journal of inequalities in pure and applied mathematics
http://jipam.vu.edu.au
issn: 1443-5756
Volume 9 (2008), Issue 3, Article 88, 23 pp.

HEISENBERG-PAULI-WEYL UNCERTAINTY PRINCIPLE FOR THE RIEMANN-LIOUVILLE OPERATOR

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Received 29 March, 2008; accepted 08 August, 2008
Communicated by J.M. Rassias

ABSTRACT. The Heisenberg-Pauli-Weyl inequality is established for the Fourier transform connected with the Riemann-Liouville operator.Also, a generalization of this inequality is proved. Lastly, a local uncertainty principle is studied.

Key words and phrases: Heisenberg-Pauli-Weyl Inequality, Riemann-Liouville operator, Fourier transform, local uncertainty principle.

2000 *Mathematics Subject Classification.* 42B10, 33C45.

1. INTRODUCTION

Uncertainty principles play an important role in harmonic analysis and have been studied by many authors and from many points of view [8].These principles state that a function f and its Fourier transform $\widehat{f}$ cannot be simultaneously sharply localized. The theorems of Hardy, Morgan, Beurling, ... are established for several Fourier transforms in [4], [9], [13] and [14]. In this context, a remarkable Heisenberg uncertainty principle [10] states, according to Weyl [25] who assigned the result to Pauli, that for all square integrable functions f on $\mathbb{R}^n$ with respect to the Lebesgue measure, we have

$$\left(\int_{\mathbb{R}^n} x_j^2|f(x)|^2dx\right)\left(\int_{\mathbb{R}^n} \xi_j^2|\widehat{f}(\xi)|^2d\xi\right) \geq \frac{1}{4}\left(\int_{\mathbb{R}^n}|f(x)|^2dx\right)^2, \quad j\in\{1,\dots,n\}.$$

This inequality is called the Heisenberg-Pauli-Weyl inequality for the classical Fourier transform.

Recently, many works have been devoted to establishing the Heisenberg-Pauli-Weyl inequality for various Fourier transforms, Rösler [21] and Shimeno [22] have proved this inequality for the Dunkl transform, in [20] Rösler and Voit have established an analogue of the Heisenberg-Pauli-Weyl inequality for the generalized Hankel transform. In the same context, Battle [3] has proved this inequality for wavelet states, and Wolf [26], has studied this uncertainty principle

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for Gelfand pairs. We cite also De Bruijn [5] who has established the same result for the classical Fourier transform by using Hermite Polynomials, and Rassias [17, 18, 19] who gave several generalized forms for the Heisenberg-Pauli-Weyl inequality.

In [2], the second author with others considered the singular partial differential operators defined by

$$\begin{cases} \Delta_1 = \frac{\partial}{\partial x}, \\ \\ \Delta_2 = \frac{\partial^2}{\partial r^2} + \frac{2\alpha+1}{r}\frac{\partial}{\partial r} - \frac{\partial^2}{\partial x^2};\ (r,x) \in]0,+\infty[\times\mathbb{R};\ \alpha \geq 0. \end{cases}$$

and they associated to Δ_1 and Δ_2 the following integral transform, called the Riemann-Liouville operator, defined on $\mathscr{C}_*(\mathbb{R}^2)$ (the space of continuous functions on $\mathbb{R}^2$, even with respect to the first variable) by

$$\mathscr{R}_\alpha(f)(r,x) = \begin{cases} \frac{\alpha}{\pi}\int_{-1}^{1}\int_{-1}^{1} f\left(rs\sqrt{1-t^2}, x+rt\right)(1-t^2)^{\alpha-\frac{1}{2}}(1-s^2)^{\alpha-1}dtds; & \text{if } \alpha > 0, \\ \\ \frac{1}{\pi}\int_{-1}^{1} f\left(r\sqrt{1-t^2}, x+rt\right)\frac{dt}{\sqrt{(1-t^2)}}; & \text{if } \alpha = 0. \end{cases}$$

In addition, a convolution product and a Fourier transform $\mathscr{F}_\alpha$ connected with the mapping $\mathscr{R}_\alpha$ have been studied and many harmonic analysis results have been established for the Fourier transform $\mathscr{F}_\alpha$ (Inversion formula, Plancherel formula, Paley-Winer and Plancherel theorems, ...).

Our purpose in this work is to study the Heisenberg-Pauli-Weyl uncertainty principle for the Fourier transform $\mathscr{F}_\alpha$ connected with $\mathscr{R}_\alpha$.More precisely, using Laguerre and Hermite polynomials we establish firstly the Heisenberg-Pauli-Weyl inequality for the Fourier transform $\mathscr{F}_\alpha$, that is

- For all $f \in L^2(d\nu_\alpha)$, we have

$$\left(\int_0^{+\infty}\int_{\mathbb{R}}(r^2+x^2)|f(r,x)|^2 d\nu_\alpha(r,x)\right)^{\frac{1}{2}} \times \left(\int\int_{\Gamma_+}(\mu^2+2\lambda^2)|\mathscr{F}_\alpha(f)(\mu,\lambda)|^2 d\gamma_\alpha(\mu,\lambda)\right)^{\frac{1}{2}} \geq \frac{2\alpha+3}{2}\left(\int_0^{+\infty}\int_{\mathbb{R}}|f(r,x)|^2 d\nu_\alpha(r,x)\right),$$

with equality if and only if

$$f(r,x) = Ce^{-\frac{r^2+x^2}{2t_0^2}};\ C \in \mathbb{C},\ t_0 > 0,$$

where

- $d\nu_\alpha(r,x)$ is the measure defined on $\mathbb{R}_+ \times \mathbb{R}$ by

$$d\nu_\alpha(r,x) = \frac{r^{2\alpha+1}}{2^\alpha\Gamma(\alpha+1)\sqrt{2\pi}}dr \otimes dx.$$

- $d\gamma_\alpha(\mu,\lambda)$ is the measure defined on the set

$$\Gamma_+ = \mathbb{R}_+ \times \mathbb{R} \cup \left\{(it,x);\ (t,x) \in \mathbb{R}_+ \times \mathbb{R};\ t \leq |x|\right\},$$

by

$$\int\int_{\Gamma_+} g(\mu,\lambda)d\gamma_\alpha(\mu,\lambda) = \frac{1}{2^\alpha\Gamma(\alpha+1)\sqrt{2\pi}}\left(\int_0^{+\infty}\int_{\mathbb{R}} g(\mu,\lambda)(\mu^2+\lambda^2)^\alpha \mu d\mu d\lambda \right.$$
$$\left. + \int_{\mathbb{R}}\int_0^{|\lambda|} g(i\mu,\lambda)(\lambda^2-\mu^2)^\alpha \mu d\mu d\lambda\right).$$

Next, we give a generalization of the Heisenberg-Pauli-Weyl inequality, that is

- For all $f \in L^2(d\nu_\alpha)$, $a, b \in \mathbb{R}$; $a, b \geq 1$ and $\eta \in \mathbb{R}$ such that $\eta a = (1-\eta)b$, we have

$$\left(\int_0^{+\infty}\int_{\mathbb{R}}(r^2+x^2)^a|f(r,x)|^2 d\nu_\alpha(r,x)\right)^{\frac{\eta}{2}}$$
$$\times\left(\int\int_{\Gamma_+}(\mu^2+2\lambda^2)^b|\mathscr{F}_\alpha(f)(\mu,\lambda)|^2 d\gamma_\alpha(\mu,\lambda)\right)^{\frac{1-\eta}{2}}$$
$$\geq \left(\frac{2\alpha+3}{2}\right)^{a\eta}\left(\int_0^{+\infty}\int_{\mathbb{R}}|f(r,x)|^2 d\nu_\alpha(r,x)\right)^{\frac{1}{2}},$$

with equality if and only if

$$a = b = 1 \quad \text{and} \quad f(r,x) = Ce^{-\frac{r^2+x^2}{2t_0^2}}; \quad C \in \mathbb{C}; \quad t_0 > 0.$$

In the last section of this paper, building on the ideas of Faris [7], and Price [15, 16], we develop a family of inequalities in their sharpest forms, which constitute the principle of local uncertainty.
Namely, we have established the following main results

- For all real numbers ξ; $0 < \xi < \frac{2\alpha+3}{2}$, there exists a positive constant $K_{\alpha,\xi}$ such that for all $f \in L^2(d\nu_\alpha)$, and for all measurable subsets $E \subset \Gamma_+$; $0 < \gamma_\alpha(E) < +\infty$, we have

$$\int\int_E |\mathscr{F}_\alpha(f)(\mu,\lambda)|^2 d\gamma_\alpha(\mu,\lambda) < K_{\alpha,\xi}\big(\gamma_\alpha(E)\big)^{\frac{2\xi}{2\alpha+3}}\int_0^{+\infty}\int_{\mathbb{R}}(r^2+x^2)^\xi|f(r,x)|^2 d\nu_\alpha(r,x).$$

- For all real number ξ; $\xi > \frac{2\alpha+3}{2}$, there exists a positive constant $M_{\alpha,\xi}$ such that for all $f \in L^2(d\nu_\alpha)$, and for all measurable subsets $E \subset \Gamma_+$; $0 < \gamma_\alpha(E) < +\infty$, we have

$$\int\int_E |\mathscr{F}_\alpha(f)(\mu,\lambda)|^2 d\gamma_\alpha(\mu,\lambda) < M_{\alpha,\xi}\gamma_\alpha(E)\left(\int_0^{+\infty}\int_{\mathbb{R}}|f(r,x)|^2 d\nu_\alpha(r,x)\right)^{\frac{2\xi-2\alpha-3}{2\xi}}$$
$$\times\left(\int_0^{+\infty}\int_{\mathbb{R}}(r^2+x^2)^\xi|f(r,x)|^2 d\nu_\alpha(r,x)\right)^{\frac{2\alpha+3}{2\xi}},$$

where $M_{\alpha,\xi}$ is the best (the smallest) constant satisfying this inequality.

2. THE FOURIER TRANSFORM ASSOCIATED WITH THE RIEMANN-LIOUVILLE OPERATOR

It is well known [2] that for all $(\mu,\lambda)\in\mathbb{C}^2$, the system

$$\begin{cases}\Delta_1 u(r,x) = -i\lambda u(r,x),\\ \Delta_2 u(r,x) = -\mu^2 u(r,x),\\ u(0,0)=1,\ \frac{\partial u}{\partial r}(0,x)=0,\ \forall x\in\mathbb{R},\end{cases}$$

admits a unique solution $\varphi_{\mu,\lambda}$, given by

$$\forall (r,x)\in\mathbb{R}^2;\quad \varphi_{\mu,\lambda}(r,x) = j_\alpha\left(r\sqrt{\mu^2+\lambda^2}\right)e^{-i\lambda x}, \tag{2.1}$$

where

$$j_\alpha(x) = 2^\alpha\Gamma(\alpha+1)\frac{J_\alpha(x)}{x^\alpha} = \Gamma(\alpha+1)\sum_{n=0}^{+\infty}\frac{(-1)^n}{n!\Gamma(\alpha+n+1)}\left(\frac{x}{2}\right)^{2n},$$

and J_α is the Bessel function of the first kind and index α [6, 11, 12, 24].

The modified Bessel function j_α has the following integral representation [1, 11], for all $\mu\in\mathbb{C}$, and $r\in\mathbb{R}$ we have

$$j_\alpha(r\mu) = \begin{cases}\frac{2\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma\left(\alpha+\frac{1}{2}\right)}\int_0^1(1-t^2)^{\alpha-\frac{1}{2}}\cos(r\mu t)dt, & \text{if } \alpha>-\frac{1}{2};\\ \cos(r\mu), & \text{if } \alpha=-\frac{1}{2}.\end{cases}$$

In particular, for all $r,s\in\mathbb{R}$, we have

$$\begin{aligned}|j_\alpha(rs)| &\le \frac{2\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma\left(\alpha+\frac{1}{2}\right)}\int_0^1(1-t^2)^{\alpha-\frac{1}{2}}|\cos(rst)|dt\\ &\le \frac{2\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}\int_0^1(1-t^2)^{\alpha-\frac{1}{2}}dt = 1.\end{aligned} \tag{2.2}$$

From the properties of the Bessel function, we deduce that the eigenfunction $\varphi_{\mu,\lambda}$ satisfies the following properties

- $$\sup_{(r,x)\in\mathbb{R}^2}|\varphi_{\mu,\lambda}(r,x)| = 1, \tag{2.3}$$

  if and only if (μ,λ) belongs to the set

  $$\Gamma = \mathbb{R}^2\cup\{(it,x);\ (t,x)\in\mathbb{R}^2;\ |t|\le|x|\}.$$

- The eigenfunction $\varphi_{\mu,\lambda}$ has the following Mehler integral representation

$$\varphi_{\mu,\lambda}(r,x) = \begin{cases}\frac{\alpha}{\pi}\int_{-1}^1\int_{-1}^1\cos(\mu rs\sqrt{1-t^2})e^{-i\lambda(x+rt)}(1-t^2)^{\alpha-\frac{1}{2}}(1-s^2)^{\alpha-1}dtds; & \text{if } \alpha>0,\\ \frac{1}{\pi}\int_{-1}^1\cos(r\mu\sqrt{1-t^2})e^{-i\lambda(x+rt)}\frac{dt}{\sqrt{1-t^2}}; & \text{if } \alpha=0.\end{cases}$$

In [2], using this integral representation, the authors have defined the Riemann-Liouville integral transform associated with Δ_1,Δ_2 by

$$\mathscr{R}_\alpha(f)(r,x) = \begin{cases}\frac{\alpha}{\pi}\int_{-1}^1\int_{-1}^1 f\left(rs\sqrt{1-t^2},x+rt\right)(1-t^2)^{\alpha-\frac{1}{2}}(1-s^2)^{\alpha-1}dtds; & \text{if } \alpha>0,\\ \frac{1}{\pi}\int_{-1}^1 f\left(r\sqrt{1-t^2},x+rt\right)\frac{dt}{\sqrt{(1-t^2)}}; & \text{if } \alpha=0.\end{cases}$$

where f is a continuous function on $\mathbb{R}^2$, even with respect to the first variable.

The transform $\mathscr{R}_\alpha$ generalizes the "mean operator" defined by

$$\mathscr{R}_0(f)(r,x) = \frac{1}{2\pi}\int_0^{2\pi} f(r\sin\theta, x + r\cos\theta)d\theta.$$

In the following we denote by

- $d\nu_\alpha$ the measure defined on $\mathbb{R}_+ \times \mathbb{R}$, by

$$d\nu_\alpha(r,x) = \frac{r^{2\alpha+1}}{2^\alpha\Gamma(\alpha+1)\sqrt{2\pi}} dr \otimes dx.$$

- $L^p(d\nu_\alpha)$ the space of measurable functions f on $\mathbb{R}_+ \times \mathbb{R}$ such that

$$\|f\|_{p,\nu_\alpha} = \left(\int_0^{+\infty}\int_{\mathbb{R}} |f(r,x)|^p d\nu_\alpha(r,x)\right)^{\frac{1}{p}} < \infty, \quad \text{if } p \in [1,+\infty[,$$

$$\|f\|_{\infty,\nu_\alpha} = \operatorname*{ess\,sup}_{(r,x)\in\mathbb{R}_+\times\mathbb{R}} |f(r,x)| < \infty, \quad \text{if } p = +\infty.$$

- $\langle \ / \ \rangle_{\nu_\alpha}$ the inner product defined on $L^2(d\nu_\alpha)$ by

$$\langle f/g\rangle_{\nu_\alpha} = \int_0^{+\infty}\int_{\mathbb{R}} f(r,x)\overline{g(r,x)}d\nu_\alpha(r,x).$$

- $\Gamma_+ = \mathbb{R}_+ \times \mathbb{R} \cup \big\{(it,x);\ (t,x) \in \mathbb{R}_+ \times \mathbb{R};\ t \le |x|\big\}$.
- $\mathscr{B}_{\Gamma_+}$ the σ-algebra defined on Γ_+ by

$$\mathscr{B}_{\Gamma_+} = \{\theta^{-1}(B),\ B \in \mathscr{B}(\mathbb{R}_+ \times \mathbb{R})\},$$

  where θ is the bijective function defined on the set Γ_+ by

$$\theta(\mu,\lambda) = \left(\sqrt{\mu^2+\lambda^2}, \lambda\right). \tag{2.4}$$

- $d\gamma_\alpha$ the measure defined on $\mathscr{B}_{\Gamma_+}$ by

$$\forall\ A \in \mathscr{B}_{\Gamma_+};\ \gamma_\alpha(A) = \nu_\alpha(\theta(A)) \tag{2.5}$$

- $L^p(d\gamma_\alpha)$ the space of measurable functions f on Γ_+, such that

$$\|f\|_{p,\gamma_\alpha} = \left(\iint_{\Gamma_+} |f(\mu,\lambda)|^p d\gamma_\alpha(\mu,\lambda)\right)^{\frac{1}{p}} < \infty, \quad \text{if } p \in [1,+\infty[,$$

$$\|f\|_{\infty,\gamma_\alpha} = \operatorname*{ess\,sup}_{(\mu,\lambda)\in\Gamma_+} |f(\mu,\lambda)| < \infty, \quad \text{if } p = +\infty.$$

- $\langle \ / \ \rangle_{\gamma_\alpha}$ the inner product defined on $L^2(d\gamma_\alpha)$ by

$$\langle f/g\rangle_{\gamma_\alpha} = \iint_{\Gamma_+} f(\mu,\lambda)\overline{g(\mu,\lambda)}d\gamma_\alpha(\mu,\lambda).$$

  Then, we have the following properties.

Proposition 2.1.

i) *For all non negative measurable functions g on Γ_+, we have*

$$\iint_{\Gamma_+} g(\mu,\lambda)d\gamma_\alpha(\mu,\lambda) = \frac{1}{2^\alpha\Gamma(\alpha+1)\sqrt{2\pi}}\left(\int_0^{+\infty}\int_{\mathbb{R}} g(\mu,\lambda)(\mu^2+\lambda^2)^\alpha \mu d\mu d\lambda + \int_{\mathbb{R}}\int_0^{|\lambda|} g(i\mu,\lambda)(\lambda^2-\mu^2)^\alpha \mu d\mu d\lambda\right). \tag{2.6}$$

In particular

$$d\gamma_{\alpha+1}(\mu,\lambda)=\frac{\mu^2+\lambda^2}{2(\alpha+1)}d\gamma_\alpha(\mu,\lambda). \tag{2.7}$$

ii) *For all measurable functions f on $\mathbb{R}_+\times\mathbb{R}$, the function $f\circ\theta$ is measurable on Γ_+. Furthermore, if f is non negative or an integrable function on $\mathbb{R}_+\times\mathbb{R}$ with respect to the measure $d\nu_\alpha$, then we have*

$$\int\int_{\Gamma_+}(f\circ\theta)(\mu,\lambda)d\gamma_\alpha(\mu,\lambda)=\int_0^{+\infty}\int_{\mathbb{R}}f(r,x)d\nu_\alpha(r,x). \tag{2.8}$$

In the following, we shall define the Fourier transform $\mathscr{F}_\alpha$ associated with the operator $\mathscr{R}_\alpha$ and we give some properties that we use in the sequel.

Definition 2.1. The Fourier transform $\mathscr{F}_\alpha$ associated with the Riemann-liouville operator $\mathscr{R}_\alpha$ is defined on $L^1(d\nu_\alpha)$ by

$$\forall(\mu,\lambda)\in\Gamma;\ \mathscr{F}_\alpha(f)(\mu,\lambda)=\int_0^{+\infty}\int_{\mathbb{R}}f(r,x)\varphi_{\mu,\lambda}(r,x)d\nu_\alpha(r,x). \tag{2.9}$$

By the relation (2.3), we deduce that the Fourier transform $\mathscr{F}_\alpha$ is a bounded linear operator from $L^1(d\nu_\alpha)$ into $L^\infty(d\gamma_\alpha)$, and that for all $f\in L^1(d\nu_\alpha)$, we have

$$\|\mathscr{F}_\alpha(f)\|_{\infty,\gamma_\alpha}\leq\|f\|_{1,\nu_\alpha}. \tag{2.10}$$

Theorem 2.2 (Inversion formula)**.** *Let $f\in L^1(d\nu_\alpha)$ such that $\mathscr{F}_\alpha(f)\in L^1(d\gamma_\alpha)$, then for almost every $(r,x)\in\mathbb{R}_+\times\mathbb{R}$, we have*

$$f(r,x)=\int\int_{\Gamma_+}\mathscr{F}_\alpha(f)(\mu,\lambda)\overline{\varphi_{\mu,\lambda}(r,x)}d\gamma_\alpha(\mu,\lambda).$$

Theorem 2.3 (Plancherel)**.** *The Fourier transform $\mathscr{F}_\alpha$ can be extended to an isometric isomorphism from $L^2(d\nu_\alpha)$ onto $L^2(d\gamma_\alpha)$.*

In particular, for all $f,g\in L^2(d\nu_\alpha)$, we have the following Parseval's equality

$$\int\int_{\Gamma_+}\mathscr{F}_\alpha(f)(\mu,\lambda)\overline{\mathscr{F}_\alpha(g)(\mu,\lambda)}d\gamma_\alpha(\mu,\lambda)=\int_0^{+\infty}\int_{\mathbb{R}}f(r,x)\overline{g(r,x)}d\nu_\alpha(r,x). \tag{2.11}$$

3. HILBERT BASIS OF THE SPACES $L^2(d\nu_\alpha)$, AND $L^2(d\gamma_\alpha)$

In this section, using Laguerre and Hermite polynomials, we construct a Hilbert basis of the spaces $L^2(d\nu_\alpha)$ and $L^2(d\gamma_\alpha)$, and establish some intermediate results that we need in the next section.

It is well known [11, 23] that for every $\alpha\geq0$, the Laguerre polynomials L_m^α are defined by the following Rodriguez formula

$$L_m^\alpha(r)=\frac{1}{m!}e^r r^{-\alpha}\frac{d^m}{dr^m}\left(r^{m+\alpha}e^{-r}\right);\qquad m\in\mathbb{N}.$$

Also, the Hermite polynomials are defined by the Rodriguez formula

$$H_n(x)=(-1)^n e^{x^2}\frac{d^n}{dx^n}\left(e^{-x^2}\right);\qquad n\in\mathbb{N}.$$

Moreover, the families

$$\left\{\sqrt{\frac{m!}{\Gamma(\alpha+m+1)}}L_m^\alpha\right\}_{m\in\mathbb{N}}\quad\text{and}\quad\left\{\sqrt{\frac{1}{2^n n!\sqrt{\pi}}}H_n\right\}_{n\in\mathbb{N}}$$

are respectively a Hilbert basis of the Hilbert spaces $L^2(\mathbb{R}_+, e^{-r}r^\alpha dr)$ and $L^2(\mathbb{R}, e^{-x^2}dx)$.

Therefore the families

$$\left\{\sqrt{\frac{2^{\alpha+1}\Gamma(\alpha+1)m!}{\Gamma(\alpha+m+1)}}e^{-\frac{r^2}{2}}L_m^\alpha(r^2)\right\}_{m\in\mathbb{N}} \quad \text{and} \quad \left\{\sqrt{\frac{1}{2^n n!\sqrt{\pi}}}e^{-\frac{x^2}{2}}H_n\right\}_{n\in\mathbb{N}}$$

are respectively a Hilbert basis of the Hilbert spaces $L^2\left(\mathbb{R}_+, \frac{r^{2\alpha+1}}{2^\alpha\Gamma(\alpha+1)}dr\right)$ and $L^2\left(\mathbb{R}, \frac{dx}{\sqrt{2\pi}}\right)$, hence the family $\left\{e_{m,n}^\alpha\right\}_{(m,n)\in\mathbb{N}^2}$ defined by

$$e_{m,n}^\alpha(r,x) = \left(\frac{2^{\alpha+1}\Gamma(\alpha+1)m!}{2^{n-\frac{1}{2}}n!\Gamma(m+\alpha+1)}\right)^{\frac{1}{2}} e^{-\frac{r^2+x^2}{2}}L_m^\alpha(r^2)H_n(x),$$

is a Hilbert basis of the space $L^2(d\nu_\alpha)$.

Using the relation (2.8), we deduce that the family $\left\{\xi_{m,n}^\alpha\right\}_{(m,n)\in\mathbb{N}^2}$, defined by

$$\xi_{m,n}^\alpha(\mu,\lambda) = (e_{m,n}^\alpha \circ \theta)(\mu,\lambda) = \left(\frac{2^{\alpha+1}\Gamma(\alpha+1)m!}{2^{n-\frac{1}{2}}n!\Gamma(m+\alpha+1)}\right)^{\frac{1}{2}} e^{-\frac{\mu^2+2\lambda^2}{2}}L_m^\alpha(\mu^2+\lambda^2)H_n(\lambda),$$

is a Hilbert basis of the space $L^2(d\gamma_\alpha)$, where θ is the function defined by the relation (2.4).

In the following, we agree that the Laguerre and Hermite polynomials with negative index are zero.

Proposition 3.1. *For all* $(m,n)\in\mathbb{N}^2$, $(r,x)\in\mathbb{R}_+\times\mathbb{R}$ *and* $(\mu,\lambda)\in\Gamma_+$, *we have*

$$x e_{m,n}^\alpha(r,x) = \sqrt{\frac{n+1}{2}}e_{m,n+1}^\alpha(r,x) + \sqrt{\frac{n}{2}}e_{m,n-1}^\alpha(r,x). \tag{3.1}$$

$$\lambda\xi_{m,n}^\alpha(\mu,\lambda) = \sqrt{\frac{n+1}{2}}\xi_{m,n+1}^\alpha(\mu,\lambda) + \sqrt{\frac{n}{2}}\xi_{m,n-1}^\alpha(\mu,\lambda). \tag{3.2}$$

$$r^2 e_{m,n}^{\alpha+1}(r,x) = \sqrt{2(\alpha+1)(\alpha+m+1)}e_{m,n}^\alpha(r,x) - \sqrt{2(\alpha+1)(m+1)}e_{m+1,n}^\alpha(r,x). \tag{3.3}$$

$$\begin{aligned}(\mu^2+\lambda^2)\xi_{m,n}^{\alpha+1}(\mu,\lambda) = \sqrt{2(\alpha+1)(\alpha+m+1)}&\xi_{m,n}^\alpha(\mu,\lambda)\\ &- \sqrt{2(\alpha+1)(m+1)}\xi_{m+1,n}^\alpha(\mu,\lambda).\end{aligned} \tag{3.4}$$

Proof. We know [11] that the Hermite polynomials satisfy the following recurrence formula

$$H_{n+1}(x) - 2xH_n(x) + 2nH_{n-1}(x) = 0; \quad n\in\mathbb{N},$$

Therefore, for all $(r,x)\in\mathbb{R}_+\times\mathbb{R}$, we have

$$\begin{aligned}
xe^{\alpha}_{m,n}(r,x) &= \left(\frac{2^{\alpha+1}\Gamma(\alpha+1)m!}{2^{n-\frac{1}{2}}n!\Gamma(m+\alpha+1)}\right)^{\frac{1}{2}} e^{-\frac{r^2+x^2}{2}} L^{\alpha}_m(r^2)xH_n(x)\\
&= \sqrt{\frac{n+1}{2}}\left(\frac{2^{\alpha+1}\Gamma(\alpha+1)m!}{2^{n+\frac{1}{2}}(n+1)!\Gamma(m+\alpha+1)}\right)^{\frac{1}{2}} e^{-\frac{r^2+x^2}{2}} L^{\alpha}_m(r^2)H_{n+1}(x)\\
&\quad + \sqrt{\frac{n}{2}}\left(\frac{2^{\alpha+1}\Gamma(\alpha+1)m!}{2^{n-\frac{3}{2}}(n-1)!\Gamma(m+\alpha+1)}\right)^{\frac{1}{2}} e^{-\frac{|(r,x)|^2}{2}} L^{\alpha}_m(r^2)H_{n-1}(x)\\
&= \sqrt{\frac{n+1}{2}}e^{\alpha}_{m,n+1}(r,x)+\sqrt{\frac{n}{2}}e^{\alpha}_{m,n-1}(r,x)
\end{aligned}$$

and it is obvious that the same relation holds for the elements $\xi^{\alpha}_{m,n}$.

On the other hand

$$r^2e^{\alpha+1}_{m,n}(r,x) = \left(\frac{2^{\alpha+2}\Gamma(\alpha+2)m!}{2^{n-\frac{1}{2}}n!\Gamma(m+\alpha+2)}\right)^{\frac{1}{2}} e^{-\frac{r^2+x^2}{2}} r^2L^{\alpha+1}_m(r^2)H_n(x).$$

However, the Laguerre polynomials satisfy the following recurrence formulas

$$(m+1)L^{\alpha}_{m+1}(r)+(r-\alpha-2m-1)L^{\alpha}_m(r)+(m+\alpha)L^{\alpha}_{m-1}(r)=0;\quad m\in\mathbb{N},$$

and

$$L^{\alpha+1}_m(r)-L^{\alpha+1}_{m-1}(r)=L^{\alpha}_m(r);\quad m\in\mathbb{N}.$$

Hence, we deduce that

$$\begin{aligned}
&r^2e^{\alpha+1}_{m,n}(r,x)\\
&= \left(\frac{2^{\alpha+2}\Gamma(\alpha+2)m!}{2^{n-\frac{1}{2}}n!\Gamma(m+\alpha+2)}\right)^{\frac{1}{2}} e^{-\frac{r^2+x^2}{2}} H_n(x)\\
&\quad\times\left((\alpha+2m+2)L^{\alpha+1}_m(r^2)-(m+1)L^{\alpha+1}_{m+1}(r^2)-(\alpha+m+1)L^{\alpha+1}_{m-1}(r^2)\right)\\
&= \left(\frac{2^{\alpha+2}\Gamma(\alpha+2)m!}{2^{n-\frac{1}{2}}n!\Gamma(m+\alpha+2)}\right)^{\frac{1}{2}} (\alpha+m+1)e^{-\frac{r^2+x^2}{2}} L^{\alpha}_m(r^2)H_n(x)\\
&\quad-\left(\frac{2^{\alpha+2}\Gamma(\alpha+2)m!}{2^{n-\frac{1}{2}}n!\Gamma(m+\alpha+2)}\right)^{\frac{1}{2}} (m+1)e^{-\frac{r^2+x^2}{2}} L^{\alpha}_{m+1}(r^2)H_n(x)\\
&= \sqrt{2(\alpha+1)(\alpha+m+1)}e^{\alpha}_{m,n}(r,x)-\sqrt{2(\alpha+1)(m+1)}e^{\alpha}_{m+1,n}(r,x).
\end{aligned}$$

□

Proposition 3.2. *For all* $(m,n)\in\mathbb{N}^2$, *and* $(\mu,\lambda)\in\Gamma_+$, *we have*

$$\mathscr{F}_{\alpha}(e^{\alpha}_{m,n})(\mu,\lambda)=(-i)^{2m+n}\xi^{\alpha}_{m,n}(\mu,\lambda). \tag{3.5}$$

Proof. It is clear that for all $(m,n) \in \mathbb{N}^2$, the function $e^{\alpha}_{m,n}$ belongs to the space $L^1(d\nu_\alpha) \cap L^2(d\nu_\alpha)$, hence by using Fubini's theorem, we get

$$\mathscr{F}_\alpha(e^{\alpha}_{m,n})(\mu,\lambda) = \left(\frac{2^{\alpha+1}\Gamma(\alpha+1)m!}{2^{n-\frac{1}{2}}n!\Gamma(m+\alpha+1)}\right)^{\frac{1}{2}} \times \left(\int_0^{+\infty} e^{-\frac{r^2}{2}} L^{\alpha}_m(r^2) j_\alpha\left(r\sqrt{\mu^2+\lambda^2}\right) \frac{r^{2\alpha+1}}{2^\alpha\Gamma(\alpha+1)}dr\right) \times \left(\int_{\mathbb{R}} e^{-\frac{x^2}{2}-i\lambda x} H_n(x) \frac{dx}{\sqrt{2\pi}}\right),$$

and then the required result follows from the following equalities [11]:

$$\forall m \in \mathbb{N};\ \int_0^{+\infty} e^{-\frac{r}{2}} L^{\alpha}_m(r) J_\alpha(\sqrt{ry}) r^{\frac{\alpha}{2}} dr = (-1)^m 2 e^{-\frac{y}{2}} y^{\frac{\alpha}{2}} L^{\alpha}_m(y),$$

and

$$\forall n \in \mathbb{N};\ \int_{\mathbb{R}} e^{ixy} e^{-\frac{x^2}{2}} H_n(x)dx = i^n \sqrt{2\pi} e^{-\frac{y^2}{2}} H_n(y),$$

where J_α denotes the Bessel function of the first kind and index α defined for all $x > 0$ by

$$J_\alpha(x) = \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!\Gamma(\alpha+n+1)} \left(\frac{x}{2}\right)^{2n+\alpha}.$$

□

Proposition 3.3. *Let* $f \in L^2(d\nu_\alpha) \cap L^2(d\nu_{\alpha+1})$ *such that* $\mathscr{F}_\alpha(f) \in L^2(d\gamma_{\alpha+1})$*, then for all* $(m,n) \in \mathbb{N}^2$*, we have*

$$\langle f/e^{\alpha+1}_{m,n}\rangle_{\nu_{\alpha+1}} = \sqrt{\frac{\alpha+m+1}{2(\alpha+1)}}\langle f/e^{\alpha}_{m,n}\rangle_{\nu_\alpha} - \sqrt{\frac{m+1}{2(\alpha+1)}}\langle f/e^{\alpha}_{m+1,n}\rangle_{\nu_\alpha}, \tag{3.6}$$

and

$$\langle \mathscr{F}_\alpha(f)/\xi^{\alpha+1}_{m,n}\rangle_{\gamma_{\alpha+1}} = \sqrt{\frac{\alpha+m+1}{2(\alpha+1)}}(-i)^{2m+n}\langle f/e^{\alpha}_{m,n}\rangle_{\nu_\alpha} + \sqrt{\frac{m+1}{2(\alpha+1)}}(-i)^{2m+n}\langle f/e^{\alpha}_{m+1,n}\rangle_{\nu_\alpha}. \tag{3.7}$$

Proof. We have

$$\begin{aligned}\langle f/e^{\alpha+1}_{m,n}\rangle_{\nu_{\alpha+1}} &= \int_0^{+\infty}\int_{\mathbb{R}} f(r,x) e^{\alpha+1}_{m,n}(r,x) d\nu_{\alpha+1}(r,x)\\ &= \frac{1}{2(\alpha+1)}\int_0^{+\infty}\int_{\mathbb{R}} f(r,x) r^2 e^{\alpha+1}_{m,n}(r,x) d\nu_\alpha(r,x)\\ &= \frac{1}{2(\alpha+1)}\langle f/r^2 e^{\alpha+1}_{m,n}\rangle_{\nu_\alpha},\end{aligned}$$

hence by using the relation (3.3), we deduce that

$$\langle f/e^{\alpha+1}_{m,n}\rangle_{\nu_{\alpha+1}} = \sqrt{\frac{\alpha+m+1}{2(\alpha+1)}}\langle f/e^{\alpha}_{m,n}\rangle_{\nu_\alpha} - \sqrt{\frac{m+1}{2(\alpha+1)}}\langle f/e^{\alpha}_{m+1,n}\rangle_{\nu_\alpha}.$$

In the same manner, and by virtue of the relation (2.7), we have

$$\begin{aligned}\langle \mathscr{F}_\alpha(f)/\xi^{\alpha+1}_{m,n}\rangle_{\gamma_{\alpha+1}} &= \int\int_{\Gamma_+} \mathscr{F}_\alpha(f)(\mu,\lambda)\xi^{\alpha+1}_{m,n}(\mu,\lambda)d\gamma_{\alpha+1}(\mu,\lambda)\\ &= \frac{1}{2(\alpha+1)}\int\int_{\Gamma_+} \mathscr{F}_\alpha(f)(\mu,\lambda)(\mu^2+\lambda^2)\xi^{\alpha+1}_{m,n}(\mu,\lambda)d\gamma_\alpha(\mu,\lambda),\end{aligned}$$

using the relations (3.4) and (3.5), we deduce that

$$\begin{aligned}\langle \mathscr{F}_\alpha(f)/\xi^{\alpha+1}_{m,n}\rangle_{\gamma_{\alpha+1}} &= \frac{1}{\sqrt{2(\alpha+1)}}\int\int_{\Gamma_+} \mathscr{F}_\alpha(f)(\mu,\lambda)\Big(\sqrt{\alpha+m+1}\xi^{\alpha}_{m,n}(\mu,\lambda)\\ &\qquad - \sqrt{m+1}\xi^{\alpha}_{m+1,n}(\mu,\lambda)\Big)d\gamma_\alpha(\mu,\lambda)\\ &= \sqrt{\frac{\alpha+m+1}{2(\alpha+1)}}\langle \mathscr{F}_\alpha(f)/(i)^{2m+n}\mathscr{F}_\alpha(e^\alpha_{m,n})\rangle_{\gamma_\alpha}\\ &\qquad - \sqrt{\frac{m+1}{2(\alpha+1)}}\langle \mathscr{F}_\alpha(f)/(i)^{2m+2+n}\mathscr{F}_\alpha(e^\alpha_{m+1,n})\rangle_{\gamma_\alpha},\end{aligned}$$

hence, according to the Parseval's equality (2.11), we obtain

$$\begin{aligned}\langle \mathscr{F}_\alpha(f)/\xi^{\alpha+1}_{m,n}\rangle_{\gamma_{\alpha+1}} &= \sqrt{\frac{\alpha+m+1}{2(\alpha+1)}}(-i)^{2m+n}\langle f/e^\alpha_{m,n}\rangle_{\nu_\alpha}\\ &\qquad + \sqrt{\frac{m+1}{2(\alpha+1)}}(-i)^{2m+n}\langle f/e^\alpha_{m+1,n}\rangle_{\nu_\alpha}.\end{aligned}$$

□

4. HEISENBERG-PAULI-WEYL INEQUALITY FOR THE FOURIER TRANSFORM $\mathscr{F}_\alpha$

In this section, we will prove the main result of this work, that is the Heisenberg-Pauli-Weyl inequality for the Fourier transform $\mathscr{F}_\alpha$ connected with the Riemann-Liouville operator $\mathscr{R}_\alpha$. Next we give a generalization of this result, for this we need the following important lemma.

Lemma 4.1. *Let $f \in L^2(d\nu_\alpha)$, such that*

$$\||(r,x)|f\|_{2,\nu_\alpha} < +\infty \quad \textit{and} \quad \|(\mu^2+2\lambda^2)^{\frac{1}{2}}\mathscr{F}_\alpha(f)\|_{2,\gamma_\alpha} < +\infty,$$

then

$$\||(r,x)|f\|^2_{2,\nu_\alpha} + \|(\mu^2+2\lambda^2)^{\frac{1}{2}}\mathscr{F}_\alpha(f)\|^2_{2,\gamma_\alpha} = \sum_{m,n=0}^{+\infty}\big(2\alpha+4m+2n+3\big)|a_{m,n}|^2, \tag{4.1}$$

where $a_{m,n} = \langle f/e^\alpha_{m,n}\rangle_{\nu_\alpha}; \quad (m,n)\in\mathbb{N}^2$.

Proof. Let $f \in L^2(d\nu_\alpha)$, such that

$$\forall (r,x)\in\mathbb{R}_+\times\mathbb{R};\quad f(r,x) = \sum_{m,n=0}^{+\infty} a_{m,n}e^\alpha_{m,n}(r,x),$$

and assume that

$$\||(r,x)|f\|_{2,\nu_\alpha} < +\infty \quad \text{and} \quad \|(\mu^2+2\lambda^2)^{\frac{1}{2}}\mathscr{F}_\alpha(f)\|_{2,\gamma_\alpha} < +\infty,$$

then the functions $(r,x)\longmapsto rf(r,x)$ and $(r,x)\longmapsto xf(r,x)$ belong to the space $L^2(d\nu_\alpha)$, in particular $f\in L^2(d\nu_\alpha)\cap L^2(d\nu_{\alpha+1})$. In the same manner, the functions

$$(\mu,\lambda)\longmapsto(\mu^2+\lambda^2)^{\frac{1}{2}}\mathscr{F}_\alpha(f)(\mu,\lambda),\quad \text{and}\quad (\mu,\lambda)\longmapsto\lambda\mathscr{F}_\alpha(f)(\mu,\lambda)$$

belong to the space $L^2(d\gamma_\alpha)$. In particular, by the relation (2.7), we deduce that $\mathscr{F}_\alpha(f)\in L^2(d\gamma_\alpha)\cap L^2(d\gamma_{\alpha+1})$, and we have

$$\begin{aligned}\|rf\|^2_{2,\nu_\alpha}&=\int_0^{+\infty}\int_{\mathbb{R}}r^2|f(r,x)|^2d\nu_\alpha(r,x)\\&=2(\alpha+1)\|f\|^2_{2,\nu_{\alpha+1}}\\&=2(\alpha+1)\sum_{m,n=0}^{+\infty}\left|\langle f/e^{\alpha+1}_{m,n}\rangle_{\nu_{\alpha+1}}\right|^2,\end{aligned}$$

hence, according to the relation (3.6), we obtain

$$\|rf\|^2_{2,\nu_\alpha}=\sum_{m,n=0}^{+\infty}\left|\sqrt{\alpha+m+1}a_{m,n}-\sqrt{m+1}a_{m+1,n}\right|^2. \tag{4.2}$$

Similarly, we have

$$\begin{aligned}\|xf\|^2_{2,\nu_\alpha}&=\int_0^{+\infty}\int_{\mathbb{R}}x^2|f(r,x)|^2d\nu_\alpha(r,x)\\&=\sum_{m,n=0}^{+\infty}\left|\langle xf/e^{\alpha}_{m,n}\rangle_{\nu_\alpha}\right|^2=\sum_{m,n=0}^{+\infty}\left|\langle f/xe^{\alpha}_{m,n}\rangle_{\nu_\alpha}\right|^2,\end{aligned}$$

and by the relation (3.1), we get

$$\|xf\|^2_{2,\nu_\alpha}=\sum_{m,n=0}^{+\infty}\left|\sqrt{\frac{n+1}{2}}a_{m,n+1}+\sqrt{\frac{n}{2}}a_{m,n-1}\right|^2. \tag{4.3}$$

By the same arguments, and using the relations (3.2), (3.7) and the Parseval's equality (2.11), we obtain

$$\|(\mu^2+\lambda^2)^{\frac{1}{2}}\mathscr{F}_\alpha(f)\|^2_{2,\gamma_\alpha}=\sum_{m,n=0}^{+\infty}\left|\sqrt{\alpha+m+1}a_{m,n}+\sqrt{m+1}a_{m+1,n}\right|^2, \tag{4.4}$$

and

$$\|\lambda\mathscr{F}_\alpha(f)\|^2_{2,\gamma_\alpha}=\sum_{m,n=0}^{+\infty}\left|\sqrt{\frac{n+1}{2}}a_{m,n+1}-\sqrt{\frac{n}{2}}a_{m,n-1}\right|^2. \tag{4.5}$$

Combining now the relations (4.2), (4.3), (4.4) and (4.5), we deduce that

$$\begin{aligned}&\||(r,x)|f\|^2_{2,\nu_\alpha}+\|(\mu^2+2\lambda^2)^{\frac{1}{2}}\mathscr{F}_\alpha(f)\|^2_{2,\gamma_\alpha}\\&=\|rf\|^2_{2,\nu_\alpha}+\|(\mu^2+\lambda^2)^{\frac{1}{2}}\mathscr{F}_\alpha(f)\|^2_{2,\gamma_\alpha}+\|xf\|^2_{2,\nu_\alpha}+\|\lambda\mathscr{F}_\alpha(f)\|^2_{2,\gamma_\alpha}\end{aligned}$$

$$= 2\sum_{m,n=0}^{+\infty}\Big((\alpha+m+1)|a_{m,n}|^2+(m+1)|a_{m+1,n}|^2\Big)$$
$$+2\sum_{m,n=0}^{+\infty}\left(\frac{n+1}{2}|a_{m,n+1}|^2+\frac{n}{2}|a_{m,n-1}|^2\right)$$
$$= 2\sum_{m,n=0}^{+\infty}(\alpha+m+1)|a_{m,n}|^2+2\sum_{m,n=0}^{+\infty}m|a_{m,n}|^2+2\sum_{m,n=0}^{+\infty}\frac{n}{2}|a_{m,n}|^2+2\sum_{m,n=0}^{+\infty}\frac{n+1}{2}|a_{m,n}|^2$$
$$= \sum_{m,n=0}^{+\infty}\big(2\alpha+4m+2n+3\big)|a_{m,n}|^2.$$

□

Remark 1. From the relation (4.1), we deduce that for all $f\in L^2(d\nu_\alpha)$, we have

$$\||(r,x)|f\|^2_{2,\nu_\alpha}+\|(\mu^2+2\lambda^2)^{\frac12}\mathscr{F}_\alpha(f)\|^2_{2,\gamma_\alpha}\geq(2\alpha+3)\|f\|^2_{2,\nu_\alpha}, \tag{4.6}$$

with equality if and only if

$$\forall(r,x)\in\mathbb{R}_+\times\mathbb{R};\quad f(r,x)=Ce^{-\frac{r^2+x^2}{2}};\quad C\in\mathbb{C}.$$

Lemma 4.2. *Let $f\in L^2(d\nu_\alpha)$ such that,*

$$\||(r,x)|f\|_{2,\nu_\alpha}<+\infty\quad\textit{and}\quad\|(\mu^2+2\lambda^2)^{\frac12}\mathscr{F}_\alpha(f)\|_{2,\gamma_\alpha}<+\infty,$$

then
$1)$ *For all $t>0$,*

$$\frac{1}{t^2}\||(r,x)|f\|^2_{2,\nu_\alpha}+t^2\|(\mu^2+2\lambda^2)^{\frac12}\mathscr{F}_\alpha(f)\|^2_{2,\gamma_\alpha}\geq(2\alpha+3)\|f\|^2_{2,\nu_\alpha}.$$

$2)$ *The following assertions are equivalent*
$i)$ $\||(r,x)|f\|_{2,\nu_\alpha}\|(\mu^2+2\lambda^2)^{\frac12}\mathscr{F}_\alpha(f)\|_{2,\gamma_\alpha}=\dfrac{2\alpha+3}{2}\|f\|^2_{2,\nu_\alpha}$.
$ii)$ *There exists $t_0>0$, such that*

$$\||(r,x)|f_{t_0}\|^2_{2,\nu_\alpha}+\|(\mu^2+2\lambda^2)^{\frac12}\mathscr{F}_\alpha(f_{t_0})\|^2_{2,\gamma_\alpha}=(2\alpha+3)\|f_{t_0}\|^2_{2,\nu_\alpha},$$

where $f_{t_0}(r,x)=f(t_0r,t_0x)$.

Proof. $1)$ Let $f\in L^2(d\nu_\alpha)$ satisfy the hypothesis.For all $t>0$ we put $f_t(r,x)=f(tr,tx)$, and then by a simple change of variables, we get

$$\|f_t\|^2_{2,\nu_\alpha}=\frac{1}{t^{2\alpha+3}}\|f\|^2_{2,\nu_\alpha}, \tag{4.7}$$

and

$$\||(r,x)|f_t\|^2_{2,\nu_\alpha}=\frac{1}{t^{2\alpha+5}}\||(r,x)|f\|^2_{2,\nu_\alpha}. \tag{4.8}$$

For all $(\mu,\lambda)\in\Gamma$,

$$\mathscr{F}_\alpha(f_t)(\mu,\lambda)=\frac{1}{t^{2\alpha+3}}\mathscr{F}_\alpha(f)\left(\frac{\mu}{t},\frac{\lambda}{t}\right), \tag{4.9}$$

and by using the relation (2.6), we deduce that

$$\|(\mu^2+2\lambda^2)^{\frac12}\mathscr{F}_\alpha(f_t)\|^2_{2,\gamma_\alpha}=\frac{1}{t^{2\alpha+1}}\|(\mu^2+2\lambda^2)^{\frac12}\mathscr{F}_\alpha(f)\|^2_{2,\gamma_\alpha}. \tag{4.10}$$

Then, the desired result follows by replacing f by f_t in the relation (4.6).

$2)$ Let $f \in L^2(d\nu_\alpha)$; $f \neq 0$.

- Assume that

$$\| |(r,x)| f\|_{2,\nu_\alpha} \|(\mu^2+2\lambda^2)^{\frac{1}{2}} \mathscr{F}_\alpha(f)\|_{2,\gamma_\alpha} = \frac{2\alpha+3}{2}\|f\|^2_{2,\nu_\alpha}.$$

By Theorem 2.2, we have $\|(\mu^2+2\lambda^2)^{\frac{1}{2}} \mathscr{F}_\alpha(f)\|_{2,\gamma_\alpha} \neq 0$, then for

$$t_0 = \sqrt{\frac{\| |(r,x)| f\|_{2,\nu_\alpha}}{\|(\mu^2+2\lambda^2)^{\frac{1}{2}} \mathscr{F}_\alpha(f)\|_{2,\gamma_\alpha}}},$$

we have

$$\frac{1}{t_0^2}\| |(r,x)| f\|^2_{2,\nu_\alpha} + t_0^2\|(\mu^2+2\lambda^2)^{\frac{1}{2}} \mathscr{F}_\alpha(f)\|^2_{2,\gamma_\alpha} = (2\alpha+3)\|f\|^2_{2,\nu_\alpha},$$

and this is equivalent to

$$\| |(r,x)| f_{t_0}\|^2_{2,\nu_\alpha} + \|(\mu^2+2\lambda^2)^{\frac{1}{2}} \mathscr{F}_\alpha(f_{t_0})\|^2_{2,\gamma_\alpha} = (2\alpha+3)\|f_{t_0}\|^2_{2,\nu_\alpha}.$$

- Conversely, suppose that there exists $t_1 > 0$, such that

$$\| |(r,x)| f_{t_1}\|^2_{2,\nu_\alpha} + \|(\mu^2+2\lambda^2)^{\frac{1}{2}} \mathscr{F}_\alpha(f_{t_1})\|^2_{2,\gamma_\alpha} = (2\alpha+3)\|f_{t_1}\|^2_{2,\nu_\alpha}.$$

This is equivalent to

$$\frac{1}{t_1^2}\| |(r,x)| f\|^2_{2,\nu_\alpha} + t_1^2\|(\mu^2+2\lambda^2)^{\frac{1}{2}} \mathscr{F}_\alpha(f)\|^2_{2,\gamma_\alpha} = (2\alpha+3)\|f\|^2_{2,\nu_\alpha}. \tag{4.11}$$

However, let h be the function defined on $]0,+\infty[$, by

$$h(t) = \frac{1}{t^2}\| |(r,x)| f\|^2_{2,\nu_\alpha} + t^2\|(\mu^2+2\lambda^2)^{\frac{1}{2}} \mathscr{F}_\alpha(f)\|^2_{2,\gamma_\alpha},$$

then, the minimum of the function h is attained at the point

$$t_0 = \sqrt{\frac{\| |(r,x)| f\|_{2,\nu_\alpha}}{\|(\mu^2+2\lambda^2)^{\frac{1}{2}} \mathscr{F}_\alpha(f)\|_{2,\gamma_\alpha}}}$$

and

$$h(t_0) = 2\| |(r,x)| f\|_{2,\nu_\alpha} \|(\mu^2+2\lambda^2)^{\frac{1}{2}} \mathscr{F}_\alpha(f)\|_{2,\gamma_\alpha}.$$

Thus by $1)$ of this lemma, we have

$$h(t_1) \geq h(t_0) = 2\| |(r,x)| f\|_{2,\nu_\alpha} \|(\mu^2+2\lambda^2)^{\frac{1}{2}} \mathscr{F}_\alpha(f)\|_{2,\gamma_\alpha} \geq (2\alpha+3)\|f\|^2_{2,\nu_\alpha}.$$

According to the relation (4.11), we deduce that

$$h(t_1) = h(t_0) = 2\| |(r,x)| f\|_{2,\nu_\alpha} \|(\mu^2+2\lambda^2)^{\frac{1}{2}} \mathscr{F}_\alpha(f)\|_{2,\gamma_\alpha} = (2\alpha+3)\|f\|^2_{2,\nu_\alpha}.$$

□

Theorem 4.3 (Heisenbeg-Pauli-Weyl inequality)**.** *For all $f \in L^2(d\nu_\alpha)$, we have*

$$\| |(r,x)| f\|_{2,\nu_\alpha} \|(\mu^2+2\lambda^2)^{\frac{1}{2}} \mathscr{F}_\alpha(f)\|_{2,\gamma_\alpha} \geq \frac{(2\alpha+3)}{2}\|f\|^2_{2,\nu_\alpha} \tag{4.12}$$

with equality if and only if

$$\forall (r,x) \in \mathbb{R}_+ \times \mathbb{R}; \qquad f(r,x) = Ce^{-\frac{r^2+x^2}{2t_0^2}}; \quad t_0 > 0,\ C \in \mathbb{C}.$$

Proof. It is obvious that if $f=0$, or if $\||(r,x)|f\|_{2,\nu_\alpha}=+\infty$, or $\|(\mu^2+2\lambda^2)^{\frac{1}{2}}\mathscr{F}_\alpha(f)\|_{2,\gamma_\alpha}=+\infty$, then the inequality (4.12) holds.

Let us suppose that $\||(r,x)|f\|_{2,\nu_\alpha}+\|(\mu^2+2\lambda^2)^{\frac{1}{2}}\mathscr{F}_\alpha(f)\|_{2,\gamma_\alpha}<+\infty$, and $f\neq 0$.

By $1)$ of Lemma 4.2, we have for all $t>0$

$$\frac{1}{t^2}\||(r,x)|f\|^2_{2,\nu_\alpha}+t^2\|(\mu^2+2\lambda^2)^{\frac{1}{2}}\mathscr{F}_\alpha(f)\|^2_{2,\gamma_\alpha}\geq(2\alpha+3)\|f\|^2_{2,\nu_\alpha},$$

and the result follows if we pick

$$t=\sqrt{\frac{\||(r,x)|f\|_{2,\nu_\alpha}}{\|(\mu^2+2\lambda^2)^{\frac{1}{2}}\mathscr{F}_\alpha(f)\|_{2,\gamma_\alpha}}}.$$

By $2)$ of Lemma 4.2, we have

$$\||(r,x)|f\|_{2,\nu_\alpha}\|(\mu^2+2\lambda^2)^{\frac{1}{2}}\mathscr{F}_\alpha(f)\|_{2,\gamma_\alpha}=\frac{2\alpha+3}{2}\|f\|^2_{2,\nu_\alpha},$$

if and only if there exists t_0, such that

$$\||(r,x)|f_{t_0}\|^2_{2,\nu_\alpha}+\|(\mu^2+2\lambda^2)^{\frac{1}{2}}\mathscr{F}_\alpha(f_{t_0})\|^2_{2,\gamma_\alpha}=(2\alpha+3)\|f_{t_0}\|^2_{2,\nu_\alpha},$$

and according to Remark 1, this is equivalent to

$$f_{t_0}(r,x)=Ce^{-\frac{r^2+x^2}{2}};\quad C\in\mathbb{C},$$

which means that

$$f(r,x)=Ce^{-\frac{r^2+x^2}{2t_0^2}};\quad C\in\mathbb{C}.$$

□

The following result gives a generalization of the Heisenberg-Pauli-Weyl inequality.

Theorem 4.4. *Let $a,b\geq 1$ and $\eta\in\mathbb{R}$ such that $\eta a=(1-\eta)b$, then for all $f\in L^2(d\nu_\alpha)$ we have*

$$\||(r,x)|^a f\|^\eta_{2,\nu_\alpha}\|(\mu^2+2\lambda^2)^{\frac{b}{2}}\mathscr{F}_\alpha(f)\|^{1-\eta}_{2,\gamma_\alpha}\geq\left(\frac{2\alpha+3}{2}\right)^{a\eta}\|f\|_{2,\nu_\alpha}$$

with equality if and only if $a=b=1$ and

$$\forall(r,x)\in\mathbb{R}_+\times\mathbb{R};\qquad f(r,x)=Ce^{-\frac{r^2+x^2}{2t_0^2}};\quad t_0>0,\ C\in\mathbb{C}.$$

Proof. Let $f\in L^2(d\nu_\alpha)$, $f\neq 0$, such that

$$\||(r,x)|^a f\|_{2,\nu_\alpha}+\|(\mu^2+2\lambda^2)^{\frac{b}{2}}\mathscr{F}_\alpha(f)\|_{2,\gamma_\alpha}<+\infty.$$

Then for all $a>1$, we have

$$\||(r,x)|^a f\|^{\frac{1}{a}}_{2,\nu_\alpha}\|f\|^{\frac{1}{a'}}_{2,\nu_\alpha}=\||(r,x)|^2|f|^{\frac{2}{a}}\|^{\frac{1}{2}}_{a,\nu_\alpha}\||f|^{\frac{2}{a'}}\|^{\frac{1}{2}}_{a',\nu_\alpha},$$

where a' is defined as usual by $a'=\frac{a}{a-1}$. By Hölder's inequality we get

$$\||(r,x)|^a f\|^{\frac{1}{a}}_{2,\nu_\alpha}\|f\|^{\frac{1}{a'}}_{2,\nu_\alpha}>\||(r,x)|f\|_{2,\nu_\alpha}.$$

The strict inequality here is justified by the fact that if $f\neq 0$, then the functions $|(r,x)|^{2a}|f|^2$ and $|f|^2$ cannot be proportional.Thus for all $a\geq 1$, we have

$$\||(r,x)|^a f\|^{\frac{1}{a}}_{2,\nu_\alpha}\geq\frac{\||(r,x)|f\|_{2,\nu_\alpha}}{\|f\|^{\frac{1}{a'}}_{2,\nu_\alpha}}. \tag{4.13}$$

with equality if and only if $a=1$.

In the same manner and using Plancherel's Theorem 2.3, we have for all $b \geq 1$

$$\|(\mu^2+2\lambda^2)^{\frac{b}{2}}\mathscr{F}_\alpha(f)\|_{2,\gamma_\alpha}^{\frac{1}{b}} \geq \frac{\|(\mu^2+2\lambda^2)^{\frac{1}{2}}\mathscr{F}_\alpha(f)\|_{2,\gamma_\alpha}}{\|\mathscr{F}_\alpha(f)\|_{2,\gamma_\alpha}^{\frac{1}{b'}}} \geq \frac{\|(\mu^2+2\lambda^2)^{\frac{1}{2}}\mathscr{F}_\alpha(f)\|_{2,\gamma_\alpha}}{\|f\|_{2,\nu_\alpha}^{\frac{1}{b'}}}. \tag{4.14}$$

with equality if and only if $b = 1$.

Let $\eta = \frac{b}{a+b}$, then by the relations (4.13), (4.14) and for all $a, b \geq 1$, we have

$$\||(r,x)|^a f\|_{2,\nu_\alpha}^{\eta}\|(\mu^2+2\lambda^2)^{\frac{b}{2}}\mathscr{F}_\alpha(f)\|_{2,\gamma_\alpha}^{1-\eta} \geq \left(\frac{\||(r,x)|f\|_{2,\nu_\alpha}\|(\mu^2+2\lambda^2)^{\frac{1}{2}}\mathscr{F}_\alpha(f)\|_{2,\gamma_\alpha}}{\|f\|_{2,\nu_\alpha}^{\frac{1}{a'}+\frac{1}{b'}}}\right)^{\eta a},$$

with equality if and only if $a = b = 1$.

Applying Theorem 4.3, we obtain

$$\||(r,x)|^a f\|_{2,\nu_\alpha}^{\eta}\|(\mu^2+2\lambda^2)^{\frac{b}{2}}\mathscr{F}_\alpha(f)\|_{2,\gamma_\alpha}^{1-\eta} \geq \left(\frac{2\alpha+3}{2}\right)^{\eta a}\|f\|_{2,\nu_\alpha},$$

with equality if and only if $a = b = 1$ and

$$\forall (r,x) \in \mathbb{R}_+ \times \mathbb{R}; \qquad f(r,x) = Ce^{-\frac{r^2+x^2}{2t_0^2}}; \quad t_0 > 0,\ C \in \mathbb{C}.$$

□

Remark 2. In the particular case when $a = b = 2$, the previous result gives us the Heisenberg-Pauli-Weyl inequality for the fourth moment of Heisenberg

$$\||(r,x)|^2 f\|_{2,\nu_\alpha}\|(\mu^2+2\lambda^2)\mathscr{F}_\alpha(f)\|_{2,\gamma_\alpha} > \left(\frac{2\alpha+3}{2}\right)^2\|f\|_{2,\nu_\alpha}^2.$$

5. THE LOCAL UNCERTAINTY PRINCIPLE

Theorem 5.1. *Let ξ be a real number such that $0 < \xi < \frac{2\alpha+3}{2}$, then for all $f \in L^2(d\nu_\alpha)$, $f \neq 0$, and for all measurable subsets $E \subset \Gamma_+$; $0 < \gamma_\alpha(E) < +\infty$, we have*

$$\iint_E |\mathscr{F}_\alpha(f)(\mu,\lambda)|^2 d\gamma_\alpha(\mu,\lambda) < K_{\alpha,\xi}\big(\gamma_\alpha(E)\big)^{\frac{2\xi}{2\alpha+3}}\||(r,x)|^\xi f\|_{2,\nu_\alpha}^2, \tag{5.1}$$

where

$$K_{\alpha,\xi} = \left(\frac{2\alpha+3-2\xi}{\xi^2 2^{\alpha+\frac{5}{2}}\Gamma\left(\alpha+\frac{3}{2}\right)}\right)^{\frac{2\xi}{2\alpha+3}}\left(\frac{2\alpha+3}{2\alpha+3-2\xi}\right)^2.$$

Proof. For all $s > 0$, we put

$$B_s = \big\{(r,x) \in \mathbb{R}_+ \times \mathbb{R}\ ;\ r^2 + x^2 < s^2\big\}.$$

Let $f \in L^2(d\nu_\alpha)$. By Minkowski's inequality, we have

$$\begin{aligned}(5.2)\qquad \left(\int\int_E |\mathscr{F}_\alpha(f)(\mu,\lambda)|^2 d\gamma_\alpha(\mu,\lambda)\right)^{\frac{1}{2}} &= \|\mathscr{F}_\alpha(f)\mathbf{1}_E\|_{2,\gamma_\alpha} \\ &\leq \|\mathscr{F}_\alpha(f\mathbf{1}_{B_s})\mathbf{1}_E\|_{2,\gamma_\alpha} + \|\mathscr{F}_\alpha(f\mathbf{1}_{B_s^c})\mathbf{1}_E\|_{2,\gamma_\alpha} \\ &\leq \big(\gamma_\alpha(E)\big)^{\frac{1}{2}}\|\mathscr{F}_\alpha(f\mathbf{1}_{B_s})\|_{\infty,\gamma_\alpha} + \|\mathscr{F}_\alpha(f1_{B_s^c})\|_{2,\gamma_\alpha}.\end{aligned}$$

Applying the relation (2.10), we deduce that for every $s>0$, we have

$$\left(\int\int_E |\mathscr{F}_\alpha(f)(\mu,\lambda)|^2 d\gamma_\alpha(\mu,\lambda)\right)^{\frac{1}{2}} \leq \big(\gamma_\alpha(E)\big)^{\frac{1}{2}}\|f\mathbf{1}_{B_s}\|_{1,\nu_\alpha} + \|\mathscr{F}_\alpha(f\mathbf{1}_{B_s^c})\|_{2,\gamma_\alpha}. \tag{5.3}$$

On the other hand, by Hölder's inequality we have

$$\|f\mathbf{1}_{B_s}\|_{1,\nu_\alpha} \leq \||(r,x)|^\xi f\|_{2,\nu_\alpha}\||(r,x)|^{-\xi}\mathbf{1}_{B_s}\|_{2,\nu_\alpha} \tag{5.4}$$

$$= \||(r,x)|^\xi f(r,x)\|_{2,\nu_\alpha}\frac{s^{\frac{2\alpha+3-2\xi}{2}}}{\left(2^{\alpha+\frac{1}{2}}\Gamma\left(\alpha+\frac{3}{2}\right)(2\alpha+3-2\xi)\right)^{\frac{1}{2}}}. \tag{5.5}$$

By Plancherel's theorem 2.3, we have also

$$\begin{aligned}\|\mathscr{F}_\alpha(f\mathbf{1}_{B_s^c})\|_{2,\gamma_\alpha} &= \|f\mathbf{1}_{B_s^c}\|_{2,\nu_\alpha}\\ &\leq \||(r,x)|^\xi f\|_{2,\nu_\alpha}\||(r,x)|^{-\xi}\mathbf{1}_{B_s^c}\|_{\infty,\nu_\alpha}\\ &= s^{-\xi}\||(r,x)|^\xi f\|_{2,\nu_\alpha}.\end{aligned} \tag{5.6}$$

Combining the relations (5.3), (5.5) and (5.6), we deduce that for all $s>0$ we have

$$\left(\int\int_E |\mathscr{F}_\alpha(f)(\mu,\lambda)|^2 d\gamma_\alpha(\mu,\lambda)\right)^{\frac{1}{2}} \leq g_{\alpha,\xi}(s)\||(r,x)|^\xi f(r,x)\|_{2,\nu_\alpha}, \tag{5.7}$$

where $g_{\alpha,\xi}$ is the function defined on $]0,+\infty[$ by

$$g_{\alpha,\xi}(s) = s^{-\xi} + \left(\frac{\gamma_\alpha(E)}{2^{\alpha+\frac{1}{2}}\Gamma\left(\alpha+\frac{3}{2}\right)(2\alpha+3-2\xi)}\right)^{\frac{1}{2}} s^{\frac{2\alpha+3-2\xi}{2}}.$$

Thus, the inequality (5.7) holds for

$$s_0 = \left(\frac{\xi^2 2^{\alpha+\frac{5}{2}}\Gamma\big(\big(\alpha+\frac{3}{2}\big)\big)}{\gamma_\alpha(E)(2\alpha+3-2\xi)}\right)^{\frac{1}{2\alpha+3}},$$

however

$$g_{\alpha,\xi}(s_0) = \big(\gamma_\alpha(E)\big)^{\frac{\xi}{2\alpha+3}} K_{\alpha,\xi}^{\frac{1}{2}},$$

where

$$K_{\alpha,\xi} = \left(\frac{2\alpha+3-2\xi}{\xi^2 2^{\alpha+\frac{5}{2}}\Gamma(\alpha+\frac{3}{2})}\right)^{\frac{2\xi}{2\alpha+3}}\left(\frac{2\alpha+3}{2\alpha+3-2\xi}\right)^2.$$

Let us prove that the equality in (5.1) cannot hold. Indeed, suppose that

$$\int\int_E |\mathscr{F}_\alpha(f)(\mu,\lambda)|^2 d\gamma_\alpha(\mu,\lambda) = K_{\alpha,\xi}\big(\gamma_\alpha(E)\big)^{\frac{2\xi}{2\alpha+3}}\||(r,x)|^\xi f\|_{2,\nu_\alpha}^2.$$

Then

$$\int\int_E |\mathscr{F}_\alpha(f)(\mu,\lambda)|^2 d\gamma_\alpha(\mu,\lambda) = g_{\alpha,\xi}(s_0)^2\||(r,x)|^\xi f\|_{2,\nu_\alpha}^2,$$

and therefore by the relations (5.2), (5.3) and (5.4), we get

$$\|\mathscr{F}_\alpha(f\mathbf{1}_{B_{s_0}})\mathbf{1}_E\|_{2,\gamma_\alpha} = \big(\gamma_\alpha(E)\big)^{\frac{1}{2}}\|\mathscr{F}_\alpha(f\mathbf{1}_{B_{s_0}})\|_{\infty,\gamma_\alpha}, \tag{5.8}$$

$$\|f\mathbf{1}_{B_{s_0}}\|_{1,\nu_\alpha} = \|\mathscr{F}_\alpha(f\mathbf{1}_{B_{s_0}})\|_{\infty,\gamma_\alpha}, \tag{5.9}$$

and

$$\|f\mathbf{1}_{B_{s_0}}\|_{1,\nu_\alpha} = \||(r,x)^\xi f\|_{2,\nu_\alpha}\||(r,x)^{-\xi}\mathbf{1}_{B_{s_0}}\|_{2,\nu_\alpha}. \tag{5.10}$$

However, if f satisfies the equality (5.10), then there exists $C > 0$, such that

$$|(r,x)|^{2\xi} f(r,x)^2 = C|(r,x)|^{-2\xi}\mathbf{1}_{B_{s_0}},$$

hence

$$\forall (r,x) \in \mathbb{R}_+ \times \mathbb{R}; \quad f(r,x) = Ce^{i\Phi(r,x)}|(r,x)|^{-2\xi}\mathbf{1}_{B_{s_0}}, \tag{5.11}$$

where Φ is a real measurable function on $\mathbb{R}_+ \times \mathbb{R}$.
But if f satisfies the relation (5.9), then there exists $(\mu_0, \lambda_0) \in \Gamma_+$, such that

$$\|f\|_{1,\nu_\alpha} = \|\mathscr{F}_\alpha(f)\|_{\infty,\gamma_\alpha} = \big|\mathscr{F}_\alpha(f)(\mu_0,\lambda_0)\big|.$$

So, there exists $\theta_0 \in \mathbb{R}$ satisfying

$$\mathscr{F}_\alpha(f)(\mu_0,\lambda_0) = e^{i\theta_0}\|f\|_{1,\nu_\alpha},$$

and therefore

$$Ce^{i\theta_0}\int_0^{+\infty}\int_{\mathbb{R}} |(r,x)|^{-2\xi}\mathbf{1}_{B_{s_0}}(r,x)\left(e^{i\Phi(r,x)-i\lambda_0 x - i\theta_0} j_\alpha\left(r\sqrt{\mu_0^2+\lambda_0^2}\right) - 1\right) d\nu_\alpha(r,x) = 0.$$

This implies that for almost every $(r,x) \in \mathbb{R}_+ \times \mathbb{R}$,

$$e^{i\Phi(r,x)-i\lambda_0 x - i\theta_0} j_\alpha\left(r\sqrt{\mu_0^2+\lambda_0^2}\right) = 1.$$

Hence, we deduce that for all $r \in \mathbb{R}_+$,

$$\left| j_\alpha\left(r\sqrt{\mu_0^2+\lambda_0^2}\right)\right| = 1.$$

Using the relation (2.2), it follows that $\mu_0^2 + \lambda_0^2 = 0$, or $\mu_0 = i|\lambda_0|$, and then

$$e^{i\Phi(r,x)} = e^{i\theta_0 + i\lambda_0 x}.$$

Replacing in (5.11), we get

$$f(r,x) = C'^{i\lambda_0 x}|(r,x)|^{-2\xi}\mathbf{1}_{B_{s_0}}(r,x).$$

Now, the relation (5.8) means that for almost every $(\mu,\lambda) \in E$, we have

$$\big|\mathscr{F}_\alpha(f)(\mu,\lambda)\big| = \|\mathscr{F}_\alpha(f)\|_{\infty,\gamma_\alpha} = \big|\mathscr{F}_\alpha(f)(\mu_0,\lambda_0)\big|,$$

which implies that for almost every $(\mu,\lambda) \in E$, we have

$$\mathscr{F}_\alpha(f)(\mu,\lambda) = e^{i\psi(\mu,\lambda)}\mathscr{F}_\alpha(f)(\mu_0,\lambda_0),$$

where ψ is a real measurable function on E, and therefore

$$C'^{i\psi(\mu,\lambda)}\int_0^{+\infty}\int_{\mathbb{R}} |(r,x)|^{-2\xi}\mathbf{1}_{B_{s_0}}(r,x)\left(e^{-i\lambda x + i\lambda_0 x - i\psi(\mu,\lambda)} j_\alpha r\sqrt{\mu^2+\lambda^2} - 1\right) d\nu_\alpha(r,x) = 0.$$

Consequently for all $(r,x) \in \mathbb{R}_+ \times \mathbb{R}$,

$$e^{-i\lambda x + i\lambda_0 x - i\psi(\mu,\lambda)} j_\alpha\left(r\sqrt{\mu^2+\lambda^2}\right) = 1,$$

which implies that $\lambda = \lambda_0$ and $\mu = \mu_0$.
However, since $\gamma_\alpha(E) > 0$, this contradicts the fact that for almost every $(\mu,\lambda) \in E$,

$$\big|\mathscr{F}_\alpha(f)(\mu,\lambda)\big| = \big|\mathscr{F}_\alpha(f)(\mu_0,\lambda_0)\big|,$$

and shows that the inequality in (5.1) is strictly satisfied. □

Lemma 5.2. *Let ξ be a real number such that $\xi > \frac{2\alpha+3}{2}$, then for all measurable function f on $\mathbb{R}_+ \times \mathbb{R}$ we have*

$$\|f\|^2_{1,\nu_\alpha} \leq M_{\alpha,\xi} \|f\|^{2-\frac{(2\alpha+3)}{\xi}}_{2,\nu_\alpha} \||(r,x)|^\xi f\|^{\frac{(2\alpha+3)}{\xi}}_{2,\nu_\alpha}, \tag{5.12}$$

where

$$M_{\alpha,\xi} = \frac{\pi}{2^{\alpha+\frac{1}{2}}\Gamma\left(\alpha+\frac{3}{2}\right)(2\xi-2\alpha-3)^{\frac{2\xi-2\alpha+3}{2\xi}}(2\alpha+3)^{\frac{2\alpha+3}{2\xi}}\sin\left(\pi\left(\frac{2\alpha+3}{2\xi}\right)\right)}.$$

with equality in (5.12) if and only if there exist $a, b > 0$ such that

$$|f(r,x)| = (a + b|(r,x)|^{2\xi})^{-1}.$$

Proof. We suppose naturally that $f \neq 0$. It is obvious that the inequality (5.12) holds if $\|f\|_{2,\nu_\alpha} = +\infty$ or $\||(r,x)|^\xi f\|_{2,\nu_\alpha} = +\infty$.

Assume that $\|f\|_{2,\nu_\alpha} + \||(r,x)|^\xi f\|_{2,\nu_\alpha} < +\infty$.From the hypothesis $2\xi > 2\alpha + 3$, we deduce that for all $a, b > 0$, the function

$$(r,x) \longmapsto (a + b|(r,x)|^{2\xi})^{-1}$$

belongs to $L^1(d\nu_\alpha) \cap L^2(d\nu_\alpha)$ and by Hölder's inequality, we have

$$\begin{aligned} \|f\|^2_{1,\nu_\alpha} &\leq \left\|(1+|(r,x)|^{2\xi})^{\frac{1}{2}} f\right\|^2_{2,\nu_\alpha} \left\|(1+|(r,x)|^{2\xi})^{-\frac{1}{2}}\right\|^2_{2,\nu_\alpha} \\ &\leq \left(\|f\|^2_{2,\nu_\alpha} + \||(r,x)|^\xi f\|^2_{2,\nu_\alpha}\right) \left\|(1+|(r,x)|^{2\xi})^{-\frac{1}{2}}\right\|^2_{2,\nu_\alpha}. \end{aligned} \tag{5.13}$$

However, by standard calculus, we have

$$\left\|(1+|(r,x)|^{2\xi})^{-\frac{1}{2}}\right\|^2_{2,\nu_\alpha} = \frac{\pi}{\xi 2^{\alpha+\frac{3}{2}}\Gamma\left(\alpha+\frac{3}{2}\right)\sin\left(\pi\left(\frac{2\alpha+3}{2\xi}\right)\right)}.$$

Thus

$$\|f\|^2_{1,\nu_\alpha} \leq \frac{\pi}{\xi 2^{\alpha+\frac{3}{2}}\Gamma\left(\alpha+\frac{3}{2}\right)\sin\left(\pi\left(\frac{2\alpha+3}{2\xi}\right)\right)} \left(\|f\|^2_{2,\nu_\alpha} + \||(r,x)|^\xi f\|^2_{2,\nu_\alpha}\right), \tag{5.14}$$

with equality in (5.14) if and only if we have equality in (5.13), that is there exists $C > 0$ satisfying

$$(1+|(r,x)|^{2\xi})^{\frac{1}{2}}|f(r,x)| = C(1+|(r,x)|^{2\xi})^{-\frac{1}{2}},$$

or

$$|f(r,x)| = C(1+|(r,x)|^{2\xi})^{-1}. \tag{5.15}$$

For $t > 0$, we put as above, $f_t(r,x) = f(tr,tx)$, then we have

$$\|f_t\|^2_{1,\nu_\alpha} = \frac{1}{t^{4\alpha+6}}\|f\|^2_{1,\nu_\alpha}, \tag{5.16}$$

and

$$\||(r,x)|^\xi f_t\|^2_{2,\nu_\alpha} = \frac{1}{t^{2\xi+2\alpha+3}}\||(r,x)|^\xi f\|^2_{2,\nu_\alpha}. \tag{5.17}$$

Replacing f by f_t in the relation (5.14), we deduce that for all $t > 0$, we have

$$\|f\|^2_{1,\nu_\alpha} \leq \frac{\pi}{\xi 2^{\alpha+\frac{3}{2}}\Gamma\left(\alpha+\frac{3}{2}\right)\sin\left(\pi\left(\frac{2\alpha+3}{2\xi}\right)\right)}\left(t^{2\alpha+3}\|f\|^2_{2,\nu_\alpha} + t^{2\alpha+3-2\xi}\||(r,x)|^\xi f\|^2_{2,\nu_\alpha}\right).$$

In particular, for

$$t = t_0 = \left(\frac{(2\xi - 2\alpha - 3)\| |(r,x)|^{\xi} f\|^2_{2,\nu_\alpha}}{(2\alpha+3)\|f\|^2_{2,\nu_\alpha}} \right)^{\frac{1}{2\xi}},$$

we get

$$\|f\|^2_{1,\nu_\alpha} \le M_{\alpha,\xi} \|f\|_{2,\nu_\alpha}^{2-\frac{(2\alpha+3)}{\xi}} \| |(r,x)|^{\xi} f\|_{2,\nu_\alpha}^{\frac{(2\alpha+3)}{\xi}},$$

where

$$M_{\alpha,\xi} = \frac{\pi}{2^{\alpha+\frac{1}{2}} \Gamma\left(\alpha+\frac{3}{2}\right) (2\xi - 2\alpha - 3)^{\frac{2\xi-2\alpha+3}{2\xi}} (2\alpha+3)^{\frac{2\alpha+3}{2\xi}} \sin\left(\pi \left(\frac{2\alpha+3}{2\xi}\right)\right)}.$$

Now suppose that we have equality in the last inequality. Then we have equality in (5.14) for f_{t_0} and by means of (5.15), we obtain

$$|f_{t_0}(r,x)| = C(1 + |(r,x)|^{2\xi})^{-1},$$

and then

$$|f(r,x)| = (a + b|(r,x)|^{2\xi})^{-1}.$$

□

Theorem 5.3. *Let ξ be a real number such that $\xi > \frac{2\alpha+3}{2}$. Then for all $f \in L^2(d\nu_\alpha)$, $f \neq 0$, and for all measurable subsets $E \subset \Gamma_+$; $0 < \gamma_\alpha(E) < +\infty$, we have*

$$\int\int_E |\mathscr{F}_\alpha(f)(\mu,\lambda)|^2 d\gamma_\alpha(\mu,\lambda) < M_{\alpha,\xi} \gamma_\alpha(E) \|f\|_{2,\nu_\alpha}^{2-\frac{2\alpha+3}{\xi}} \| |(r,x)|^{\xi} f\|_{2,\nu_\alpha}^{\frac{2\alpha+3}{\xi}}, \tag{5.18}$$

where

$$M_{\alpha,\xi} = \frac{\pi}{2^{\alpha+\frac{1}{2}} \Gamma\left(\alpha+\frac{3}{2}\right) (2\xi - 2\alpha - 3)^{\frac{2\xi-2\alpha+3}{2\xi}} (2\alpha+3)^{\frac{2\alpha+3}{2\xi}} \sin\left(\pi \left(\frac{2\alpha+3}{2\xi}\right)\right)}.$$

Moreover, $M_{\alpha,\xi}$ is the best (the smallest) constant satisfying (5.18).

Proof. • Suppose that the right-hand side of (5.18) is finite. Then, according to Lemma 5.2, the function f belongs to $L^1(d\nu_\alpha)$ and we have

$$\begin{aligned} \int\int_E |\mathscr{F}_\alpha(f)(\mu,\lambda)|^2 d\gamma_\alpha(\mu,\lambda) &\le \gamma_\alpha(E) \|\mathscr{F}_\alpha(f)\|^2_{\infty,\gamma_\alpha} \\ &\le \gamma_\alpha(E) \|f\|^2_{1,\nu\alpha} \\ &\le \gamma_\alpha(E) M_{\alpha,\xi} \|f\|_{2,\nu_\alpha}^{2-\frac{(2\alpha+3)}{\xi}} \| |(r,x)|^{\xi} f\|_{2,\nu_\alpha}^{\frac{(2\alpha+3)}{\xi}}, \end{aligned}$$

where

$$M_{\alpha,\xi} = \frac{\pi}{2^{\alpha+\frac{1}{2}} \Gamma\left(\alpha+\frac{3}{2}\right) (2\xi - 2\alpha - 3)^{\frac{2\xi-2\alpha+3}{2\xi}} (2\alpha+3)^{\frac{2\alpha+3}{2\xi}} \sin\left(\pi \left(\frac{2\alpha+3}{2\xi}\right)\right)}.$$

• Let us prove that the equality in (5.18) cannot hold.

Indeed, suppose that

$$\int\int_E |\mathscr{F}_\alpha(f)(\mu,\lambda)|^2 d\gamma_\alpha(\mu,\lambda) = M_{\alpha,\xi} \gamma_\alpha(E) \|f\|_{2,\nu_\alpha}^{2-\frac{2\alpha+3}{\xi}} \| |(r,x)|^{\xi} f\|_{2,\nu_\alpha}^{\frac{2\alpha+3}{\xi}}.$$

Then

$$\int\int_{\Gamma_+} |\mathscr{F}_\alpha(f)(\mu,\lambda)|^2 d\gamma_\alpha(\mu,\lambda) = \gamma_\alpha(E) \|\mathscr{F}_\alpha(f)\|^2_{\infty,\gamma_\alpha}, \tag{5.19}$$

$$\|\mathscr{F}_\alpha(f)\|_{\infty,\gamma_\alpha} = \|f\|_{1,\nu\alpha}, \tag{5.20}$$

and

$$\|f\|^2_{1,\nu_\alpha} = M_{\alpha,\xi}\|f\|_{2,\nu_\alpha}^{2-\frac{2\alpha+3}{\xi}} \||(r,x)|^\xi f\|_{2,\nu_\alpha}^{\frac{(2\alpha+3)}{\xi}}. \tag{5.21}$$

Applying Lemma 5.2 and the relation (5.21), we deduce that

$$\forall (r,x)\in\mathbb{R}_+\times\mathbb{R};\quad f(r,x)=\varphi(r,x)(a+b|(r,x)|^{2\xi})^{-1}, \tag{5.22}$$

with $|\varphi(r,x)|=1$; $a,b>0$.

On the other hand, there exists $(\mu_0,\lambda_0)\in\Gamma_+$ such that

$$\|\mathscr{F}_\alpha(f)\|_{\infty,\gamma_\alpha} = |\mathscr{F}_\alpha(f)(\mu_0,\lambda_0)| = e^{i\theta_0}\mathscr{F}_\alpha(f)(\mu_0,\lambda_0);\quad \theta_0\in\mathbb{R}. \tag{5.23}$$

Combining now the relations (5.20), (5.22) and (5.23), we get

$$\int_0^{+\infty}\int_{\mathbb{R}}\left[1-e^{i\theta_0}\varphi(r,x)j_\alpha\left(r\sqrt{\mu_0^2+\lambda_0^2}\right)e^{-i\lambda_0 x}\right](a+b|(r,x)|^{2\xi})^{-1}d\nu_\alpha(r,x)=0.$$

This implies that for almost every $(r,x)\in\mathbb{R}_+\times\mathbb{R}$,

$$e^{-i\lambda_0 x}e^{i\theta_0}\varphi(r,x)j_\alpha\left(r\sqrt{\mu_0^2+\lambda_0^2}\right)=1.$$

Since $|\varphi(r,x)|=1$, we deduce that for all $r\in\mathbb{R}_+$,

$$\left|j_\alpha\left(r\sqrt{\mu_0^2+\lambda_0^2}\right)\right|=1.$$

Using the relation (2.2), it follows that $\mu_0^2+\lambda_0^2=0$, or $\mu_0=i|\lambda_0|$, and therefore

$$\varphi(r,x)=e^{-i\theta_0}e^{i\lambda_0 x}.$$

Replacing in (5.22), we get

$$f(r,x)=Ce^{i\lambda_0 x}(a+b|(r,x)|^{2\xi})^{-1};\qquad |C|=1.$$

Now, the relation (5.19) means that

$$\int\int_E\left(\|\mathscr{F}_\alpha(f)\|^2_{\infty,\gamma_\alpha}-|\mathscr{F}_\alpha(f)(\mu,\lambda)|^2\right)d\gamma_\alpha(\mu,\lambda)=0.$$

Hence, for almost every $(\mu,\lambda)\in E$,

$$|\mathscr{F}_\alpha(f)(\mu,\lambda)|=\|\mathscr{F}_\alpha(f)\|_{\infty,\gamma_\alpha}=e^{i\theta_0}\mathscr{F}_\alpha(f)(\mu_0,\lambda_0). \tag{5.24}$$

Let $\Psi(\mu,\lambda)\in\mathbb{R}$, such that

$$|\mathscr{F}_\alpha(f)(\mu,\lambda)|=e^{i\Psi(\mu,\lambda)}\mathscr{F}_\alpha(f)(\mu,\lambda).$$

Then from (5.24), for almost every $(\mu,\lambda)\in E$,

$$e^{i\Psi(\mu,\lambda)}\mathscr{F}_\alpha(f)(\mu,\lambda)=e^{i\theta_0}\mathscr{F}_\alpha(f)(\mu_0,\lambda_0),$$

and therefore

$$\int_0^{+\infty}\int_{\mathbb{R}}(a+b|(r,x)|^{2\xi})^{-1}\left[1-e^{i(\lambda_0-\lambda)x}e^{i\Psi(\mu,\lambda)}j_\alpha\left(r\sqrt{\mu^2+\lambda^2}\right)\right]d\nu_\alpha(r,x)=0.$$

Consequently, for all $(r,x)\in\mathbb{R}_+\times\mathbb{R}$,

$$e^{i(\lambda_0-\lambda)x}e^{i\Psi(\mu,\lambda)}j_\alpha\left(r\sqrt{\mu^2+\lambda^2}\right)=1,$$

which implies that $\lambda=\lambda_0$ and $\mu=\mu_0$.
However, since $\gamma_\alpha(E)>0$, this contradicts the fact that for almost every $(\mu,\lambda)\in E$,

$$e^{i\Psi(\mu,\lambda)}\mathscr{F}_\alpha(f)(\mu,\lambda)=e^{i\theta_0}\mathscr{F}_\alpha(f)(\mu_0,\lambda_0),$$

and shows that the inequality in (5.18) is strictly satisfied.

- Let us prove that the constant $M_{\alpha,\xi}$ is the best one satisfying (5.18).

Let A be a positive constant, such that for all $f \in L^2(d\nu_\alpha)$,

$$\int\int_E |\mathscr{F}_\alpha(f)(\mu,\lambda)|^2 d\gamma_\alpha(\mu,\lambda) \leq A\gamma_\alpha(E)\|f\|_{2,\nu_\alpha}^{2-\frac{2\alpha+3}{\xi}} \||(r,x)|^\xi f\|_{2,\nu_\alpha}^{\frac{2\alpha+3}{\xi}}. \tag{5.25}$$

We assume that $\||(r,x)|^\xi f\|_{2,\nu_\alpha} < +\infty$. Then by Lemma 5.2, f belongs to $L^1(d\nu_\alpha)$. Replacing f by $f_t(r,x) = f(tr,tx)$ in (5.25), and using the relations (4.7), (4.9) and (5.17), we deduce that for all $t > 0$

$$\int\int_E \left|\mathscr{F}_\alpha(f)\left(\frac{\mu}{t},\frac{\lambda}{t}\right)\right|^2 d\gamma_\alpha(\mu,\lambda) \leq A\|f\|_{2,\nu_\alpha}^{2-\frac{2\alpha+3}{\xi}} \||(r,x)|^\xi f\|_{2,\nu_\alpha}^{\frac{2\alpha+3}{\xi}}.$$

Using the dominate convergence theorem and the relations (2.1), (2.3), (2.9) and (2.10), we deduce that

$$\lim_{t\to+\infty}\int\int_E \left|\mathscr{F}_\alpha(f)\left(\frac{\mu}{t},\frac{\lambda}{t}\right)\right|^2 d\gamma_\alpha(\mu,\lambda) = |\mathscr{F}_\alpha(f)(0,0)|^2\gamma_\alpha(E).$$

Consequently, for all $f \in L^2(d\nu_\alpha)$, such that $\||(r,x)|^\xi f\|_{2,\nu_\alpha} < +\infty$,

$$|\mathscr{F}_\alpha(f)(0,0)|^2 \leq A\|f\|_{2,\nu_\alpha}^{2-\frac{2\alpha+3}{\xi}} \||(r,x)|^\xi f\|_{2,\nu_\alpha}^{\frac{2\alpha+3}{\xi}}. \tag{5.26}$$

Now, let $f \in L^2(d\nu_\alpha)$, such that $\||(r,x)|^\xi f\|_{2,\nu_\alpha} < +\infty$, and let $(\mu,\lambda) \in \Gamma$. Putting

$$g(r,x) = j_\alpha\left(r\sqrt{\mu^2+\lambda^2}\right)e^{-i\lambda x}f(r,x),$$

then $g \in L^2(d\nu_\alpha)$, and $\||(r,x)|^\xi g\|_{2,\nu_\alpha} < +\infty$. Moreover, $\mathscr{F}_\alpha(g)(0,0) = \mathscr{F}_\alpha(f)(\mu,\lambda)$, and by (5.26), it follows that

$$|\mathscr{F}_\alpha(f)(\mu,\lambda)|^2 \leq A\|f\|_{2,\nu_\alpha}^{2-\frac{2\alpha+3}{\xi}} \||(r,x)|^\xi f\|_{2,\nu_\alpha}^{\frac{2\alpha+3}{\xi}}.$$

Thus, for all $f \in L^2(d\nu_\alpha)$ such that $\||(r,x)|^\xi f\|_{2,\nu_\alpha} < +\infty$, we have

$$\|\mathscr{F}_\alpha(f)\|_{\infty,\gamma_\alpha}^2 \leq A\|f\|_{2,\nu_\alpha}^{2-\frac{2\alpha+3}{\xi}} \||(r,x)|^\xi f\|_{2,\nu_\alpha}^{\frac{2\alpha+3}{\xi}}. \tag{5.27}$$

Taking $f_0(r,x) = (1+|(r,x)|^{2\xi})^{-1}$, we have

$$\|\mathscr{F}_\alpha(f_0)\|_{\infty,\gamma_\alpha}^2 = \|f_0\|_{1,\nu_\alpha}^2 = \left(\frac{\pi}{\xi 2^{\alpha+\frac{3}{2}}\Gamma\left(\alpha+\frac{3}{2}\right)\sin\left(\pi\left(\frac{2\alpha+3}{2\xi}\right)\right)}\right)^2,$$

$$\|f_0\|_{2,\nu_\alpha}^2 = \frac{(2\xi-2\alpha-3)\pi}{\xi^2 2^{\alpha+\frac{5}{2}}\Gamma\left(\alpha+\frac{3}{2}\right)\sin\left(\pi\left(\frac{2\alpha+3}{2\xi}\right)\right)},$$

$$\||(r,x)|^\xi f_0\|_{2,\nu_\alpha}^2 = \frac{(2\alpha+3)\pi}{\xi^2 2^{\alpha+\frac{5}{2}}\Gamma\left(\alpha+\frac{3}{2}\right)\sin\left(\pi\left(\frac{2\alpha+3}{2\xi}\right)\right)}.$$

Replacing f by f_0 in the relation (5.27), we obtain $A \geq M_{\alpha,\xi}$.
This completes the proof of Theorem 5.3. □

REFERENCES

[1] G. ANDREWS, R. ASKEY AND R. ROY, *Special Functions*. Cambridge University Press, 1999.

[2] C. BACCAR, N.B. HAMADI AND L.T. RACHDI, Inversion formulas for the Riemann-Liouville transform and its dual associated with singular partial differential operators, *International Journal of Mathematics and Mathematical Sciences*, **2006** (2006), Article ID 86238, 1-26.

[3] G. BATTLE, Heisenberg inequalities for wavelet states, *Appl. and Comp. Harmonic Anal.*, **4**(2) (1997), 119–146.

[4] M.G. COWLING AND J.F. PRICE, Generalizations of Heisenberg's inequality in Harmonic analysis (Cortona, 1982), *Lecture Notes in Math.*, **992**, Springer, Berlin (1983), 443–449. MR0729369 (86g:42002b).

[5] N.G. DE BRUIJN, Uncertainty principles in Fourier analysis, *Inequalities* (Proc. Sympos. Wright-Patterson Air Force Base, Ohio, 1965), Academic Press, New York, (1967), 57–71.

[6] A. ERDELY, *Asymptotic Expansions*. Dover Publications, Inc.New-York, 1956.

[7] W.G. FARIS, Inequalities and uncertainty principles, *J. Math. Phys.*, **19** (1978), 461–466.

[8] G.B. FOLLAND AND A. SITARAM, The uncertainty principle: a mathematical survey, *J. Fourier Anal. Appl.*, **3** (1997), 207–238.

[9] G.H. HARDY, A theorem concerning Fourier transform, *J. London Math. Soc.*, **8** (1933), 227–231.

[10] W. HEISENBERG, Über den anschaulichen Inhalt der quantentheoretischen Kinematic und Mechanik. Zeit. Physik, 43, 172 (1927); *The Physical Principles of the Quantum Theory* (Dover, New York, 1949; The Univ. Chicago Press, 1930).

[11] N.N. LEBEDEV, *Special Functions and their Applications*, Dover Publications Inc., New-York, 1972.

[12] W. MAGNUS, F. OBERHETTINGER AND F.G. TRICOMI, *Tables of Integral Transforms*, McGraw-Hill Book Compagny, Inc. New York/London/Toronto, 1954.

[13] G.W. MORGAN, A note on Fourier transforms, *J. London. Math. Soc.*, **9** (1934), 178–192.

[14] S. OMRI AND L.T. RACHDI, An $L^p - L^q$ version of Morgan's theorem associated with Riemann-Liouville transform, *International Journal of Mathematical Analysis*, **1**(17) (2007), 805–824.

[15] J.F. PRICE, Inequalities and local uncertainty principles, *J. Math. Phys.*, **24** (1983), 1711–1714.

[16] J.F. PRICE, Sharp local uncertainty inequalities, *Studia Math.*, **85** (1987), 37–45.

[17] J.M. RASSIAS, On the Heisenberg-Pauli-Weyl inequality, *J. Inequ. Pure and Appl. Math.*, **5**(1) (2004), Art. 4. [ONLINE: `http://jipam.vu.edu.au/article.php?sid=356`].

[18] J.M. RASSIAS, On the Heisenberg-Weyl inequality, *J. Inequ. Pure and Appl. Math.*, **6**(1) (2005), Art. 11. [ONLINE: `http://jipam.vu.edu.au/article.php?sid=480`].

[19] J.M. RASSIAS, On the sharpened Heisenberg-Weyl inequality, *J. Inequ. Pure and Appl. Math.*, **7**(3) (2006), Art. 80. [ONLINE: `http://jipam.vu.edu.au/article.php?sid=727`].

[20] M. RÖSLER and M. VOIT, An uncertainty principle for Hankel transforms, *American Mathematical Society*, **127**(1) (1999), 183–194.

[21] M. RÖSLER, An uncertainty principle for the Dunkl transform, *Bull. Austral. Math. Soc.*, **59** (1999), 353–360.

[22] N. SHIMENO, A note on the uncertainty principle for the Dunkl transform, *J. Math. Sci. Univ. Tokyo*, **8** (2001), 33–42.

[23] G. SZEGÖ, Orthogonal polynomials, *Amer. Math. Soc. Colloquium*, Vol. **23**, New York, 1939.

[24] G.N. WATSON, *A Treatise on the Theory of Bessel Functions*, 2nd ed, Cambridge Univ. Press, Inc. Cambridge, 1959.

[25] H. WEYL, *Gruppentheorie und Quantenmechanik*. S. Hirzel, Leipzig, 1928; and Dover edition, New York, 1950.

[26] J.A. WOLF, The Uncertainty principle for Gelfand pairs, *Nova J. Alg. Geom.*, **1** (1992), 383–396.