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AN IDENTITY IN REAL INNER PRODUCT SPACES

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ABSTRACT. We obtain an identity in real inner product spaces that leads to the Grüss inequality and an inequality of Ostrowski.

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1. INTRODUCTION

The Grüss inequality was generalized by S.S. Dragomir to the inner product spaces in [1]. It turned out to be an inequality relative to the inner products and norms of vectors in inner product space, that is,

"Let $(H; \langle\cdot,\cdot\rangle)$ be an inner product space over $\mathbb{K}(\mathbb{K} = \mathbb{C}, \mathbb{R})$ and $e \in H$, $\|e\| = 1$. if $\phi, \gamma, \Phi, \Gamma$ are real or complex numbers and x, y are vectors in H such that the condition

$$\operatorname{Re}\langle \Phi e - x, x - \phi e\rangle \geq 0, \quad \operatorname{Re}\langle \Gamma e - y, y - \gamma e\rangle \geq 0 \tag{1.1}$$

holds, then

$$|\langle x, y\rangle - \langle x, e\rangle\langle e, y\rangle| \leq \frac{1}{4}|\Phi - \phi||\Gamma - \gamma|.\text{"} \tag{1.2}$$

In this paper, we give an identity that yields the inequality

$$\left|\langle x, y\rangle - \frac{1}{\|z\|^2}\langle x, z\rangle\langle y, z\rangle\right|^2 \leq \left[\|x\|^2 - \frac{1}{\|z\|^2}\langle x, z\rangle^2\right]\left[\|y\|^2 - \frac{1}{\|z\|^2}\langle y, z\rangle^2\right] \tag{1.3}$$

here $x, y, z \in H$, H is a real inner product space.

From inequality (1.3), we obtain the Grüss inequality and an inequality by A. Ostrowski.

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2. MAIN RESULT

Let x, y, z be three vectors in real inner product spaces. Denote by $Z := \operatorname{span}\{z\}$ the linear subspace spanned by z, and $W := \operatorname{span}\{x, z\}$ the linear subspace spanned by x and z, denote by $\operatorname{dist}(x, \operatorname{span}\{z\}) = \inf_{-\infty<s<+\infty} \|x - sz\|$ for the distance between x and $\operatorname{span}\{z\}$, and $\operatorname{dist}(z, \operatorname{span}\{x, y\}) = \inf_{-\infty<s,t<+\infty} \|z - (sx + ty)\|$. The main result of this paper is:

Theorem 2.1. *Suppose x, y, z are three non-zero vectors in a real inner product space, then*

$$\operatorname{dist}^2(x, \operatorname{span}\{z\}) \operatorname{dist}^2(y, \operatorname{span}\{z\}) - \left|\langle x, y\rangle - \frac{1}{\|z\|^2} \langle x, z\rangle \langle y, z\rangle\right|^2 = \frac{\|y\|^2}{\|z\|^2} \operatorname{dist}^2(x, \operatorname{span}\{y\}) \operatorname{dist}^2(z, \operatorname{span}\{x, y\}).$$

Proof. Let $D = \operatorname{dist}^2(x, \operatorname{span}\{y\})\|y\|^2$. It is easy to see that

$$D = \|x\|^2\|y\|^2 - \langle x, y\rangle^2 . \tag{2.1}$$

When $D \neq 0$, we determine the infimum of $J(s, t) = \|z - (sx + ty)\|^2$ by discovering critical points of $J(s, t)$. Simple calculus yields

$$J(s, t) = \|z\|^2 - 2\langle x, z\rangle s - 2\langle y, z\rangle t + \|x\|^2 s^2 + 2\langle x, y\rangle st + \|y\|^2 t^2,$$

thus partial derivatives of $J(s, t)$ are

$$\begin{aligned} \frac{\partial J}{\partial s} &= 2\|x\|^2 s + 2\langle x, y\rangle t - 2\langle x, z\rangle \\ \frac{\partial J}{\partial t} &= 2\langle x, y\rangle s + 2\|y\|^2 t - 2\langle y, z\rangle . \end{aligned} \tag{2.2}$$

Let $\frac{\partial J}{\partial s} = 0$ and $\frac{\partial J}{\partial t} = 0$, we obtain

$$\begin{aligned} s &= \frac{1}{D}(\|y\|^2 \langle x, z\rangle - \langle y, z\rangle \langle x, y\rangle) \\ t &= \frac{1}{D}(\|x\|^2 \langle y, z\rangle - \langle x, z\rangle \langle x, y\rangle). \end{aligned} \tag{2.3}$$

Substituting for s and t in

$$J(s, t) = \|z\|^2 - 2\langle x, z\rangle s - 2\langle y, z\rangle t + \|x\|^2 s^2 + 2\langle x, y\rangle st + \|y\|^2 t^2,$$

by (2.3), we obtain

$$\begin{aligned} &\operatorname{dist}^2(z, \operatorname{span}\{x, y\}) \\ &= \frac{\|x\|^2\|y\|^2\|z\|^2}{D}\left(1 - \frac{\langle x, z\rangle^2}{\|x\|^2\|z\|^2} - \frac{\langle y, z\rangle^2}{\|y\|^2\|z\|^2} - \frac{\langle x, y\rangle^2}{\|x\|^2\|y\|^2} + 2\frac{\langle x, z\rangle \langle y, z\rangle \langle x, y\rangle}{\|x\|^2\|y\|^2\|z\|^2}\right). \end{aligned} \tag{2.4}$$

On the other hand, we have

$$\begin{aligned} &\operatorname{dist}^2(x, \operatorname{span}\{z\}) \operatorname{dist}^2(y, \operatorname{span}\{z\}) - \left|\langle x, y\rangle - \frac{1}{\|z\|^2} \langle x, z\rangle \langle y, z\rangle\right|^2 \\ &= \left(\|x\|^2 - \frac{\langle x, z\rangle^2}{\|z\|^2}\right)\left(\|y\|^2 - \frac{\langle y, z\rangle}{\|z\|^2}\right) - \left|\langle x, y\rangle - \frac{1}{\|z\|^2} \langle x, z\rangle \langle y, z\rangle\right|^2 \end{aligned} \tag{2.5}$$

$$= \|x\|^2\|y\|^2 \left(1 - \frac{\langle x, z\rangle^2}{\|x\|^2\|z\|^2} - \frac{\langle y, z\rangle^2}{\|y\|^2\|z\|^2} - \frac{\langle x, y\rangle^2}{\|x\|^2\|y\|^2} + 2\frac{\langle x, z\rangle \langle y, z\rangle \langle x, y\rangle}{\|x\|^2\|y\|^2\|z\|^2}\right).$$

Comparing (2.4) and (2.5), and taking note that $D = \operatorname{dist}^2(x, \operatorname{span}\{y\})\|y\|^2$, we finish our proof for the case $D \neq 0$.

When $D = 0$, then x and y are linearly dependent. in this case we can prove the theorem by straightforward verification. □

We point out that Theorem 2.1 is true also for complex inner product spaces.

3. APPLICATIONS

An application of Theorem 2.1 is the well known Grüss inequality [2] (see also [3]).

Theorem 3.1 (G. Grüss)**.** *Let f and g be two Lebesque integrable functions on (a, b). m, M and n, N are four real numbers such that*

$$m \leq f(x) \leq M, \quad n \leq g(x) \leq N \tag{3.1}$$

for each $x \in (a, b)$, then we have the Grüss inequality

$$\left|\frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{(b-a)^2}\int_a^b f(x)dx \int_a^b g(x)dx\right| \leq \frac{1}{4}(M-m)(N-n). \tag{3.2}$$

Proof. We consider the Hilbert space $L^2(a, b)$ equipped with an inner product defined by

$$\langle f, g\rangle = \frac{1}{b-a}\int_a^b f(x)g(x)dx. \tag{3.3}$$

According to Theorem 2.1, we have

$$\left|\langle x, y\rangle - \frac{1}{\|z\|^2}\langle x, z\rangle \langle y, z\rangle\right| \leq \operatorname{dist}(x, \operatorname{span}\{z\}) \operatorname{dist}(y, \operatorname{span}\{z\}). \tag{3.4}$$

This inequality yields inequality (1.3) by (2.1).

Let $x = f$, $y = g$ and $z = 1$. Note that by $m \leq f(x) \leq M$ and $n \leq g(x) \leq N$, it is easy to see that

$$\left(f(x) - \frac{m+M}{2}\right)^2 \leq \frac{(M-m)^2}{4} \tag{3.5}$$

and

$$\left(g(x) - \frac{n+N}{2}\right)^2 \leq \frac{(N-n)^2}{4}. \tag{3.6}$$

Therefore,

$$\operatorname{dist}(f, \operatorname{span}\{1\}) \leq \left(\frac{1}{b-a}\int_a^b (f(x) - \frac{M+m}{2})^2 dx\right)^{\frac{1}{2}} \leq \frac{M-m}{2}. \tag{3.7}$$

An identical argument yields

$$\operatorname{dist}(g, \operatorname{span}\{1\}) \leq \frac{N-n}{2}. \tag{3.8}$$

Substitute x, y and z in (3.4), and by f, g and 1, we obtain (3.2). □

Theorem 2.1 also contains a useful inequality of A. Ostrowski [4] (see also [3]).

Theorem 3.2 (Ostrowski). *Let $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$ be two linearly independent vectors. If the vector $x = (x_1, \dots, x_n)$ satisfies*

$$\sum_{i=1}^{n} a_i x_i = 0, \quad \sum_{i=1}^{n} b_i x_i = 1, \tag{3.9}$$

then

$$\sum_{i=1}^{n} x_i^2 \geq \frac{\sum_{i=1}^{n} a_i^2}{\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) - \left(\sum_{i=1}^{n} a_i b_i\right)^2}. \tag{3.10}$$

The equality holds if and only if

$$x_k = \frac{b_k \sum_{i=1}^{n} a_i^2 - a_k \sum_{i=1}^{n} a_i b_i}{\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) - \left(\sum_{i=1}^{n} a_i b_i\right)^2}, \quad k = 1, 2, \dots, n. \tag{3.11}$$

Proof. Substituting x, y, z in inequality (1.3), by vectors x, a, b, we have

$$\left(\|x\|^2 - \frac{1}{\|b\|^2}\right)\left(\|a\|^2 - \frac{\langle a, b\rangle^2}{\|b\|^2}\right) \geq \frac{1}{\|b\|^2}\langle a, b\rangle^2. \tag{3.12}$$

Simple calculation shows that

$$\|x\|^2 \geq \frac{\|a\|^2}{\|a\|^2\|b\|^2 - \langle a, b\rangle^2}, \tag{3.13}$$

that is, (3.10). According to Theorem 2.1, equality in (3.13) holds if and only if x, a, b are linearly dependent, that is, there exist constants λ, μ such that $x = \lambda a + \mu b$. Taking the inner product of a and b, we get $\|a\|^2\lambda + \langle a, b\rangle \mu = 0$ and $\langle a, b\rangle \lambda + \|b\|^2\mu = 1$. Solutions of the last two equations are

$$\lambda = \frac{-\langle a, b\rangle}{\|a\|^2\|b\|^2 - \langle a, b\rangle^2}, \quad \mu = \frac{\|a\|^2}{\|a\|^2\|b\|^2 - \langle a, b\rangle^2}, \tag{3.14}$$

thus

$$x = \frac{\|a\|^2 b - \langle a, b\rangle a}{\|a\|^2\|b\|^2 - \langle a, b\rangle^2}, \tag{3.15}$$

that is, (3.11). □

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