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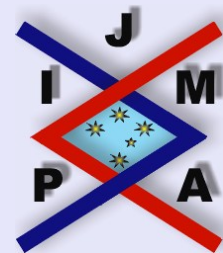
A SUFFICIENT CONDITION FOR STARLIKENESS OF ANALYTIC FUNCTIONS OF KOEBE TYPE

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[Abstract](#)

[Contents](#)



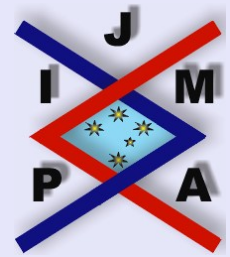
[Home Page](#)

[Go Back](#)

[Close](#)

[Quit](#)





Abstract

By making use of Jack's Lemma as well as several differential and other inequalities (and parametric constraints), the authors derive sufficient conditions for starlikeness of a certain class of n -fold symmetric analytic functions of Koebe type. Relevant connections of the results presented here with those given in earlier works are also indicated.

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Contents

1	Introduction, Definitions and Preliminaries	3
2	The Main Result and Its Consequences	6
3	Applications of Differential Inequalities	11
	References	

A Sufficient Condition for Starlikeness of Analytic Functions of Koebe Type

Muhammet Kamali and
H.M. Srivastava

Title Page

Contents



Go Back

Close

Quit

Page 2 of 17

1. Introduction, Definitions and Preliminaries

Let \mathcal{A} denote the class of functions f which are analytic in the *open* unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$$

and *normalized* by

$$f(0) = f'(0) - 1 = 0.$$

Also, as usual, let

$$(1.1) \quad \mathcal{S}^* = \left\{ f : f \in \mathcal{A} \text{ and } \Re \left(\frac{zf'(z)}{f(z)} \right) > 0 \quad (z \in \mathbb{U}) \right\}$$

and

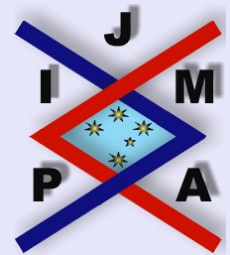
$$(1.2) \quad \tilde{\mathcal{S}}^*(\alpha) = \left\{ f : f \in \mathcal{A} \text{ and } \left| \arg \left(\frac{zf'(z)}{f(z)} \right) \right| < \frac{\alpha\pi}{2} \quad (z \in \mathbb{U}; 0 < \alpha \leq 1) \right\}$$

be the familiar classes of *starlike functions* in \mathbb{U} and *strongly starlike functions of order* α in \mathbb{U} ($0 < \alpha \leq 1$), respectively. We note that

$$\tilde{\mathcal{S}}^*(\alpha) \subset \mathcal{S}^* \quad (0 < \alpha < 1) \quad \text{and} \quad \tilde{\mathcal{S}}^*(1) \equiv \mathcal{S}^*.$$

We denote by $\mathcal{H}(\alpha)$ the class of functions $f \in \mathcal{A}$ defined by

$$(1.3) \quad \mathcal{H}(\alpha) := \left\{ f : f \in \mathcal{A} \text{ and } \Re \left(\alpha z^2 \frac{f''(z)}{f(z)} + z \frac{f'(z)}{f(z)} \right) > 0 \right. \\ \left. \left(\frac{f(z)}{z} \neq 0; z \in \mathbb{U}; \alpha \geq 0 \right) \right\},$$



A Sufficient Condition for Starlikeness of Analytic Functions of Koebe Type

Muhammet Kamali and H.M. Srivastava

Title Page

Contents



Go Back

Close

Quit

Page 3 of 17

so that, as already observed by Ramesha *et al.* [6], we have the following inclusion relationships (*cf.* [6]):

$$(1.4) \quad \mathcal{H}(\alpha) \subset \mathcal{S}^* \quad \text{and} \quad \mathcal{H}(1) \subset \tilde{\mathcal{S}}^*\left(\frac{1}{2}\right).$$

In fact, a sharper inclusion relationship than the second one in (1.4) was given subsequently by Nunokawa *et al.* [4] as follows:

$$(1.5) \quad \mathcal{H}(1) \subset \tilde{\mathcal{S}}^*(\beta) \quad \left(\beta < \frac{1}{2}\right).$$

Obradović and Joshi [5], on the other hand, made use of the method of differential inequalities in order to derive several other related results for classes of strongly starlike functions in \mathbb{U} .

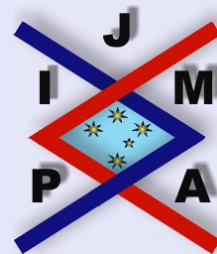
Motivated essentially by the aforementioned earlier works, we aim here at deriving sufficient conditions for starlikeness of an n -fold symmetric function $f_b(z)$ of Koebe type, defined by

$$(1.6) \quad f_b(z) := \frac{z}{(1-z^n)^b} \quad (b \geq 0; n \in \mathbb{N} := \{1, 2, 3, \dots\}),$$

which obviously corresponds to the familiar Koebe function when

$$n = 1 \quad \text{and} \quad b = 2.$$

The following result (popularly known as *Jack's Lemma*) will also be required in the derivation of our main result (Theorem 1 below).



**A Sufficient Condition for
Starlikeness of Analytic
Functions of Koebe Type**

Muhammet Kamali and
H.M. Srivastava

Title Page

Contents



Go Back

Close

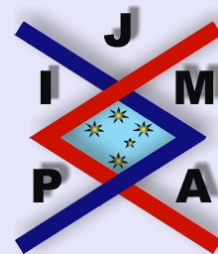
Quit

Page 4 of 17

Lemma 1 (Jack [2]). *Let the (nonconstant) function $w(z)$ be analytic in $|z| < \rho$ with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r < \rho$ at a point z_0 , then*

$$z_0 w'(z_0) = kw(z_0),$$

where k is a real number and $k \geq 1$.



**A Sufficient Condition for
Starlikeness of Analytic
Functions of Koebe Type**

Muhammet Kamali and
H.M. Srivastava

Title Page

Contents



Go Back

Close

Quit

Page 5 of 17

2. The Main Result and Its Consequences

We begin by proving a stronger result than what we indicated in the preceding section.

Theorem 1. *Let the n -fold symmetric function $f_b(z)$, defined by (1.6), be analytic in \mathbb{U} with*

$$\frac{f_b(z)}{z} \neq 0 \quad (z \in \mathbb{U}).$$

(i) *If $f_b(z)$ satisfies the inequality:*

$$(2.1) \quad \Re \left(\alpha z^2 \frac{f_b''(z)}{f_b(z)} + \frac{z f_b'(z)}{f_b(z)} \right) > -\frac{\alpha nb}{4} + \left(1 - \frac{nb}{2}\right) \left(1 - \frac{\alpha nb}{2}\right) \quad (z \in \mathbb{U}),$$

then $f_b(z)$ is starlike in \mathbb{U} for

$$\alpha > 0 \quad \text{and} \quad \frac{3\alpha + 2 - \sqrt{\Delta}}{2\alpha} \leq nb \leq \frac{3\alpha + 2 + \sqrt{\Delta}}{2\alpha} \\ (\Delta := 9\alpha^2 - 4\alpha + 4).$$

(ii) *If $f_b(z)$ satisfies the inequality (2.1) with $\alpha = 0$, that is, if*

$$(2.2) \quad \Re \left(\frac{z f_b'(z)}{f_b(z)} \right) > 1 - \frac{nb}{2} \quad (z \in \mathbb{U}),$$

then $f_b(z)$ is starlike in \mathbb{U} for $0 \leq nb \leq 2$.



A Sufficient Condition for Starlikeness of Analytic Functions of Koebe Type

Muhammet Kamali and
H.M. Srivastava

Title Page

Contents



Go Back

Close

Quit

Page 6 of 17

Proof. (i) Let $\alpha > 0$ and $f_b(z)$ satisfy the hypotheses of Theorem 1. We put

$$\frac{zf'_b(z)}{f_b(z)} = \frac{1 + (nb - 1)w(z)}{1 - w(z)},$$

where $w(z)$ is analytic in \mathbb{U} with

$$w(0) = 0 \quad \text{and} \quad w(z) \neq 1 \quad (z \in \mathbb{U}).$$

Then we have

$$\begin{aligned} & \frac{\{f'_b(z) + zf''_b(z)\}f_b(z) - z\{f'_b(z)\}^2}{\{f_b(z)\}^2} \\ &= \frac{(nb - 1)w'(z)\{1 - w(z)\} + w'(z)\{1 + (nb - 1)w(z)\}}{\{1 - w(z)\}^2}, \end{aligned}$$

which implies that

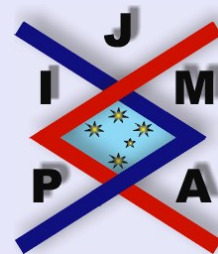
$$(2.3) \quad z \frac{f''_b(z)}{f_b(z)} + \frac{f'_b(z)}{f_b(z)} - z \left(\frac{f'_b(z)}{f_b(z)} \right)^2 = \frac{nbw'(z)}{\{1 - w(z)\}^2}.$$

On the other hand, we can write

$$z^2 \frac{f''_b(z)}{f_b(z)} = \frac{nbzw'(z)}{\{1 - w(z)\}^2} - \frac{1 + (nb - 1)w(z)}{1 - w(z)} + \left(\frac{1 + (nb - 1)w(z)}{1 - w(z)} \right)^2,$$

that is,

$$\alpha z^2 \frac{f''_b(z)}{f_b(z)} = \alpha \left[\frac{nbzw'(z)}{\{1 - w(z)\}^2} + \left(\frac{1 + (nb - 1)w(z)}{1 - w(z)} \right)^2 \right] - \alpha \cdot \frac{1 + (nb - 1)w(z)}{1 - w(z)},$$



A Sufficient Condition for Starlikeness of Analytic Functions of Koebe Type

Muhammet Kamali and
H.M. Srivastava

Title Page

Contents



Go Back

Close

Quit

Page 7 of 17

which, in turn, implies that

$$(2.4) \quad \alpha z^2 \frac{f_b''(z)}{f_b(z)} + z \frac{f_b'(z)}{f_b(z)} = \alpha \left[\frac{nbzw'(z)}{\{1-w(z)\}^2} + \left(\frac{1+(nb-1)w(z)}{1-w(z)} \right)^2 \right] + (1-\alpha) \frac{1+(nb-1)w(z)}{1-w(z)}.$$

Now we claim that $|w(z)| < 1$ ($z \in \mathbb{U}$). If there exists a $z_0 \in \mathbb{U}$ such that $|w(z_0)| = 1$, then (by Jack's Lemma) we have

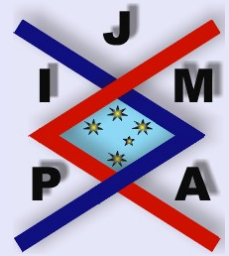
$$z_0 w'(z_0) = kw(z_0) \quad (k \geq 1).$$

By setting

$$w(z_0) = e^{i\theta} \quad (0 \leq \theta < 2\pi),$$

we thus find that

$$\begin{aligned} & \Re \left(\alpha z_0^2 \frac{f_b''(z_0)}{f_b(z_0)} + z_0 \frac{f_b'(z_0)}{f_b(z_0)} \right) \\ &= \Re \left(\alpha \left[\frac{nbz_0 w'(z_0)}{(1-w(z_0))^2} + \left(\frac{1+(nb-1)w(z_0)}{1-w(z_0)} \right)^2 \right] \right. \\ & \quad \left. + (1-\alpha) \frac{1+(nb-1)w(z_0)}{1-w(z_0)} \right) \\ &= \Re \left(\alpha \left[\frac{nbke^{i\theta}}{(1-e^{i\theta})^2} + \left(\frac{1+(nb-1)e^{i\theta}}{1-e^{i\theta}} \right)^2 \right] + (1-\alpha) \frac{1+(nb-1)e^{i\theta}}{1-e^{i\theta}} \right) \end{aligned}$$



A Sufficient Condition for Starlikeness of Analytic Functions of Koebe Type

Muhammet Kamali and
H.M. Srivastava

Title Page

Contents



Go Back

Close

Quit

Page 8 of 17

$$\begin{aligned}
&= \alpha \left[\frac{-nbk}{4 \sin^2 \left(\frac{\theta}{2} \right)} + \left(1 - \frac{nb}{2} \right)^2 - \frac{n^2 b^2}{4} \left(\frac{1 + \cos \theta}{1 - \cos \theta} \right) \right] + (1 - \alpha) \left(1 - \frac{nb}{2} \right) \\
&= -\frac{\alpha nb}{4} \left(\frac{k + nb \cos^2 \left(\frac{\theta}{2} \right)}{\sin^2 \left(\frac{\theta}{2} \right)} \right) + \left(1 - \frac{nb}{2} \right) \left(1 - \frac{\alpha nb}{2} \right) \\
&\leq -\frac{\alpha nb}{4} + \left(1 - \frac{nb}{2} \right) \left(1 - \frac{\alpha nb}{2} \right) \quad (z \in \mathbb{U}),
\end{aligned}$$

since $k \geq 1$.

If we let

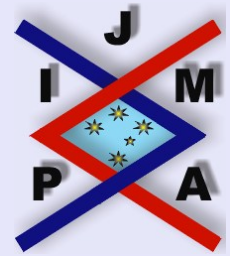
$$\begin{aligned}
(2.5) \quad \Re \left(\alpha z_0^2 \frac{f_b''(z_0)}{f_b(z_0)} + z_0 \frac{f_b'(z_0)}{f_b(z_0)} \right) &\leq -\frac{\alpha nb}{4} + \left(1 - \frac{nb}{2} \right) \left(1 - \frac{\alpha nb}{2} \right) \\
&= \frac{1}{4} [\alpha (nb)^2 - (3\alpha + 2)(nb) + 4] \\
&=: \vartheta(nb) \quad (z \in \mathbb{U}),
\end{aligned}$$

then

$$\vartheta(nb) \leq 0 \quad \left(\frac{3\alpha + 2 - \sqrt{\Delta}}{2\alpha} \leq nb \leq \frac{3\alpha + 2 + \sqrt{\Delta}}{2\alpha}; \Delta := 9\alpha^2 - 4\alpha + 4 \right).$$

Thus we have

$$(2.6) \quad \Re \left(\alpha z_0^2 \frac{f_b''(z_0)}{f_b(z_0)} + z_0 \frac{f_b'(z_0)}{f_b(z_0)} \right) \leq 0 \quad (z \in \mathbb{U})$$



A Sufficient Condition for Starlikeness of Analytic Functions of Koebe Type

Muhammet Kamali and
H.M. Srivastava

Title Page

Contents



Go Back

Close

Quit

Page 9 of 17

$$\left(\frac{3\alpha + 2 - \sqrt{\Delta}}{2\alpha} \leq nb \leq \frac{3\alpha + 2 + \sqrt{\Delta}}{2\alpha}; \Delta := 9\alpha^2 - 4\alpha + 4 \right),$$

which is a contradiction to the hypotheses of Theorem 2.

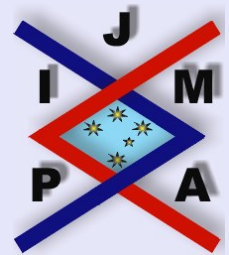
Therefore, $|w(z)| < 1$ for all z in \mathbb{U} . Hence $f_b(z)$ is starlike in \mathbb{U} , thereby proving the assertion (i) of Theorem 1.

(ii) The proof of the assertion (ii) of Theorem 1 was given by Fukui *et al.* [1], and so we omit the details here. \square

Corollary 1. *The following inclusion relationship holds true:*

$$\mathcal{H}_b(\alpha) := \left\{ f_b : f_b \in \mathcal{A} \text{ and } \Re \left(\alpha z^2 \frac{f_b''(z)}{f_b(z)} + z \frac{f_b'(z)}{f_b(z)} \right) > 0 \right. \\ \left. \left(\frac{f_b(z)}{z} \neq 0; z \in \mathbb{U}; \alpha \geq 0 \right) \right\} \subset \mathcal{S}^*$$

for the n -fold symmetric function $f_b(z)$ defined by (1.6).



A Sufficient Condition for Starlikeness of Analytic Functions of Koebe Type

Muhammet Kamali and
H.M. Srivastava

Title Page

Contents



Go Back

Close

Quit

Page 10 of 17

3. Applications of Differential Inequalities

In this section, we apply the following known result involving differential inequalities with a view to deriving several further sufficient conditions for starlikeness of the n -fold symmetric function $f_b(z)$ defined by (1.6).

Lemma 2 (Miller and Mocanu [3]). *Let $\Theta(u, v)$ be a complex-valued function such that*

$$\Theta : \mathbb{D} \rightarrow \mathbb{C} \quad (\mathbb{D} \subset \mathbb{C} \times \mathbb{C}),$$

\mathbb{C} being (as usual) the complex plane, and let

$$u = u_1 + iu_2 \quad \text{and} \quad v = v_1 + iv_2.$$

Suppose that the function $\Theta(u, v)$ satisfies each of the following conditions:

- (i) $\Theta(u, v)$ is continuous in \mathbb{D} ;
- (ii) $(1, 0) \in \mathbb{D}$ and $\Re(\Theta(1, 0)) > 0$;
- (iii) $\Re(\Theta(iu_2, v_1)) \leq 0$ for all $(iu_2, v_1) \in \mathbb{D}$ such that

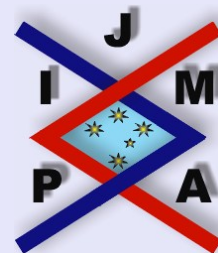
$$v_1 \leq -\frac{1}{2}(1 + u_2^2).$$

Let

$$p(z) = 1 + p_1z + p_2z^2 + \dots$$

be analytic (regular) in \mathbb{U} such that

$$(p(z), zp'(z)) \in \mathbb{D} \quad (z \in \mathbb{U}).$$



A Sufficient Condition for
Starlikeness of Analytic
Functions of Koebe Type

Muhammet Kamali and
H.M. Srivastava

Title Page

Contents



Go Back

Close

Quit

Page 11 of 17

If $\Re(\Theta(p(z), zp'(z))) > 0 \quad (z \in \mathbb{U}),$

then $\Re(p(z)) > 0 \quad (z \in \mathbb{U}).$

Let us now consider the following implication:

$$(3.1) \quad \Re\left(\alpha z^2 \frac{f_b''(z)}{f_b(z)} + z \frac{f_b'(z)}{f_b(z)}\right) > -\frac{\alpha nb}{4} + \left(1 - \frac{nb}{2}\right) \left(1 - \frac{\alpha nb}{2}\right) \\ \Rightarrow \Re\left(\left(z \frac{f_b'(z)}{f_b(z)}\right)^\mu\right) > 0$$

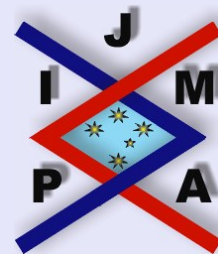
$$\left(z \in \mathbb{U}; -\frac{\alpha nb}{4} + \left(1 - \frac{nb}{2}\right) \left(1 - \frac{\alpha nb}{2}\right) < 1; \alpha \geq 0; \mu \geq 1\right).$$

If we put

$$p(z) = \left(z \frac{f_b'(z)}{f_b(z)}\right)^\mu,$$

then (3.1) is equivalent to

$$(3.2) \quad \Re\left(\frac{\alpha}{\mu} \{p(z)\}^{(1-\mu)/\mu} zp'(z) + \alpha \{p(z)\}^{2/\mu} \right. \\ \left. + (1 - \alpha) \{p(z)\}^{1/\mu} + \frac{\alpha nb}{4} - \left(1 - \frac{nb}{2}\right) \left(1 - \frac{\alpha nb}{2}\right)\right) > 0 \\ \Rightarrow \Re(p(z)) > 0 \quad (z \in \mathbb{U}).$$



A Sufficient Condition for Starlikeness of Analytic Functions of Koebe Type

Muhammet Kamali and
H.M. Srivastava

Title Page

Contents



Go Back

Close

Quit

Page 12 of 17

By setting

$$p(z) = u \quad \text{and} \quad zp'(z) = v,$$

and letting

$$\Theta(u, v) = \frac{\alpha}{\mu} u^{(1-\mu)/\mu} v + \alpha u^{2/\mu} + (1-\alpha)u^{1/\mu} + \frac{\alpha nb}{4} - \left(1 - \frac{nb}{2}\right) \left(1 - \frac{\alpha nb}{2}\right),$$

it is easy to show that, for

$$\alpha \geq 0 \quad \text{and} \quad \mu \geq 1,$$

we have

(i) $\Theta(u, v)$ is continuous in $\mathbb{D} = (\mathbb{C} \setminus \{0\}) \times \mathbb{C}$;

(ii) $(1, 0) \in \mathbb{D}$ and

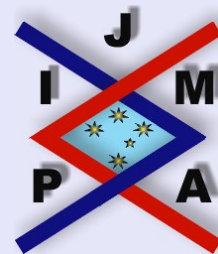
$$\Re(\Theta(1, 0)) = \frac{3\alpha nb}{4} + \frac{nb}{2} - \frac{\alpha n^2 b^2}{4} > 0,$$

since

$$-\frac{\alpha nb}{4} + \left(1 - \frac{nb}{2}\right) \left(1 - \frac{\alpha nb}{2}\right) < 1.$$

Thus the conditions (i) and (ii) of Lemma 2 are satisfied. Moreover, for

$$(iu_2, v_1) \in \mathbb{D} \quad \text{such that} \quad v_1 \leq -\frac{1}{2} (1 + u_2^2),$$



A Sufficient Condition for Starlikeness of Analytic Functions of Koebe Type

Muhammet Kamali and
H.M. Srivastava

Title Page

Contents



Go Back

Close

Quit

Page 13 of 17

we obtain

$$\begin{aligned} \Re(\theta(iu_2, v_1)) &= \frac{\alpha}{\mu} |u_2|^{(1-\mu)/\mu} v_1 \cos\left(\frac{(1-\mu)\pi}{2\mu}\right) + \alpha |u_2|^{2/\mu} \cos\left(\frac{\pi}{\mu}\right) \\ &\quad + (1-\alpha) |u_2|^{1/\mu} \cos\left(\frac{\pi}{2\mu}\right) + \frac{\alpha nb}{4} - \left(1 - \frac{nb}{2}\right) \left(1 - \frac{\alpha nb}{2}\right) \\ &\leq -\frac{\alpha}{2\mu} (1+u_2^2) |u_2|^{(1-\mu)/\mu} \sin\left(\frac{\pi}{2\mu}\right) + \alpha |u_2|^{2/\mu} \cos\left(\frac{\pi}{\mu}\right) \\ &\quad + (1-\alpha) |u_2|^{1/\mu} \cos\left(\frac{\pi}{2\mu}\right) + \frac{\alpha nb}{4} - \left(1 - \frac{nb}{2}\right) \left(1 - \frac{\alpha nb}{2}\right), \end{aligned}$$

which, upon putting $|u_2| = s$ ($s > 0$), yields

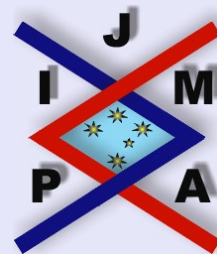
$$(3.3) \quad \Re(\Theta(iu_2, v_1)) \leq \Phi(s),$$

where

$$(3.4) \quad \begin{aligned} \Phi(s) &:= -\frac{\alpha}{2\mu} (1+s^2) s^{(1-\mu)/\mu} \sin\left(\frac{\pi}{2\mu}\right) + \alpha s^{2/\mu} \cos\left(\frac{\pi}{\mu}\right) \\ &\quad + (1-\alpha) s^{1/\mu} \cos\left(\frac{\pi}{2\mu}\right) + \frac{\alpha nb}{4} - \left(1 - \frac{nb}{2}\right) \left(1 - \frac{\alpha nb}{2}\right). \end{aligned}$$

Remark. If, for some choices of the parameters α , μ , and nb , we find that

$$\Phi(s) \leq 0 \quad (s > 0),$$



**A Sufficient Condition for
Starlikeness of Analytic
Functions of Koebe Type**

Muhammet Kamali and
H.M. Srivastava

Title Page

Contents



Go Back

Close

Quit

Page 14 of 17

then we can conclude from (3.3) and Lemma 2 that the corresponding implication (3.1) holds true.

First of all, for the choice:

$$\mu = 1 \quad \text{and} \quad nb = 2,$$

we obtain

Theorem 2. *If the n -fold symmetric function $f_b(z)$, defined by (1.6) and analytic in \mathbb{U} with*

$$\frac{f_b(z)}{z} \neq 0 \quad (z \in \mathbb{U}),$$

satisfies the following inequality:

$$(3.5) \quad \Re \left(\alpha z^2 \frac{f_b''(z)}{f_b(z)} + z \frac{f_b'(z)}{f_b(z)} \right) > -\frac{\alpha}{2} \quad (z \in \mathbb{U}),$$

then $f_b \in \mathcal{S}^$ for any real $\alpha \geq 0$.*

Proof. For $\mu = 1$ and $nb = 2$, we find from (3.4) that

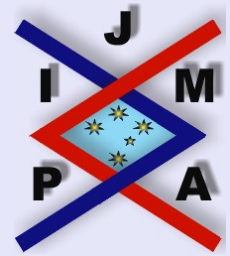
$$\Phi(s) = -\frac{3}{2}\alpha s^2 \leq 0 \quad (s \in \mathbb{R}),$$

which implies Theorem 2 in view of the above remark. □

Next, for

$$\alpha = \frac{2}{3}, \quad nb = 3 \pm \sqrt{3}, \quad \text{and} \quad \mu = 2,$$

we get



A Sufficient Condition for Starlikeness of Analytic Functions of Koebe Type

Muhammet Kamali and
H.M. Srivastava

Title Page

Contents



Go Back

Close

Quit

Page 15 of 17

Theorem 3. If the n -fold symmetric function $f_b(z)$, defined by (1.6) and analytic in \mathbb{U} with

$$\frac{f_b(z)}{z} \neq 0 \quad (z \in \mathbb{U}),$$

satisfies the following inequality:

$$(3.6) \quad \Re \left(\frac{2}{3} z^2 \frac{f_b''(z)}{f_b(z)} + z \frac{f_b'(z)}{f_b(z)} \right) > 0 \quad (z \in \mathbb{U}),$$

then

$$\left| \arg \left(\frac{z f_b'(z)}{f_b(z)} \right) \right| < \frac{\pi}{4} \quad (z \in \mathbb{U})$$

or, equivalently,

$$\mathcal{H}_b \left(\frac{2}{3} \right) \subset \tilde{\mathcal{S}}^* \left(\frac{1}{2} \right).$$

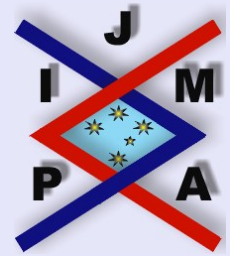
Proof. By setting

$$\alpha = \frac{2}{3}, \quad nb = 3 \pm \sqrt{3}, \quad \text{and} \quad \mu = 2$$

in (3.4), we have

$$\Phi(s) = - \frac{(1-s)^2}{6\sqrt{2s}} \leq 0 \quad (s > 0),$$

which leads us to Theorem 3 just as in the proof of Theorem 2. □



A Sufficient Condition for Starlikeness of Analytic Functions of Koebe Type

Muhammet Kamali and
H.M. Srivastava

Title Page

Contents



Go Back

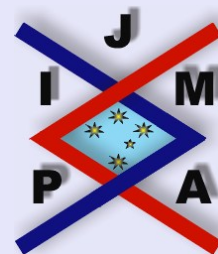
Close

Quit

Page 16 of 17

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**A Sufficient Condition for
Starlikeness of Analytic
Functions of Koebe Type**

Muhammet Kamali and
H.M. Srivastava

Title Page

Contents



Go Back

Close

Quit

Page 17 of 17