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NEW WEIGHTED MULTIVARIATE GRÜSS TYPE INEQUALITIES

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ABSTRACT. In this paper we establish some new weighted multidimensional Grüss type integral and discrete inequalities by using a fairly elementary analysis .

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1. INTRODUCTION

In 1935, G. Grüss [3] proved the following classical integral inequality (see, also [4, p. 296]):

$$\left|\frac{1}{b-a}\int_a^b f(x)\,g(x)-\left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b g(x)\,dx\right)\right| \leq \frac{1}{4}(P-p)(Q-q),$$

provided that f and g are two integrable functions on $[a,b]$ such that

$$p\leq f(x)\leq P, \qquad q\leq g(x)\leq Q,$$

for all $x\in[a,b]$, where p,P,q,Q are constants.

A large number of generalizations, extensions and variants of this inequality have been given by several authors since its discovery, see [1, 2], [4] – [6] and the references given therein. The main purpose of this paper is to establish new weighted integral and discrete inequalities of the Grüss type involving functions of several independent variables. The analysis used in the proofs is elementary and our results provide new estimates on inequalities of this type.

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2. STATEMENT OF RESULTS

In what follows, $\mathbb{R}$ and $\mathbb{N}$ denote the set of real and natural numbers respectively.

Let $D_i\left[a,b\right]=\{x_i : a_i < x_i < b_i\}$ for $i=1,\dots,n$, $a_i,b_i\in\mathbb{R}$, $D=\prod_{i=1}^{n} D_i\left[a_i,b_i\right]$ and $\bar{D}$ be the closure of D. For a differentiable function $u\left(x\right):\bar{D}\to\mathbb{R}$, we denote the first order partial derivatives by $\frac{\partial u(x)}{\partial x_i}$ $\left(i=1,\dots,n\right)$ and $\int_D u\left(x\right)dx$ the n-fold integral

$$\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n} u\left(x_1,\dots,x_n\right)dx_1\dots dx_n.$$

If

$$\left\|\frac{\partial u}{\partial x_i}\right\|_{\infty}=\sup_{x\in D}\left|\frac{\partial u\left(x\right)}{\partial x_i}\right|<\infty,$$

then we say that the partial derivatives $\frac{\partial u(x)}{\partial x_i}$ are bounded. Let $N_i\left[0,a_i\right]=\{0,1,2,\dots,a_i\}$, $a_i\in\mathbb{N}$, $\left(i=1,\dots,n\right)$ and $B=\prod_{i=1}^{n} N_i\left[0,a_i\right]$. For a function $z\left(x\right):B\to\mathbb{R}$ we define the first order forward difference operators as

$$\Delta_1 z\left(x\right)=z\left(x_1+1,x_2,\dots,x_n\right)-z\left(x\right),\dots,\Delta_n z\left(x\right)=z\left(x_1,\dots,x_{n-1},x_n+1\right)-z\left(x\right)$$

and denote the n-fold sum over B with respect to the variable $y=\left(y_1,\dots,y_n\right)\in B$ by

$$\sum_{y} z\left(y\right)=\sum_{y_1=0}^{a_1-1}\cdots\sum_{y_n=0}^{a_n-1} z\left(y_1,\dots,y_n\right).$$

Clearly,

$$\sum_{y} z\left(y\right)=\sum_{x} z\left(x\right)\qquad\text{for } x,y\in B.$$

If $\|\Delta_i z\|_{\infty}=\sup_{x\in B}\left|\Delta_i z\left(x\right)\right|<\infty$, then we say that the partial differences $\Delta_i z\left(x\right)$ are bounded. The notation

$$\sum_{t_i=y_i}^{x_i-1}\Delta_i z\left(y_1,\dots,y_{i-1},t_i,x_{i+1},\dots,x_n\right),\ x_i,y_i\in N_i\ \left[0,a_i\right]\quad\left(i=1,\dots,n\right),$$

we mean for $i=1$ it is $\sum_{t_1=y_1}^{x_1-1}\Delta_1 z\left(t_1,x_2,\dots,x_n\right)$ and so on, and for $i=n$ it is $\sum_{t_n=y_n}^{x_n-1}\Delta_n \times z\left(y_1,\dots,y_{n-1},t_n\right)$. We use the usual convention that the empty sum is taken to be zero. We use the following notations to simplify the details of presentation.

For continuous functions p,q defined on $\bar{D}$ and differentiable on D, $w\left(x\right)$ a real-valued non-negative and integrable function for every $x\in D$ with $\int_D w\left(x\right)dx>0$ and $x_i,y_i\in D_i\left[a_i,b_i\right]$, we set

$$A\left[w,p,q\right]=\int_D w\left(x\right)p\left(x\right)q\left(x\right)dx-\frac{1}{\int_D w\left(x\right)dx}\left(\int_D w\left(x\right)p\left(x\right)dx\right)\left(\int_D w\left(x\right)q\left(x\right)dx\right),$$

$$H\left[p,x_i,y_i\right]=\sum_{i=1}^{n}\left\|\frac{\partial p}{\partial x_i}\right\|_{\infty}\left|x_i-y_i\right|.$$

For the functions $p, q : B \rightarrow \mathbb{R}$ whose forward differences $\Delta_i p$, $\Delta_i q$ exist, $w(x)$ a real-valued nonnegative function defined on B and $\sum_x w(x) > 0$ and $x_i, y_i \in N_i[0, a_i]$, we set

$$L[w,p,q] = \sum_x w(x) p(x) q(x) - \frac{1}{\sum_x w(x)} \left(\sum_x w(x) p(x) \right) \left(\sum_x w(x) q(x) \right),$$

$$M[p, x_i, y_i] = \sum_{i=1}^{n} \|\Delta_i p\|_\infty |x_i - y_i|.$$

Our main results on weighted Grüss type integral inequalities involving functions of many independent variables are embodied in the following theorem.

Theorem 2.1. *Let f, g be real-valued continuous functions on $\bar{D}$ and differentiable on D whose derivatives $\frac{\partial f}{\partial x_i}, \frac{\partial g}{\partial x_i}$ are bounded. Let $w(x)$ be a real-valued, nonnegative and integrable function for $x \in D$ and $\int_D w(x)\, dx > 0$. Then*

$$|A[w,f,g]| \leq \frac{1}{2\int_D w(x)\, dx} \int_D w(x) \left[|g(x)| \int_D H[f, x_i, y_i] w(y)\, dy + |f(x)| \int_D H[g, x_i, y_i] w(y)\, dy \right] dx, \tag{2.1}$$

$$|A[w,f,g]| \leq \frac{1}{\left(\int_D w(x)\, dx\right)^2} \int_D w(x) \left(\int_D H[f, x_i, y_i] w(y)\, dy \right) \times \left(\int_D H[g, x_i, y_i] w(y)\, dy \right) dx. \tag{2.2}$$

Remark 2.2. If we take $n = 1$ and $D = I = \{a < x < b\}$ in (2.1), then we get

$$\left| \int_a^b w(t) f(t) g(t)\, dt - \frac{1}{\int_a^b w(t)\, dt} \left(\int_a^b w(t) f(t)\, dt \right) \left(\int_a^b w(t) g(t)\, dt \right) \right|$$
$$\leq \frac{1}{2\int_a^b w(t)\, dt} \int_a^b w(t) \left[|g(t)| \int_a^b \|f'\|_\infty |t-s| w(s)\, ds + |f(t)| \int_a^b \|g'\|_\infty |t-s| w(s)\, ds \right] dt.$$

Similarly, one can obtain the special version of (2.2). It is easy to see that the upper bound given on the right side in the above inequality (when $w(t) = 1$) is different from those given by Grüss in [3].

The next theorem deals with the discrete versions of the inequalities in Theorem 2.1.

Theorem 2.3. *Let f, g be real-valued functions defined on B and $\Delta_i f, \Delta_i g$ are bounded. Let $w(x)$ be a real-valued nonnegative function defined on B and $\sum_x w(x) > 0$. Then*

$$|L[w,f,g]| \leq \frac{1}{2\sum_x w(x)} \sum_x w(x) \left[|g(x)| \sum_y M[f,x_i,y_i] w(y) + |f(x)| \sum_y M[g,x_i,y_i] w(y) \right], \tag{2.3}$$

$$|L(w,f,g)| \leq \frac{1}{\left(\sum_x w(x)\right)^2} \sum_x w(x) \left(\sum_y M[f,x_i,y_i] w(y) \right) \left(\sum_y M[g,x_i,y_i] w(y) \right). \tag{2.4}$$

Remark 2.4. In a recent paper [6] the author gave multidimensional Grüss type finite difference inequalities whose proofs were based on a certain finite difference identity. Here we note that the inequalities established in (2.3) and (2.4) are of more general type and can be considered as the weighted generalizations of the similar inequalities given in [6].

3. Proof of Theorem 2.1

Let $x = (x_1, \ldots, x_n) \in \bar{D},\ y = (y_1, \ldots, y_n) \in D$. From the n-dimensional version of the mean value theorem we have (see [7, p. 174])

$$f(x) - f(y) = \sum_{i=1}^{n} \frac{\partial f(c)}{\partial x_i} (x_i - y_i) \tag{3.1}$$

and

$$g(x) - g(y) = \sum_{i=1}^{n} \frac{\partial g(c)}{\partial x_i} (x_i - y_i), \tag{3.2}$$

where $c = (y_1 + \alpha(x_1 - y_1), \ldots, y_n + \alpha(x_n - y_n))$ $(0 < \alpha < 1)$. Multiplying both sides of (3.1) and (3.2) by $g(x)$ and $f(x)$ respectively and adding we get

$$2f(x)g(x) - g(x)f(y) - f(x)g(y) = g(x) \sum_{i=1}^{n} \frac{\partial f(c)}{\partial x_i} (x_i - y_i) + f(x) \sum_{i=1}^{n} \frac{\partial g(c)}{\partial x_i} (x_i - y_i). \tag{3.3}$$

Multiplying both sides of (3.3) by $w(y)$ and integrating the resulting identity with respect to y over D we have

$$2\left(\int_D w(y)\,dy\right) f(x)g(x) - g(x)\int_D w(y)f(y)\,dy - f(x)\int_D w(y)g(y)\,dy = g(x)\int_D \sum_{i=1}^{n} \frac{\partial f(c)}{\partial x_i}(x_i - y_i) w(y)\,dy + f(x)\int_D \sum_{i=1}^{n} \frac{\partial g(c)}{\partial x_i}(x_i - y_i) w(y)\,dy. \tag{3.4}$$

Next, multiplying both sides of (3.4) by $w(x)$ and integrating the resulting identity with respect to x on D we get

$$2\left(\int_D w(y)\,dy\right)\int_D w(x) f(x) g(x)\,dx - \left(\int_D w(x) g(x)\,dx\right)\left(\int_D w(y) f(y)\,dy\right) - \left(\int_D w(x) f(x)\,dx\right)\left(\int_D w(y) g(y)\,dy\right)$$
$$= \int_D w(x) g(x)\left(\int_D \sum_{i=1}^{n} \frac{\partial f(c)}{\partial x_i}(x_i - y_i) w(y)\,dy\right)dx + \int_D w(x) f(x)\left(\int_D \sum_{i=1}^{n} \frac{\partial g(c)}{\partial x_i}(x_i - y_i) w(y)\,dy\right)dx. \tag{3.5}$$

From (3.5) and using the properties of modulus we have

$$\begin{aligned}
|A[w,f,g]| &\le \frac{1}{2\int_D w(x)\,dx}\left[\int_D w(x)|g(x)|\left(\int_D \sum_{i=1}^{n}\left|\frac{\partial f(c)}{\partial x_i}\right||x_i - y_i| w(y)\,dy\right)dx\right.\\
&\quad\left. + \int_D w(x)|f(x)|\left(\int_D \sum_{i=1}^{n}\left|\frac{\partial g(c)}{\partial x_i}\right||x_i - y_i| w(y)\,dy\right)dx\right]\\
&\le \frac{1}{2\int_D w(x)\,dx}\int_D w(x)\left[|g(x)|\int_D \sum_{i=1}^{n}\left\|\frac{\partial f}{\partial x_i}\right\|_\infty |x_i - y_i| w(y)\,dy\right.\\
&\quad\left. + |f(x)|\int_D \sum_{i=1}^{n}\left\|\frac{\partial g}{\partial x_i}\right\|_\infty |x_i - y_i| w(y)\,dy\right]dx\\
&= \frac{1}{2\int_D w(x)\,dx}\int_D w(x)\left[|g(x)|\int_D H[f,x_i,y_i] w(y)\,dy\right.\\
&\quad\left. + |f(x)|\int_D H[g,x_i,y_i] w(y)\,dy\right]dx.
\end{aligned}$$

This is the required inequality in (2.1).

Multiplying both sides of (3.1) and (3.2) by $w(y)$ and integrating the resulting identities with respect to y on D we get

$$\left(\int_D w(y)\,dy\right) f(x) - \int_D w(y) f(y)\,dy = \int_D \sum_{i=1}^{n} \frac{\partial f(c)}{\partial x_i}(x_i - y_i) w(y)\,dy \tag{3.6}$$

and

$$\left(\int_D w(y)\,dy\right) g(x) - \int_D w(y) g(y)\,dy = \int_D \sum_{i=1}^{n} \frac{\partial g(c)}{\partial x_i}(x_i - y_i) w(y)\,dy. \tag{3.7}$$

Multiplying the left sides and right sides of (3.6) and (3.7) we get

$$\begin{aligned}(3.8)\quad &\left(\int_D w(y)\,dy\right)^2 f(x)\,g(x) - \left(\int_D w(y)\,dy\right) f(x)\int_D w(y)\,g(y)\,dy \\ &- \left(\int_D w(y)\,dy\right) g(x)\int_D w(y)\,f(y)\,dy + \left(\int_D w(y)\,f(y)\,dy\right)\left(\int_D w(y)\,g(y)\,dy\right) \\ &= \left(\int_D \sum_{i=1}^n \frac{\partial f(c)}{\partial x_i}(x_i - y_i)\,w(y)\,dy\right)\left(\int_D \sum_{i=1}^n \frac{\partial g(c)}{\partial x_i}(x_i - y_i)\,w(y)\,dy\right).\end{aligned}$$

Multiplying both sides of (3.8) by $w(x)$ and integrating the resulting identity with respect to x on D, by simple calculations we obtain

$$\begin{aligned}(3.9)\quad &\int_D w(x)\,f(x)\,g(x)\,dx - \frac{1}{\int_D w(y)\,dy}\left(\int_D w(x)\,f(x)\,dx\right)\left(\int_D w(x)\,g(x)\,dx\right) \\ &= \frac{1}{\left(\int_D w(y)\,dy\right)^2}\int_D w(x)\left(\int_D \sum_{i=1}^n \frac{\partial f(c)}{\partial x_i}(x_i - y_i)\,w(y)\,dy\right) \\ &\qquad\times\left(\int_D \sum_{i=1}^n \frac{\partial g(c)}{\partial x_i}(x_i - y_i)\,w(y)\,dy\right)dx.\end{aligned}$$

From (3.9) and following the proof of the inequality (2.1) with suitable modifications we get the required inequality in (2.2). The proof is complete. □

Remark 3.1. Multiplying the left sides and right sides of (3.1) and (3.2), then multiplying the resulting identity by $w(y)$, integrating it with respect to y on D, again multiplying the resulting identity by $w(x)$, integrating it with respect to x over D and following the similar arguments as in the proofs of (2.1), (2.2) we have

$$(3.10)\quad |A[w,f,g]| \le \frac{1}{2\int_D w(x)\,dx}\int_D w(x)\left(\int_D H[f,x_i,y_i]\,H[g,x_i,y_i]\,w(y)\,dy\right)dx.$$

4. Proof of Theorem 2.3

For $x = (x_1, \ldots, x_n)\,, y = (y_1, \ldots, y_n)$ in B, it is easy to observe that the following identities hold (see [6]):

$$(4.1)\quad f(x) - f(y) = \sum_{i=1}^n \left\{\sum_{t_i = y_i}^{x_i - 1} \Delta_i f(y_1, \ldots, y_{i-1}, t_i, x_{i+1}, \ldots, x_n)\right\}$$

and

$$(4.2)\quad g(x) - g(y) = \sum_{i=1}^n \left\{\sum_{t_i = y_i}^{x_i - 1} \Delta_i g(y_1, \ldots, y_{i-1}, t_i, x_{i+1}, \ldots, x_n)\right\}.$$

Multiplying both sides of (4.1) and (4.2) by $g(x)$ and $f(x)$ respectively, and adding we obtain

$$\begin{aligned}(4.3)\quad & 2f(x)g(x) - g(x)f(y) - f(x)g(y) \\ &= g(x)\sum_{i=1}^{n}\left\{\sum_{t_i=y_i}^{x_i-1}\Delta_i f(y_1,\ldots,y_{i-1},t_i,x_{i+1},\ldots,x_n)\right\} \\ &\quad + f(x)\sum_{i=1}^{n}\left\{\sum_{t_i=y_i}^{x_i-1}\Delta_i g(y_1,\ldots,y_{i-1},t_i,x_{i+1},\ldots,x_n)\right\}.\end{aligned}$$

Multiplying both sides of (4.3) by $w(y)$ and summing both sides of the resulting identity with respect to y over B, we have

$$\begin{aligned}(4.4)\quad & 2\sum_y w(y)f(x)g(x) - g(x)\sum_y w(y)f(y) - f(x)\sum_y w(y)g(y) \\ &= g(x)\sum_y\left(\sum_{i=1}^{n}\left\{\sum_{t_i=y_i}^{x_i-1}\Delta_i f(y_1,\ldots,y_{i-1},t_i,x_{i+1},\ldots,x_n)\right\}\right)w(y) \\ &\quad + f(x)\sum_y\left(\sum_{i=1}^{n}\left\{\sum_{t_i=y_i}^{x_i-1}\Delta_i g(y_1,\ldots,y_{i-1},t_i,x_{i+1},\ldots,x_n)\right\}\right)w(y).\end{aligned}$$

Now, multiplying both sides of (4.4) by $w(x)$ and summing the resulting identity with respect to x on B we have

$$\begin{aligned}(4.5)\quad & 2\left(\sum_y w(y)\right)\sum_x w(x)f(x)g(x) \\ & - \left(\sum_x w(x)g(x)\right)\left(\sum_y w(y)f(y)\right) - \left(\sum_x w(x)f(x)\right)\left(\sum_y w(y)g(y)\right) \\ &= \sum_x w(x)g(x)\left[\sum_y\left(\sum_{i=1}^{n}\left\{\sum_{t_i=y_i}^{x_i-1}\Delta_i f(y_1,\ldots,y_{i-1},t_i,x_{i+1},\ldots,x_n)\right\}\right)w(y)\right] \\ &\quad + \sum_x w(x)f(x)\left[\sum_y\left(\sum_{i=1}^{n}\left\{\sum_{t_i=y_i}^{x_i-1}\Delta_i g(y_1,\ldots,y_{i-1},t_i,x_{i+1},\ldots,x_n)\right\}\right)w(y)\right].\end{aligned}$$

From (4.5) and using the properties of modulus we have

$$\begin{aligned} & |L(w,f,g)|| \\ &\le \frac{1}{2\sum_x w(x)}\left[\sum_x w(x)|g(x)|\right. \\ &\quad \times\sum_y\left(\sum_{i=1}^{n}\left|\left\{\sum_{t_i=y_i}^{x_i-1}|\Delta_i f(y_1,\ldots,y_{i-1},t_i,x_{i+1},\ldots,x_n)|\right\}\right|\right)w(y) \\ &\quad + \sum_x w(x)|f(x)| \\ &\quad \left.\times\sum_y\left(\sum_{i=1}^{n}\left|\left\{\sum_{t_i=y_i}^{x_i-1}|\Delta_i g(y_1,\ldots,y_{i-1},t_i,x_{i+1},\ldots,x_n)|\right\}\right|\right)w(y)\right]\end{aligned}$$

$$\leq \frac{1}{2\sum_x w(x)} \sum_x w(x) \left[|g(x)| \sum_y \left(\sum_{i=1}^n \left| \left\{ \|\Delta_i f\|_\infty \sum_{t_i=y_i}^{x_i-1} 1 \right\} \right| \right) w(y) \right.$$
$$\left. + |f(x)| \sum_y \left(\sum_{i=1}^n \|\Delta_i g\|_\infty |x_i - y_i| \right) w(y) \right]$$
$$= \frac{1}{2\sum_x w(x)} \sum_x w(x) \left[\sum_x w(x) |g(x)| \sum_y \left(\sum_{i=1}^n \|\Delta_i f\|_\infty |x_i - y_i| \right) w(y) \right.$$
$$\left. + |f(x)| \sum_y \left(\sum_{i=1}^n \|\Delta_i g\|_\infty |x_i - y_i| \right) w(y) \right]$$
$$= \frac{1}{2\sum_x w(x)} \sum_x w(x) \left[|g(x)| \sum_y M[f, x_i, y_i] w(y) \right.$$
$$\left. + |f(x)| \sum_y M[g, x_i, y_i] w(y) \right],$$

which is the required inequality in (2.3).

The proof of the inequality (2.4) can be completed by following the proof of (2.3) and closely looking at the proof of (2.2). Here we omit the details. □

Remark 4.1. Multiplying the left sides and right sides of (4.1) and (4.2), then multiplying the resulting identity by $w(y)$, summing it with respect to y over B, again multiplying the resulting identity by $w(x)$, summing it with respect to x over B and closely looking at the proof of the inequality (2.3) we get

$$|L(w, f, g)| \leq \frac{1}{2\sum_x w(x)} \sum_x w(x) \left(\sum_y M[f, x_i, y_i] M[g, x_i, y_i] w(y) \right). \tag{4.6}$$

In concluding we note that in [2] Fink has given some Grüss type inequalities for measures other than the Lebesgue measure, including signed measures which provide different upper bounds. In addition, in [2] new proofs to some old results are also given. However, the inequalities established here are different and cannot be compared with those of given in [2].

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