journal of inequalities in pure and applied mathematics
http://jipam.vu.edu.au
issn: 1443-5756
Volume 9 (2008), Issue 3, Article 73, 7 pp.


# ON A GENERALIZATION OF LIPSCHITZ'S CLASSES

XH. Z. KRASNIQI
DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCES,
AVENUE "MOTHER TERESA" 5,
PRISHTINË, 10000, REPUBLIC OF KOSOVO
xheki00@hotmail.com

*Received 26 March, 2008; accepted 06 August, 2008*
*Communicated by H. Bor*

ABSTRACT. In this paper we obtain a generalization of Lipschitz's classes $\Lambda^m(\beta, p, r)$ defined in [1]. We give necessary conditions for even or odd functions with Fourier series to belong to the classes $\Lambda^m(p, r, \alpha)$. We also give sufficient conditions for even or odd functions with Fourier series to belong to the same classes.

*Key words and phrases:* Lipschitz classes, Fourier series.

2000 *Mathematics Subject Classification.* 42A20, 42A32.

## 1. DEFINITIONS AND USEFUL STATEMENTS

We consider the series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \tag{1.1}$$

or

$$\sum_{n=1}^{\infty} a_n \sin nx \tag{1.2}$$

where $a_n$ are Fourier coefficients of integrable function $f$.

**Definition 1.1.** We say that a function $f$ belongs to $WA^p$, $(1 < p < \infty)$ if

$$\sum_{n=1}^{\infty} n^{p-2} \left( \sum_{k=n}^{\infty} |\Delta a_k| \right)^p < +\infty$$

where $\Delta a_k = a_k - a_{k+1}$ (see [1]).

We say that any function $\alpha(t)$ is a function of type $\sigma$ (see [4]) if it is measurable in $[0, 1]$, integrable in $[\delta, 1]$ for each $\delta \in (0, 1)$, and there exist real numbers $C_{1,\alpha} > 0$, $\sigma$ and $\delta_0 \in (0, 1)$ such that

(1) $\alpha(t) \geq C_{1,\alpha}$, for all $t \in [0, 1]$;

096-08

(2) $\int_0^\delta \alpha(t)t^s dt < \infty$ for each $s > \sigma$ and $\delta \in (0, \delta_0)$;
(3) $\int_0^\delta \alpha(t)t^s dt = \infty$ for each $s < \sigma$ and $\delta \in (0, \delta_0)$, and

$$\int_0^\delta \alpha(t)t^\sigma dt \leq C_2\delta^\sigma \int_\delta^{2\delta} \alpha(t)dt.$$

In [1], the classes $\Lambda^m(\beta, p, r, )$ are defined in the following way

**Definition 1.2.** $f \in \Lambda^m(\beta, p, r, )$, if

$$\|f\|_{\beta,p,r}^{(m)} \equiv \left\{ \int_0^1 \left[ \int_0^{2\pi} \frac{|\Delta_m f(x,t)|^p}{t^{\beta p}} dx \right]^{\frac{r}{p}} \frac{dt}{t} \right\}^{\frac{1}{r}} < +\infty,$$

where $1 < p < +\infty$, $1 \leq r < +\infty$, $\beta > 0$, $m \in \mathbb{N}$ and

$$\Delta_m f(x,t) = \sum_{i=1}^m (-1)^i C_m^i f[x + (m-2i)t].$$

Now we define classes $\Lambda^m(p, r, \alpha)$ as follows:

**Definition 1.3.** We say $f \in \Lambda^m(p, r, \alpha)$, if

$$\|f\|_{p,r,\alpha}^{(m)} \equiv \left\{ \int_0^1 \alpha(t) \left[ \int_0^{2\pi} |\Delta_m f(x,t)|^p \, dx \right]^{\frac{r}{p}} dt \right\}^{\frac{1}{r}} < +\infty,$$

where $\alpha(t)$ is function of the type $\sigma$.

For $\alpha(t) = t^{-r\beta-1}, \beta > 0$, we get the classes $\Lambda^m(\beta, p, r)$, considered in [1]. Therefore classes $\Lambda^m(p, r, \alpha)$ are generalizations of classes $\Lambda^m(\beta, p, r)$.

We need some auxiliary statements.

**Lemma 1.1** ([2]). *Let $a_\nu$, $b_\nu$ and $\beta_n$ be numbers such that $a_\nu \geq 0$, $b_\nu \geq 0$ and $\sum_{\nu=n}^\infty a_\nu = a_n\beta_n$:*

(1) *For $0 < p \leq 1$ the following inequality is valid*

$$\sum_{\nu=1}^\infty a_\nu \left( \sum_{\mu=1}^\nu b_\mu \right)^p \geq p^p \sum_{\nu=1}^\infty a_\nu \left(b_\nu \beta_\nu\right)^p;$$

(2) *For $1 \leq p < \infty$ we have*

$$\sum_{\nu=1}^\infty a_\nu \left( \sum_{\mu=1}^\nu b_\mu \right)^p \leq p^p \sum_{\nu=1}^\infty a_\nu \left(b_\nu \beta_\nu\right)^p.$$

**Lemma 1.2** ([2]). *Let $a_\nu$, $b_\nu$ and $\gamma_n$ be numbers such that $a_\nu \geq 0$, $b_\nu \geq 0$ and $\sum_{\nu=1}^n a_\nu = b_n\gamma_n$:*

(1) *For $0 < p \leq 1$, we have*

$$\sum_{\nu=1}^\infty a_\nu \left( \sum_{\mu=\nu}^\infty b_\mu \right)^p \geq p^p \sum_{\nu=1}^\infty a_\nu \left(b_\nu \gamma_\nu\right)^p;$$

(2) *For $1 \leq p < \infty$, we have*

$$\sum_{\nu=1}^\infty a_\nu \left( \sum_{\mu=\nu}^\infty b_\mu \right)^p \leq p^p \sum_{\nu=1}^\infty a_\nu \left(b_\nu \gamma_\nu\right)^p.$$

**Lemma 1.3** ([3]). *Let $\mu$, $\tau$ and $a_\nu$ be numbers such that $0<\mu<\tau<\infty$ and $a_\nu \geq 0$. Then*

$$\left(\sum_{\nu=1}^{\infty} a_\nu^\tau\right)^{\frac{1}{\tau}} \leq \left(\sum_{\nu=1}^{\infty} a_\nu^\mu\right)^{\frac{1}{\mu}}.$$

We denote by $C$ a constant that depends only on $m, p, r$ and may be different in different relations.

**Theorem 1.4** ([1]). *If $f \in WA^p$, $1<p<+\infty$, then*

$$\left\{\omega_p^{(m)}(h;f)\right\}^p \leq Ch^{mp} \sum_{n\leq\left[\frac{1}{h}\right]} n^{(m+1)p-2}\left(\sum_{k=n}^{\infty}|\Delta a_k|\right)^p + C\sum_{n>\left[\frac{1}{h}\right]} n^{p-2}\left(\sum_{k=n}^{\infty}|\Delta a_k|\right)^p,$$

*where $\omega_p^{(m)}(h;f)$ is the integral modulus of smoothness of order $m$.*

## 2. MAIN RESULTS

Let us denote

$$A(n) := \int_{1/(n+1)}^{1/n} \alpha(t)dt,$$

$$b(n) := b_1(n) + b_2(n) = n^{mr}\int_0^{1/n}\alpha(t)t^{mr}dt + \int_{1/(n+1)}^{1}\alpha(t)dt.$$

We have the following first main result.

**Theorem 2.1.** *Let $m$ be any natural number and*

$$f \in AW^p, \quad 1<p<+\infty, \quad 1\leq r<+\infty.$$

*If for the coefficients of series (1.1) or (1.2) we have $\sum_{k=1}^{\infty}|\Delta a_k| < +\infty$, then:*

(1) *For $p\leq r$ we have*

$$\|f\|_{p,r,\alpha}^{(m)} \leq C\left\{\sum_{n=1}^{\infty} n^{r\left(1-\frac{2}{p}\right)}\left(\sum_{k=n}^{\infty}|\Delta a_k|\right)^r b(n)\left(\frac{b(n)}{A(n)}\right)^{\frac{r}{p}-1}\right\}^{\frac{1}{r}};$$

(2) *For $p>r$ we have*

$$\|f\|_{p,r,\alpha}^{(m)} \leq C\left\{\sum_{n=1}^{\infty} n^{r\left(1-\frac{2}{p}\right)}\left(\sum_{k=n}^{\infty}|\Delta a_k|\right)^r b(n)\right\}^{\frac{1}{r}}.$$

*Proof.* Using the characteristics of the integral modulus of smoothness we have

$$\begin{aligned}\left\{\|f\|_{p,r,\alpha}^{(m)}\right\}^r &= \int_0^1 \alpha(t)\left[\int_0^{2\pi}|\Delta_m f(x,t)|^p dx\right]^{\frac{r}{p}} dt \\ &\leq \sum_{N=1}^{\infty}\int_{1/(N+1)}^{1/N}\alpha(t)\left[\omega_p^{(m)}(f;t)\right]^r dt \\ &\leq \sum_{N=1}^{\infty}\left[\omega_p^{(m)}(f;1/N)\right]^r\int_{1/(N+1)}^{1/N}\alpha(t)dt \\ &= \sum_{N=1}^{\infty} A(N)\left[\omega_p^{(m)}(f;1/N)\right]^r.\end{aligned}$$

According to the Theorem 1.4, we have

$$\begin{aligned}\left\{\|f\|_{p,r,\alpha}^{(m)}\right\}^r &\leq C\sum_{N=1}^{\infty}A(N)N^{-mr}\left\{\sum_{n=1}^{N}n^{(m+1)p-2}\left(\sum_{k=n}^{\infty}|\Delta a_k|\right)^p\right\}^{\frac{r}{p}}\\ &\quad + C\sum_{N=1}^{\infty}A(N)\left\{\sum_{n=N+1}^{\infty}n^{p-2}\left(\sum_{k=n}^{\infty}|\Delta a_k|\right)^p\right\}^{\frac{r}{p}}\\ &= I_1+I_2.\end{aligned}$$

Now we estimate $I_1$ and $I_2$. Let $r/p\geq 1$. Then according to Lemma 1.1 we have

$$I_1\leq C\sum_{N=1}^{\infty}A(N)N^{-mr}\left\{N^{(m+1)p-2}\left(\sum_{k=N}^{\infty}|\Delta a_k|\right)^p\beta_N\right\}^{\frac{r}{p}}.$$

Now we estimate the quantity $\beta_N$:

$$\begin{aligned}A(N)N^{-mr}\beta_N &= \sum_{i=N}^{\infty}A(i)i^{-mr}\\ &= \sum_{i=N}^{\infty}\left(\frac{i+1}{i}\right)^{mr}\cdot\frac{1}{(i+1)^{mr}}\int_{1/(i+1)}^{1/i}\alpha(t)dt\\ &\leq 2^{mr}\int_0^{1/N}\alpha(t)t^{mr}dt,\end{aligned}$$

or

$$\beta_N\leq C\frac{b_1(N)}{A(N)}.$$

Consequently

$$I_1\leq C\sum_{N=1}^{\infty}A(N)N^{r\left(1-\frac{2}{p}\right)}\left(\sum_{k=N}^{\infty}|\Delta a_k|\right)^r\left\{\frac{b_1(N)}{A(N)}\right\}^{\frac{r}{p}}. \tag{2.1}$$

According to Lemma 1.2, for $r/p\geq 1$ we have

$$I_2\leq C\sum_{N=1}^{\infty}A(N)\left\{N^{p-2}\left(\sum_{k=N}^{\infty}|\Delta a_k|\right)^p\gamma_N\right\}^{\frac{r}{p}}.$$

We estimate the quantity $\gamma_N$:

$$A(N)\gamma_N=\sum_{i=1}^{N}A(i)=\int_{1/(N+1)}^{1}\alpha(t)dt\Rightarrow\gamma_N=\frac{b_2(N)}{A(N)}.$$

Consequently

$$I_2\leq C\sum_{N=1}^{\infty}A(N)N^{r\left(1-\frac{2}{p}\right)}\left(\sum_{k=N}^{\infty}|\Delta a_k|\right)^r\left\{\frac{b_2(N)}{A(N)}\right\}^{\frac{r}{p}}. \tag{2.2}$$

By (2.1) and (2.2) we get

$$\left\{\|f\|_{p,r,\alpha}^{(m)}\right\}^r\leq C\sum_{N=1}^{\infty}A(N)N^{r\left(1-\frac{2}{p}\right)}\left(\sum_{k=N}^{\infty}|\Delta a_k|\right)^r\left\{\left(\frac{b_1(N)}{A(N)}\right)^{\frac{r}{p}}+\left(\frac{b_2(N)}{A(N)}\right)^{\frac{r}{p}}\right\}.$$

Finally, according to Lemma 1.3, for $r \geq p$ we have

$$\left\|f\right\|_{p,r,\alpha}^{(m)} \leq C \left\{ \sum_{N=1}^{\infty} A(N) N^{r\left(1-\frac{2}{p}\right)} \left( \sum_{k=N}^{\infty} |\Delta a_k| \right)^r \left( \frac{b(N)}{A(N)} \right)^{\frac{r}{p}} \right\}^{\frac{1}{r}}.$$

Now let $r/p < 1$. Then, according to Lemma 1.3, we have

$$I_1 \leq C \sum_{N=1}^{\infty} A(N) N^{-mr} \sum_{n=1}^{N} n^{(m+1)r-\frac{2r}{p}} \left( \sum_{k=n}^{\infty} |\Delta a_k| \right)^r.$$

If we change the order of summation we get

$$\begin{aligned} I_1 &\leq C \sum_{n=1}^{\infty} n^{(m+1)r-\frac{2r}{p}} \left( \sum_{k=n}^{\infty} |\Delta a_k| \right)^r \sum_{N=n}^{\infty} A(N) N^{-mr} \\ &\leq C \sum_{n=1}^{\infty} n^{r\left(1-\frac{2}{p}\right)} \left( \sum_{k=n}^{\infty} |\Delta a_k| \right)^r b_1(n). \qquad (2.3) \end{aligned}$$

Now we estimate $I_2$. Using Lemma 1.3 and changing the order of summation we have:

$$\begin{aligned} I_2 &\leq C \sum_{N=1}^{\infty} A(N) \sum_{n=N}^{\infty} n^{r\left(1-\frac{2}{p}\right)} \left( \sum_{k=n}^{\infty} |\Delta a_k| \right)^r \\ &= C \sum_{n=1}^{\infty} n^{r\left(1-\frac{2}{p}\right)} \left( \sum_{k=n}^{\infty} |\Delta a_k| \right)^r \sum_{N=1}^{n} A(N) \\ &= C \sum_{n=1}^{\infty} n^{r\left(1-\frac{2}{p}\right)} \left( \sum_{k=n}^{\infty} |\Delta a_k| \right)^r b_2(n). \qquad (2.4) \end{aligned}$$

From (2.3) and (2.4) we deduce

$$\left\|f\right\|_{p,r,\alpha}^{(m)} \leq C \left\{ \sum_{n=1}^{\infty} n^{r\left(1-\frac{2}{p}\right)} \left( \sum_{k=n}^{\infty} |\Delta a_k| \right)^r b(n) \right\}^{\frac{1}{r}},$$

which fully demonstrates Theorem 2.1.

□

**Theorem 2.2.** *Let $m$ be any natural number and*

$$1 < p \leq 2, \quad 1 \leq r < +\infty, \quad 1/p + 1/q = 1.$$

*If $a_n$ are the coefficients of series $(1.1)$ or $(1.2)$, then:*

(1) *For $r \leq q$ we have*

$$\left\|f\right\|_{p,r,\alpha}^{(m)} \geq C \left\{ \sum_{n=1}^{\infty} n^{-mr} \left|a_n\right|^r b_1(n) \left[ \frac{b_1(n)}{A(n)} \right]^{\frac{r}{q}-1} \right\}^{\frac{1}{r}};$$

(2) *For $r > q$ we have*

$$\left\|f\right\|_{p,r,\alpha}^{(m)} \geq C \left\{ \sum_{n=1}^{\infty} n^{-mr} \left|a_n\right|^r b_1(n) \right\}^{\frac{1}{r}}.$$

*Proof.* Let $f$ be an even function. If $f$ is an odd function then the proof of the theorem is analogous to the even case. It is not difficult to see that the Fourier series of $\Delta_m f(x,t)$ is

$$\Delta_m f(x,t) \sim \begin{cases} (-1)^{\frac{m}{2}} 2^m \sum_{n=1}^{\infty} a_n \cos nx \sin^m nt, & \textit{for } m \textit{ even} \\ (-1)^{\frac{m-1}{2}-1} 2^m \sum_{n=1}^{\infty} a_n \sin nx \sin^m nt, & \textit{for } m \textit{ odd}. \end{cases}$$

According to the well-known Hausdorf-Young's theorem we find

$$C\left(\int_0^{2\pi} |\Delta_m f(x,t)|^p \, dx\right)^{\frac{r}{p}} \geq \left(\sum_{n=1}^{\infty} |a_n|^q |\sin nt|^{mq}\right)^{\frac{r}{q}},$$

and then

$$\left\{\|f\|_{p,r,\alpha}^{(m)}\right\}^r \geq C\sum_{\nu=1}^{\infty}\int_{1/(\nu+1)}^{1/\nu} \alpha(t)\left(\sum_{n=1}^{\nu} |a_n|^q |\sin nt|^{mq}\right)^{\frac{r}{q}} dt.$$

Using the well-known inequality $\sin B \geq \frac{2}{\pi}B$ for $0 \leq B \leq \frac{\pi}{2}$, we get

$$\left\{\|f\|_{p,r,\alpha}^{(m)}\right\}^r \geq C\sum_{\nu=1}^{\infty} A(\nu)\left(\sum_{n=1}^{\nu} n^{-mq} |a_n|^q\right)^{\frac{r}{q}}.$$

Let $r \leq q$, then according to Lemma 1.1 we have

$$\left\{\|f\|_{p,r,\alpha}^{(m)}\right\}^r \geq C\sum_{\nu=1}^{\infty} A(\nu)\left[\nu^{-mq} |a_\nu|^q \beta_\nu\right]^{\frac{r}{q}}.$$

It is easy to prove that $\beta_\nu \geq \frac{b_1(\nu)}{A(\nu)}$, from which we get

$$\left\{\|f\|_{p,r,\alpha}^{(m)}\right\}^r \geq C\sum_{\nu=1}^{\infty} \nu^{-mr} |a_\nu|^r b_1(\nu)\left[\frac{b_1(\nu)}{A(\nu)}\right]^{\frac{r}{q}-1}. \tag{2.5}$$

Let $q < r$, then according to Lemma 1.3 and with the change of the order of summation we have

$$\begin{aligned} \left\{\|f\|_{p,r,\alpha}^{(m)}\right\}^r &\geq C\sum_{\nu=1}^{\infty} A(\nu)\sum_{n=1}^{\nu} n^{-mr} |a_n|^r \\ &= C\sum_{n=1}^{\infty} n^{-mr} |a_n|^r \sum_{\nu=n}^{\infty} A(\nu) \\ &\geq C\sum_{n=1}^{\infty} n^{-mr} |a_n|^r b_1(n). \end{aligned} \tag{2.6}$$

Relations (2.5) and (2.6) prove Theorem 2.2. □

We can deduce three corollaries from *Theorem 2.1* and *Theorem 2.2*.

**Corollary 2.3.** *Under the conditions of Theorem 2.1 and with $b(n) \leq CA(n)$, we have*

$$\|f\|_{p,r,\alpha}^{(m)} \leq C\left\{\sum_{n=1}^{\infty} n^{r\left(1-\frac{2}{p}\right)}\left(\sum_{k=n}^{\infty} |\Delta a_k|\right)^r b(n)\right\}^{\frac{1}{r}}.$$

**Corollary 2.4.** *Under the conditions of Theorem 2.2 and with* $b_1(n) \le CA(n)$*, we have*

$$\|f\|_{p,r,\alpha}^{(m)} \ge C \left\{ \sum_{n=1}^{\infty} n^{-mr} |a_n|^r \, b_1(n) \right\}^{\frac{1}{r}}.$$

As a special case, for $\alpha(t) = t^{-\beta r - 1}$, it is easy to prove the estimates:

$$A(n) \le Cn^{\beta r - 1} \quad \text{and} \qquad b(n) \le Cn^{\beta r}.$$

From Theorem 2.1 and the last estimates we can deduce the following result proved in [1].

**Corollary 2.5** ([1])**.** *Let* $m$ *be any natural number and*

$$0 < \beta \le m, \quad 1 < p < +\infty, \quad 1 \le r < +\infty, \quad 1/p + 1/q = 1.$$

*If the coefficients of series (1.1) or (1.2) satisfy* $\sum_{k=1}^{\infty} |\Delta a_k| < +\infty$*, then*

$$\|f\|_{\beta,p,r}^{(m)} \le C \left\{ \sum_{n=1}^{\infty} n^{r\left(\beta + \frac{1}{q}\right) - 1} \left( \sum_{k=n}^{\infty} |\Delta a_k| \right)^r \right\}^{\frac{1}{r}}.$$

## REFERENCES


[1] T.Sh. TEVZADZE, Some classes of functions and trigonometric Fourier series, *Some Questions of Function Theory*, v. II, 31–92, Tbilisi University Press, 1981 (in Russian).

[2] M.K. POTAPOV AND M. BERISHA, Moduli of smoothnes and Fourier coefficients of functions of one variable, *Publ. Inst. Math.* (Beograd) (N.S.), **26**(40) (1979), 215–228 (in Russian).

[3] B. HARDY, E. LITLEWOOD AND G. POLYA, *Inequalities*, GIIL Moscow, 1948, 1–456 (in Russian).

[4] M.K. POTAPOV, A certain imbedding theorem, *Mathematica* (Cluj), **14**(37) (1972), 123–146.