q^n(1 - qt)) \\ &\leq \omega(f, q^n) \left(1 + \frac{(1 - qt)}{q^n}\right). \end{aligned}$$

Thus by using Lemma 2.1, we get

$$\begin{aligned}
 |A_{n,k,q}(f)| &\leq A_{n,k,q}(|f|) \\
 &= \int_0^1 b_{n,k,q}(t) |f(qt)| d_q t \\
 &\leq \omega(f, q^n) \int_0^1 b_{n,k,q}(t) \left(1 + \frac{(1-qt)}{q^n}\right) d_q t \\
 &= \omega(f, q^n) \left(\left(1 + \frac{1}{q^n}\right) \int_0^1 b_{n,k,q}(t) d_q t - \frac{1}{q^n} \int_0^1 b_{n,k,q}(t)(qt) d_q t \right) \\
 &= \omega(f, q^n) \left(\left(1 + \frac{1}{q^n}\right) - \frac{1}{q^{n-1}} \frac{[k+1]}{[k+n+1]} \right) \\
 &\leq \omega(f, q^n) (1 + q^{-n}).
 \end{aligned}$$

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Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 19 of 21

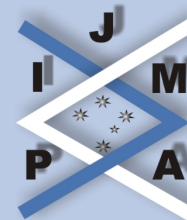
Go Back

Full Screen

Close

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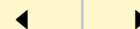
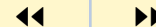
q -Meyer-König-Zeller
Durrmeyer Operators

Honey Sharma

vol. 10, iss. 4, art. 105, 2009

Title Page

Contents



Page 20 of 21

Go Back

Full Screen

Close

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q -Meyer-König-Zeller
Durrmeyer Operators

Honey Sharma

vol. 10, iss. 4, art. 105, 2009

Title Page

Contents



Page 21 of 21

Go Back

Full Screen

Close

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