

Journal of Inequalities in Pure and Applied Mathematics

GEOMETRIC INEQUALITIES FOR A SIMPLEX

SHIGUO YANG

Department of Mathematics
Anhui Institute of Education
Hefei, 230061, P.R. China.

EMail: sxx@ahieedu.net.cn

©2000 Victoria University
ISSN (electronic): 1443-5756
096-04



volume 6, issue 3, article 76,
2005.

*Received 18 October, 2004;
accepted 11 March, 2005.*

Communicated by: J. Sándor

[Abstract](#)

[Contents](#)

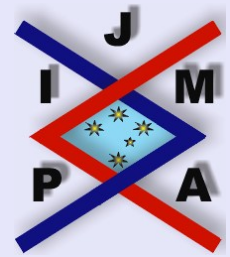


[Home Page](#)

[Go Back](#)

[Close](#)

[Quit](#)



Abstract

In this paper, we study a problem of geometric inequalities for an n -simplex. Some new geometric inequalities for a simplex are established. As special cases, some known inequalities are deduced.

2000 Mathematics Subject Classification: 52A40, 51K16.

Key words: Simplex, Volume, Inradius, Circumradius, Inequality.

The author would like to express his thanks to the editor for his kind help and invaluable suggestions in the formatting and writing of this paper.

Contents

1	Introduction	3
2	Main Results	4
3	Lemmas and Proofs of Theorems	7
	References	

Geometric Inequalities for a Simplex

Shiguo Yang

Title Page

Contents



Go Back

Close

Quit

Page 2 of 16

1. Introduction

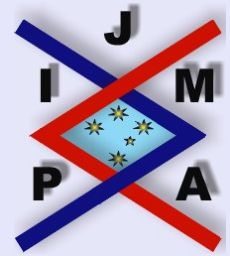
Let σ_n be an n -dimensional simplex in the n -dimensional Euclidean space E^n , $\tau = \{A_0, A_1, \dots, A_n\}$ denote the vertex set of σ_n , V the volume of σ_n , R and r the circumradius and inradius of σ_n , respectively. For $i = 0, 1, \dots, n$, let r_i be the radius of i th escribed sphere of σ_n , F_i the area of the i th face $f_i = A_0 \cdots A_{i-1}A_{i+1} \cdots A_n$ of σ_n . Let P be an arbitrary interior point of the simplex σ_n , d_i the distance from the point P to the i th face f_i of σ_n , h_i the altitude of σ_n from vertex A_i for $i = 0, 1, \dots, n$.

Let a_0 , a_1 and a_2 denote the edge-lengths of triangle $A_0A_1A_2$ (2-dimensional simplex). An important inequality for a triangle was established by Janić (see [1]) as follows:

$$(1.1) \quad \frac{a_0^2}{r_1r_2} + \frac{a_1^2}{r_2r_0} + \frac{a_2^2}{r_0r_1} \geq 4.$$

Let P be an arbitrary interior point of the triangle $A_0A_1A_2$. Gerasimov (see [2]) obtained an inequality for the triangle $A_0A_1A_2$ as follows:

$$(1.2) \quad \frac{d_1d_2}{a_1a_2} + \frac{d_2d_0}{a_2a_0} + \frac{d_0d_1}{a_0a_1} \leq \frac{1}{4}.$$



Geometric Inequalities for a
Simplex

Shiguo Yang

Title Page

Contents



Go Back

Close

Quit

Page 3 of 16

2. Main Results

We will extend inequalities (1.1) and (1.2) to an n -dimensional simplex. Our main results are contained in the following theorem:

Theorem 2.1. *For the n -dimensional simplex σ_n we have*

$$(2.1) \quad \sum_{i=0}^n \frac{F_i^{n/(n-1)}}{r_0 \cdots r_{i-1} r_{i+1} \cdots r_n} \geq \frac{(n-1)^n n^{3n^2/2(n-1)}}{n^n (n+1)^{(n-2)/2} (n!)^{n/(n-1)}},$$

with equality iff the simplex σ_n is regular.

By letting $n = 2$ in relation (2.1), inequality (1.1) is reobtained.

Theorem 2.2. *Let P be an arbitrary interior point of the simplex σ_n , and let $\theta \in (0, 1]$ be a real number. Then we have*

$$(2.2) \quad \sum_{i=0}^n \frac{d_0 \cdots d_{i-1} d_{i+1} \cdots d_n}{(F_0 \cdots F_{i-1} F_{i+1} \cdots F_n)^{2\theta-1}} \leq \frac{(n!)^{2\theta}}{(n+1)^{(n-1)(1-\theta)} n^{n(3\theta-1)}} V^{n-2(n-1)\theta},$$

with equality iff the simplex σ_n is regular and the point P is the circumcenter of σ_n .

If we take $\theta = \frac{n}{2(n-1)}$ in inequality (2.2), we obtain the following corollary:



Geometric Inequalities for a Simplex

Shiguo Yang

Title Page

Contents



Go Back

Close

Quit

Page 4 of 16

Corollary 2.3. Let P be an arbitrary interior point of the simplex σ_n . Then we have

$$(2.3) \quad \sum_{i=0}^n \frac{d_0 \cdots d_{i-1} d_{i+1} \cdots d_n}{(F_0 \cdots F_{i-1} F_{i+1} \cdots F_n)^{1/(n-1)}} \leq \frac{(n!)^{n/(n-1)}}{(n+1)^{(n-2)/2} n^{n(n+2)/2(n-1)}},$$

with equality iff the simplex σ_n is regular and the point P is the circumcenter of σ_n .

If $n = 2$ in inequality (2.3), then inequality (1.2) follows from inequality (2.3).

By taking $\theta = \frac{1}{2}$ in inequality (2.2), we obtain a generalization of Gerber's inequality as follows:

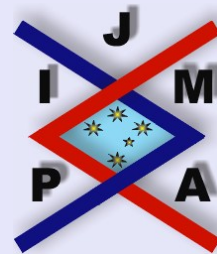
Corollary 2.4. Let P be arbitrary interior point of the simplex σ_n . Then

$$(2.4) \quad \sum_{i=0}^n d_0 \cdots d_{i-1} d_{i+1} \cdots d_n \leq \frac{n!}{(n+1)^{(n-1)/2} n^{n/2}} V,$$

with equality iff the simplex σ_n is regular.

Using inequality (2.4) and the arithmetic-geometric mean inequality we get Gerber's inequality [3] as follows:

$$(2.5) \quad \prod_{i=0}^n d_i \leq \frac{(n!)^{(n+1)/n}}{n^{(n+1)/2} (n+1)^{1/2n}} V^{(n+1)/n}.$$



Geometric Inequalities for a Simplex

Shiguo Yang

Title Page

Contents



Go Back

Close

Quit

Page 5 of 16

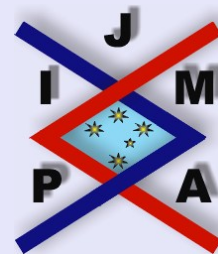
Theorem 2.5. *Let P be an arbitrary interior point of the simplex σ_n . Then we have*

$$(2.6) \quad \sum_{i=0}^n \frac{1}{d_0 \cdots d_{i-1} d_{i+1} \cdots d_n} \geq (n+1)n^{n+1} \cdot \frac{r}{R^{n+1}},$$

with equality iff the simplex σ_n is regular and the point P is the circumcenter of σ_n .

If the point P is the incenter I of the simplex σ_n , i.e. $d_i = r (i = 0, 1, \dots, n)$, then the following n -dimensional Euler inequality stated in [4] is obtained from (2.6):

$$(2.7) \quad R \geq nr.$$



Geometric Inequalities for a Simplex

Shiguo Yang

Title Page

Contents



Go Back

Close

Quit

Page 6 of 16

3. Lemmas and Proofs of Theorems

To prove the theorems stated above, we need some lemmas as follows.

Let m_i ($i = 0, 1, \dots, n$) be positive numbers, $V_{i_0 i_1 \dots i_k}$ denote the k -dimensional volume of the k -dimensional simplex $A_{i_0} A_{i_1} \dots A_{i_k}$ for $A_{i_0}, A_{i_1}, \dots, A_{i_k} \in \tau$.

Put

$$M_k = \sum_{0 \leq i_0 < i_1 < \dots < i_k \leq n} m_{i_0} m_{i_1} \dots m_{i_k} V_{i_0 i_1 \dots i_k}^2, \quad (1 \leq k \leq n),$$

$$M_0 = \sum_{i=0}^n m_i.$$

Lemma 3.1. For positive numbers m_i ($i = 0, 1, \dots, n$) and the n -dimensional simplex σ_n , we have

$$(3.1) \quad M_k^l \geq \frac{[(n-l)!(l!)^3]^k}{[(n-k)!(k!)^3]^l} (n! \cdot M_0)^{l-k} M_l^k, \quad (1 \leq k < l \leq n),$$

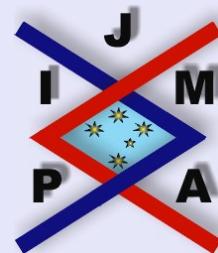
with equality iff the simplex σ_n is regular and $m_0 = m_1 = \dots = m_n$.

Lemma 3.2.

$$(3.2) \quad \left(\prod_{i=0}^n F_i \right)^{\frac{n}{n^2-1}} \geq \frac{1}{(n+1)^{1/2}} \left(\frac{n^{3n}}{n!^2} \right)^{\frac{1}{2(n-1)}} V^n,$$

with equality iff the simplex σ_n is regular.

For the proof of Lemmas 3.1 and 3.2, the reader is referred to [5] or [1].



Geometric Inequalities for a
Simplex

Shiguo Yang

Title Page

Contents



Go Back

Close

Quit

Page 7 of 16

Lemma 3.3.

$$(3.3) \quad \sum_{i=0}^n \frac{h_0 \cdots h_{i-1} h_{i+1} \cdots h_n}{r_0 \cdots r_{i-1} r_{i+1} \cdots r_n} \geq (n+1)(n_1)^n,$$

with equality iff the simplex σ_n is regular.

For the proof of Lemma 3.3, see [5].

Lemma 3.4.

$$(3.4) \quad V \geq \frac{n^{n/2}(n+1)^{(n+1)/2}}{n!} r^n,$$

with equality iff the simplex σ_n is regular.

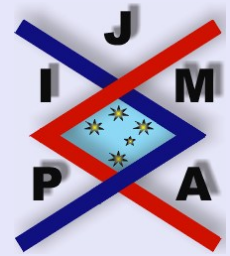
This is also known, see [5] or [1].

Proof of Theorem 2.1. Without loss of generality, let $F_0 \leq F_1 \leq \cdots \leq F_n$. By the known formula ([1])

$$(3.5) \quad r_i = \frac{nV}{\sum_{j=0}^n F_j - 2F_i}, \quad (i = 0, 1, \dots, n),$$

it follows that $r_0 \leq r_1 \leq \cdots \leq r_n$ and

$$\frac{1}{\prod_{j=1}^n F_j r_j} \leq \frac{1}{\prod_{\substack{j=0 \\ j \neq i}}^n F_j r_j} \leq \cdots \leq \frac{1}{\prod_{\substack{j=0 \\ j \neq n}}^n F_j r_j}.$$



Title Page

Contents



Go Back

Close

Quit

Page 8 of 16

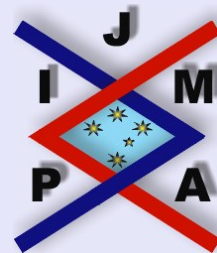
Using the Chebyshev inequality, we have

$$(3.6) \quad \sum_{i=0}^n \frac{F_i^{n/(n-1)}}{\prod_{\substack{j=0 \\ j \neq i}} r_j} = \left(\prod_{i=0}^n F_i \right) \sum_{i=0}^n \frac{F_i^{1/(n-1)}}{\prod_{\substack{j=0 \\ j \neq i}} F_j r_j} \\ \geq \frac{1}{n+1} \left(\prod_{i=0}^n F_i \right) \left(\sum_{i=0}^n F_i^{1/(n-1)} \right) \left(\sum_{i=0}^n \frac{1}{\prod_{\substack{j=0 \\ j \neq i}} F_j r_j} \right).$$

Substituting $F_j = \frac{nV}{h_j}$ ($j = 0, 1, \dots, n$) into the right side of inequality (3.6) and using the arithmetic-geometric mean inequality we get

$$(3.7) \quad \sum_{i=0}^n \frac{F_i^{n/(n-1)}}{\prod_{\substack{j=0 \\ j \neq i}} r_j} \geq \frac{1}{n+1} \left(\prod_{i=0}^n F_i \right) \left(\sum_{i=0}^n F_i^{1/(n-1)} \right) \\ \times \frac{1}{(nV)^n} \sum_{i=0}^n \frac{h_0 \cdots h_{i-1} h_{i+1} \cdots h_n}{r_0 \cdots r_{i-1} r_{i+1} \cdots r_n} \\ \geq \frac{\left(\prod_{i=0}^n F_i \right)^{\frac{n^2}{(n^2-1)}}}{(nV)^n} \sum_{i=0}^n \frac{h_0 \cdots h_{i-1} h_{i+1} \cdots h_n}{r_0 \cdots r_{i-1} r_{i+1} \cdots r_n}.$$

By inequalities (3.7), (3.2) and (3.3) we obtain relation (2.1). It is easy to see that equality in (2.1) holds iff the simplex σ_n is regular. The proof of Theorem 2.1 is thus complete. \square



Geometric Inequalities for a Simplex

Shiguo Yang

Title Page

Contents



Go Back

Close

Quit

Page 9 of 16

Proof of Theorem 2.2. Taking $k = n - 1$, $l = n$ in inequality (3.1), we can write

$$(3.8) \quad \left(\sum_{i=0}^n m_0 \cdots m_{i-1} m_{i+1} \cdots m_n F_i^2 \right)^n \geq \frac{n^{3n}}{n!^2} \left(\sum_{i=0}^n m_i \right) \left(\prod_{i=0}^n m_i \right)^{n-1} V^{2(n-1)}.$$

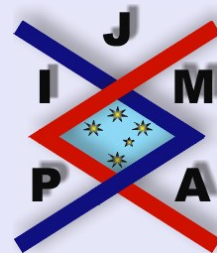
By putting $m_0 \cdots m_{i-1} m_{i+1} \cdots m_n = \lambda_i F_i^{n-2}$ ($i = 0, 1, \dots, n$) in equality (3.8), we get

$$(3.9) \quad \left(\frac{1}{n} \sum_{i=0}^n \lambda_i \right)^n \left(\prod_{i=0}^n F_i^2 \right) \geq \frac{(nV)^{2(n-1)}}{(n-1)!^2} \left(\prod_{i=0}^n \lambda_i \right) \left(\sum_{i=0}^n \frac{F_i^2}{\lambda_i} \right).$$

We now prove that the following inequality (3.10) is valid for any number $\theta \in (0, 1]$:

$$(3.10) \quad \left(\frac{1}{n} \sum_{i=0}^n \lambda_i \right)^n \prod_{i=0}^n F_i^{2\theta} \geq \left(\sum_{i=0}^n \lambda_i \right)^n \left(\sum_{i=0}^n \frac{F_i^{2\theta}}{\lambda_i} \right) \frac{(n+1)^{2(n-1)\theta}}{n^{n(1-\theta)}} \cdot \frac{(nV)^{2(n-1)\theta}}{(n-1)!^{2\theta}}.$$

When $\theta = 1$, inequalities (3.10) and (3.9) are the same, so inequality (3.10) is



Geometric Inequalities for a Simplex

Shiguo Yang

Title Page	
Contents	
◀◀	▶▶
◀	▶
Go Back	
Close	
Quit	
Page 10 of 16	

valid for $\theta = 1$. For $\theta \in (0, 1)$, using inequality (3.9) we have

$$\begin{aligned}
 (3.11) \quad & \left(\frac{1}{n} \sum_{i=0}^n \lambda_i\right)^n \prod_{i=0}^n F_i^{2\theta} \\
 &= \left[\left(\frac{1}{n} \sum_{i=0}^n \lambda_i\right)^n \prod_{i=0}^n F_i^2\right]^\theta \cdot \left[\left(\frac{1}{n} \sum_{i=0}^n \lambda_i\right)^n\right]^{1-\theta} \\
 &\geq \left[\frac{(nV)^{2(n-1)}}{(n-1)!^2} \left(\prod_{i=0}^n \lambda_i\right) \left(\sum_{i=0}^n \frac{F_i^2}{\lambda_i}\right)\right]^\theta \cdot \left[\left(\frac{1}{n} \sum_{i=0}^n \lambda_i\right)^n\right]^{1-\theta}.
 \end{aligned}$$

By Maclaurin's inequality ([1]) we have

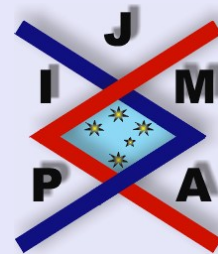
$$\left(\frac{1}{n+1} \sum_{i=0}^n \lambda_0 \cdots \lambda_{i-1} \lambda_{i+1} \cdots \lambda_n\right)^{\frac{1}{n}} \leq \frac{1}{n+1} \sum_{i=0}^n \lambda_i,$$

i.e.

$$(3.12) \quad \left(\frac{1}{n} \sum_{i=0}^n \lambda_i\right)^n \geq \frac{(n+1)^{n-1}}{n^n} \left(\prod_{i=0}^n \lambda_i\right) \left(\sum_{i=0}^n \frac{1}{\lambda_i}\right).$$

From (3.11) and (3.12) we can write

$$(3.13) \quad \left(\frac{1}{n} \sum_{i=0}^n \lambda_i\right)^n \prod_{i=0}^n F_i^{2\theta}$$



Geometric Inequalities for a Simplex

Shiguo Yang

Title Page

Contents

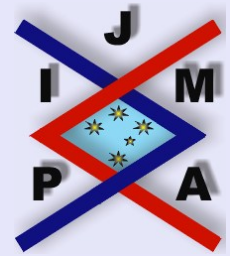


Go Back

Close

Quit

Page 11 of 16



$$\geq \left(\sum_{i=0}^n \lambda_i \right) \left[\sum_{i=0}^n \left(\frac{F_i^{2\theta}}{\lambda_i^\theta} \right)^{\frac{1}{\theta}} \right]^\theta \cdot \left[\sum_{i=0}^n \left(\frac{1}{\lambda_i^{1-\theta}} \right)^{\frac{1}{1-\theta}} \right]^{1-\theta} \\ \times \left[\frac{(n+1)^{n-1}}{n^n} \right]^{1-\theta} \left[\frac{(nV)^{2(n-1)}}{(n-1)!^2} \right]^\theta.$$

By Hölder's inequality ([1]) we have

$$(3.14) \quad \left[\sum_{i=0}^n \left(\frac{F_i^{2\theta}}{\lambda_i^\theta} \right)^{\frac{1}{\theta}} \right]^\theta \cdot \left[\sum_{i=0}^n \left(\frac{1}{\lambda_i^{1-\theta}} \right)^{\frac{1}{1-\theta}} \right]^{1-\theta} \geq \sum_{i=0}^n \frac{F_i^{2\theta}}{\lambda_i}.$$

Using (3.13) and (3.14) we get relation (3.9).

Taking $\lambda_i = d_i F_i$ ($i = 0, 1, \dots, n$) in equality (3.9) and noting the fact that $\sum_{i=0}^n d_i F_i = nV$, we get inequality (2.2). It is easy to prove that equality in (2.2) holds iff the simplex σ_n is regular and the point P is the circumcenter of σ_n . The proof of Theorem 2.2 is thus complete. \square

Proof of Theorem 2.5. Inequality (3.9) can be written also as

$$(3.15) \quad \frac{n^{3n}}{n!^2} V^{2(n-1)} \sum_{i=0}^n \lambda_0 \cdots \lambda_{i-1} \lambda_{i+1} \cdots \lambda_n F_i^2 \leq \left(\sum_{i=0}^n \lambda_i \right)^n \prod_{i=0}^n F_i^2.$$

Let V' denote the volume of the n -dimensional simplex $\sigma'_n = A'_0 A'_1 \cdots A'_n$, F'_i being the area of the i th face f'_i of σ'_n . By Cauchy's inequality and inequality

(3.15), we have

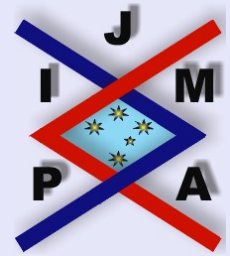
$$\begin{aligned}
 (3.16) \quad & \frac{n^{3n}}{n!^2} V^{n-1} (V')^{n-1} \sum_{i=0}^n \lambda_0 \cdots \lambda_{i-1} \lambda_{i+1} \cdots \lambda_n F_i F'_i \\
 & \leq \left[\frac{n^{3n}}{n!^2} V^{2(n-1)} \sum_{i=0}^n \lambda_0 \cdots \lambda_{i-1} \lambda_{i+1} \cdots \lambda_n F_i^2 \right]^{\frac{1}{2}} \\
 & \quad \times \left[\frac{n^{3n}}{n!^2} (V')^{2(n-1)} \sum_{i=0}^n \lambda_0 \cdots \lambda_{i-1} \lambda_{i+1} \cdots \lambda_n (F'_i)^2 \right]^{\frac{1}{2}} \\
 & \leq \left(\sum_{i=0}^n \lambda_i \right)^n \left(\prod_{i=0}^n F_i \right) \left(\prod_{i=0}^n F'_i \right).
 \end{aligned}$$

If we suppose that σ'_n is a regular simplex with $F'_0 = F'_1 = \cdots = F'_n = 1$. then

$$V' = (n+1)^{1/2} \left(\frac{n!^2}{n^{3n}} \right)^{\frac{1}{2(n-1)}},$$

so inequality (3.16) becomes

$$\begin{aligned}
 (3.17) \quad & \frac{(n+1)^{(n-1)/2} n^{3n/2}}{n!} V^{n-1} \sum_{i=0}^n \lambda_0 \cdots \lambda_{i-1} \lambda_{i+1} \cdots \lambda_n F_i \\
 & \leq \left(\sum_{i=0}^n \lambda_i \right)^n \prod_{i=0}^n F_i.
 \end{aligned}$$



Geometric Inequalities for a Simplex

Shiguo Yang

Title Page

Contents



Go Back

Close

Quit

Page 13 of 16

By letting $\lambda_0 = \lambda_1 = \dots = \lambda_n = 1$ in inequality (3.17), we get

$$(3.18) \quad \frac{1}{V} \geq \frac{n^{3n/2(n-1)}}{n!^{1/(n-1)}(n+1)^{(n+1)/2(n-1)}} \times \left(\sum_{i=0}^n \frac{1}{F_0 \cdots F_{i-1} F_{i+1} \cdots F_n} \right)^{\frac{1}{(n-1)}}.$$

Now by Cauchy's inequality we have

$$\left(\sum_{i=0}^n d_0 \cdots d_{i-1} d_{i+1} \cdots d_n \right) \left(\sum_{i=0}^n \frac{1}{d_0 \cdots d_{i-1} d_{i+1} \cdots d_n} \right) \geq (n+1)^2,$$

i.e.

$$(3.19) \quad \sum_{i=0}^n \frac{1}{d_0 \cdots d_{i-1} d_{i+1} \cdots d_n} \geq \frac{(n+1)^2}{\sum_{i=0}^n d_0 \cdots d_{i-1} d_{i+1} \cdots d_n}.$$

Using (3.19), (2.4) and (3.18), we get

$$(3.20) \quad \begin{aligned} & \sum_{i=0}^n \frac{1}{d_0 \cdots d_{i-1} d_{i+1} \cdots d_n} \\ & \geq \frac{(n+1)^{(n+3)/2} n^{n/2}}{n!} \cdot \frac{1}{V} \\ & \geq \frac{(n+1)^{(n^2+n-4)/2(n-1)} n^{n^2/2(n-1)}}{(n-1)!^{n/(n-1)}} \left(\sum_{i=0}^n F_i \right)^{\frac{1}{(n-1)}} \left(\prod_{i=0}^n F_i \right)^{\frac{1}{(n-1)}}. \end{aligned}$$



Geometric Inequalities for a Simplex

Shiguo Yang

Title Page

Contents



Go Back

Close

Quit

Page 14 of 16

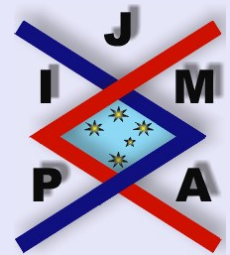
By inequality (3.20), formula $\sum_{i=0}^n F_i = \frac{nV}{r}$ and the known inequality ([1]):

$$(3.21) \quad \prod_{i=0}^n F_i \leq \frac{(n+1)^{(n^2-1)/2}}{n!^{n+1} n^{(n^2-3n-4)/2}} R^{n^2-1},$$

we get

$$(3.22) \quad \sum_{i=0}^n \frac{1}{d_0 \cdots d_{i-1} d_{i+1} \cdots d_n} \geq (n+1)^{(n-3)/2(n-1)} n^{(2n^2-n-2)/2(n-1)} \cdot n!^{1/(n-1)} \left(\frac{V}{r}\right)^{\frac{1}{(n-1)}} \cdot \frac{1}{R^{n+1}}.$$

Relations (3.22) and (3.4) imply inequality (2.6). It is easy to prove that equality in (2.6) holds iff the simplex σ_n is regular and the point P is the circumcenter of σ_n . The proof of Theorem 2.5 is thus complete. \square



Geometric Inequalities for a Simplex

Shiguo Yang

Title Page

Contents



Go Back

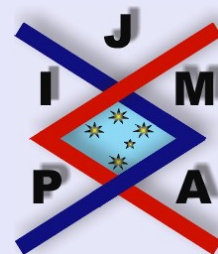
Close

Quit

Page 15 of 16

References

- [1] D.S. MITRINOVIĆ, J.E. PEČARIĆ AND V. VOLENEC, *Recent Advances in Geometric Inequalities*, Kluwer Acad. Publ., Dordrecht, Boston, London, 1989, 434–547.
- [2] Ju.I. GERASIMOV, Problem 848, *Mat. v Škole*, **4** (1971), 86.
- [3] L. GERBER, The orthocentric simplex as an extreme simplex, *Pacific. J. Math.*, **56** (1975), 97–111.
- [4] M.S. KLAMKIN, Problem 85–26, *SIAM. Rev.*, **27**(4) (1985), 576.
- [5] J.Zh. ZHANG AND L. YANG, A class of geometric inequalities concerning the masspoint system, *J. China Univ. Sci. Technol.*, **11**(2) (1981), 1–8.



Geometric Inequalities for a Simplex

Shiguo Yang

Title Page

Contents



Go Back

Close

Quit

Page 16 of 16