Journal of Inequalities in Pure and Applied Mathematics

A NOTE ON COMMUTATIVE BANACH ALGEBRAS

TAKASHI SANO

Department of Mathematical Sciences
Faculty of Science
Yamagata University
Yamagata 990-8560, Japan.

EMail: sano@sci.kj.yamagata-u.ac.jp

volume 7, issue 2, article 68, 2006.

Received 09 March, 2006; accepted 06 April, 2006.

Communicated by: C.-K. Li

Abstract
Contents
Home Page
Go Back
Close
Quit

ISSN (electronic): 1443-5756
104-06

Abstract

Let $\mathcal{A}$ be a unital Banach algebra over $\mathbb{C}$ with norm $\|\cdot\|$. In this note, several characterizations of commutativity of $\mathcal{A}$ are given. For instance, it is shown that $\mathcal{A}$ is commutative if

$$\|AB\| = \|BA\|$$

for all $A, B \in \mathcal{A}$, or if the spectral radius on $\mathcal{A}$ is a norm.

2000 Mathematics Subject Classification: 46J99, 47A30.
Key words: Commutative Banach algebra; Norm; Similarity transformation; Spectral radius.

The author is grateful to the referee for careful reading of the manuscript and for helpful comments.

Let $\mathcal{A}$ be a unital Banach algebra over $\mathbb{C}$ with norm $\|\cdot\|$. In this note, several characterizations of the commutativity of $\mathcal{A}$ are studied.

The following theorem is a simple characterization of commutativity in terms of norm inequalities, whose proof depends on complex analysis as the well-known one for the Fuglede-Putnum theorem, for instance, see [2, p. 278].

Theorem 1. *Let $\mathcal{A}$ be a unital Banach algebra over $\mathbb{C}$ with norm $\|\cdot\|_0$. If there is a norm $\|\cdot\|$ on $\mathcal{A}$ and positive constants γ, κ such that*

$$\|A\| \leqq \gamma\|A\|_0, \quad \|AB\| \leqq \kappa\|BA\|$$

for all $A, B \in \mathcal{A}$, then $\mathcal{A}$ is commutative, that is, $AB = BA$ for all $A, B \in \mathcal{A}$.

Before giving a proof, we recall the definition of e^A for $A \in \mathcal{A}$:

$$e^A := \sum_{n=0}^{\infty} \frac{1}{n!} A^n \in \mathcal{A}.$$

The assumption that $\mathcal{A}$ is a complete, unital normed algebra with a submultiplicative norm guarantees the convergence of this infinite series in $\mathcal{A}$ and implies

$$\frac{d}{dz} e^{zA} = A e^{zA} \quad (z \in \mathbb{C}).$$

Proof. Let $A, B \in \mathcal{A}$. Let us consider the normed space $(\mathcal{A}, \|\cdot\|)$. For each bounded linear functional φ on this normed space, we define a complex-valued function f on $\mathbb{C}$ by

$$f(z) := \varphi(e^{zA} B e^{-zA}) \quad (z \in \mathbb{C}).$$

Then the first assumption of $\|\cdot\|$ guarantees that f is an entire analytic function. f is also bounded: in fact, by the second assumption

$$\begin{aligned} |f(z)| &\leqq \|\varphi\| \|e^{zA} B e^{-zA}\| \\ &\leqq \kappa \|\varphi\| \|B e^{-zA} \cdot e^{zA}\| \\ &= \kappa \|\varphi\| \|B\| < \infty \qquad (z \in \mathbb{C}). \end{aligned}$$

Thus, by the Liouville theorem, f is constant. Hence,

$$0 = f'(z) = \varphi\left((A e^{zA}) B e^{-zA} + e^{zA} B (-A e^{-zA})\right).$$

Putting $z = 0$ yields

$$\varphi(AB - BA) = 0$$

for each bounded linear functional φ on $\mathcal{A}$. By the Hahn-Banach theorem, $AB = BA$ and the proof is completed. □

Remark 1.

1. *By considering completion, we find it sufficient to assume in Theorem 1 that $\mathcal{A}$ is a unital normed algebra over $\mathbb{C}$ with submultiplicative norm $\|\cdot\|_0$.*

2. *The assumption that*

$$\|AB\| \leqq \kappa\|BA\|$$

for all $A, B \in \mathcal{A}$ can be replaced with a weaker one

$$\|SAS^{-1}\| \leqq \kappa\|A\|$$

for all $A \in \mathcal{A}$ and all invertible $S \in \mathcal{A}$, or even further

$$\|e^{zA}Be^{-zA}\| \leqq \kappa\|B\|$$

for all $A, B \in \mathcal{A}$ and all $z \in \mathbb{C}$. In fact, it is essential to the proof of Theorem 1 that for given A, B

$$\sup\{\|e^{zA}Be^{-zA}\| : z \in \mathbb{C}\} < \infty.$$

Theorem 1 and Remark 1 (2) yield:

Corollary 2. *Let $\mathcal{A}$ be a unital Banach algebra over $\mathbb{C}$ with norm $\|\cdot\|$. Suppose that there is a positive constant γ such that*

$$\|AB\| \leqq \gamma \|BA\|$$

for all $A, B \in \mathcal{A}$. Then $\mathcal{A}$ is commutative. In particular, if $\|AB\| = \|BA\|$ for all $A, B \in \mathcal{A}$, then $\mathcal{A}$ is commutative.

Corollary 3 ([1, Exercise IV 4.1]). *On the set of all complex n-square matrices for $n \geqq 2$ no norm is invariant under all similarity transformations.*

See [1, p.102] for similarity transformations.

Corollary 4. *Let $\mathcal{A}$ be a unital Banach algebra over $\mathbb{C}$ with norm $\|\cdot\|$. If the spectral radius is a norm on $\mathcal{A}$, then $\mathcal{A}$ is commutative.*

This follows from Theorem 1 and the properties of the spectral radius $r(A)$ that $r(AB) = r(BA)$ and $r(A) \leqq \|A\|$ for $A, B \in \mathcal{A}$.

Remark 2. *There is a unital Banach algebra whose spectral radius is not a norm but a semi-norm. This semi-norm condition is not sufficient for commutativity.*

In fact, let $\mathcal{A}$ $(\subseteqq M_n(\mathbb{C}))$ be the set of upper triangular matrices whose diagonal entries are identical; $\mathcal{A}$ consists of $A := (a_{ij}) \in M_n(\mathbb{C})$ such that $a_{11} = a_{22} = \cdots = a_{nn} (=: \alpha)$ and $a_{ij} = 0 \quad (i > j)$. For this A, $r(A) = |\alpha|$ and the spectral radius on $\mathcal{A}$ is a semi-norm. Therefore, the unital Banach algebra $\mathcal{A}$ is a non-commutative example.

References

[1] R. BHATIA, *Matrix Analysis*, Springer-Verlag (1996).

[2] J.B. CONWAY, *A Course in Functional Analysis, 2nd Ed.*, Springer-Verlag, (1990).