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ANDERSSON'S INEQUALITY AND BEST POSSIBLE INEQUALITIES

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ABSTRACT. We investigate the notion of 'best possible inequality' in the context of Andersson's Inequality.

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Andersson [1] proved that if for each i, $f_i(0) = 0$ and f_i is convex and increasing, then

$$\int_0^1 \prod_1^n f_i(x)dx \geq \frac{2^n}{n+1} \prod_1^n \int_0^1 f_i(x)dx \tag{1}$$

with equality when each f_i is linear.

Elsewhere [2] we have proved that if $f_i \in M = \{f | f(0) = 0$ and $\frac{f(x)}{x}$ is increasing and bounded$\}$ and

$$d\sigma \in \widehat{M} = \left\{ d\sigma \,\middle|\, \int_0^t x d\sigma(x) \geq 0,\ \int_t^1 x d\sigma(x) \geq 0 \text{ for } t \in [0,1], \text{ and } \int_0^1 x d\sigma(x) > 0 \right\}$$

then

$$\int_0^1 \prod_1^n f_i(x) d\sigma(x) \geq \frac{\int_0^1 x^n d\sigma(x)}{\left(\int_0^1 x d\sigma(x)\right)^n} \prod_1^n \int_0^1 f_i(x) d\sigma(x). \tag{2}$$

One notices that if f is convex and increasing with $f(0) = 0$ then $f \in M$. For $\frac{f(x)}{x} = \int_0^1 f'(xt)dt$ when f' exists. The question arises if in fact Andersson's inequality can be extended beyond (2).

Lemma 1 (Andersson). *If $f_i(0) = 0$, increasing and convex, $i = 1, 2$ and $f_2^* = \alpha_2 x$ where α_2 is chosen so that $\int_0^1 f_2 = \int_0^1 f_2^*$ then $\int_0^1 f_1 f_2 \geq \int_0^1 f_1 f_2^*$.*

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We will examine whether Andersson's Lemma is best possible. We now discuss the notion of best possible.

An (integral) inequality $I(f, d\mu) \geq 0$ is best possible if the following situation holds. We consider both the functions and measures as 'variables'. Let the functions be in some universe U usually consisting of continuous functions and the measures in some universe $\widehat{U}$, usually regular Borel measures. Suppose we can find $M \subset U$ and $\widehat{M} \subset \widehat{U}$ so that $I(f, d\mu) \geq 0$ for all $f \in M$ if and only if $\mu \in \widehat{M}$ (given that $\mu \in \widehat{U}$) and $I(f, d\mu) \geq 0$ for all $\mu \in \widehat{M}$ if and only if $f \in M$ (given that $f \in U$). We then say the pair $(M, \widehat{M})$ give us a best possible inequality.

As an historical example, Chebyshev [3] in 1882 submitted a paper in which he proved that

$$\int_a^b f(x)g(x)p(x)dx \int_a^b p(x)dx \geq \int_a^b f(x)p(x)dx \int_a^b g(x)p(x)dx \tag{3}$$

provided that $p \geq 0$ and f and g were monotone in the same sense. Even before this paper appeared in 1883, it was shown to be not best possible since the pairs f, g for which (3) holds can be expanded. Consider the identity

$$\frac{1}{2}\int_a^b \int_a^b (f(x)-f(y))[g(x)-g(y)]p(x)p(y)dxdy = \int_a^b fgp \int_a^b p - \int_a^b fp \int_a^b gp. \tag{4}$$

So (3) holds if f and g are similarly ordered, i.e.

$$[f(x)-f(y)][g(x)-g(y)] \geq 0,\ x, y \in [a,b]. \tag{5}$$

For example x^2 and x^4 are similarly ordered but not monotone.

Jodeit and Fink [4] invented the notion of 'best possible' in a manuscript circulated in 1975 and published in parts in [3] and [4]. They showed that if we take U to be pairs of continuous functions and $\widehat{U}$ to be regular Borel measures μ with $\int_a^b d\mu > 0$, then

$$\int_a^b fg\, d\mu \int_a^b d\mu \geq \int_a^b f\, d\mu \int_a^b g\, d\mu \tag{6}$$

is a best possible inequality if $M_1 = \{(f,g) |$ (5) holds$\} \subset U$ and $\widehat{M_1} = \{\mu | \mu \geq 0\}$ i.e.
(6) holds for all pairs in M_1 if and only if $\mu \in \widehat{M_1}$, and
(6) holds for all $\mu \in \widehat{M_1}$ if and only if $(f,g) \in M_1$.

The sufficiency in both cases is the identity corresponding to (4). If $d\mu = \delta_x + \delta_y$ where x and $y \in [a,b]$, the inequality (6) gives (5), and if $f = g = x_A,\ A \subset [a,b]$, then (6) is $\mu(A)\mu(a,b) \geq \mu(A)^2$ which gives $\mu(A) \geq 0$. Strictly speaking this pair is not in M_1, but can be approximated in L_1 by continous functions.

If we return to Chebyshev's hypothesis that f and g are monotone in the same sense, let us take U be the class of pairs of continuous functions, neither of which is a constant and $\widehat{U}$ as above, $M_0 = \{f, g \in U |\ f$ and g are simularly monotone$\}$ and

$$\widehat{M_0} = \left\{\mu \,\middle|\, \int_a^t d\mu \geq 0,\ \int_t^b d\mu \geq 0 \text{ for } a \leq t \leq b \right\}.$$

Lemma 2. *The inequality (6) holds for all* $(f,g) \in M_0$ *if and only if* $\mu \in \widehat{M_0}$.

Proof. There exist measures $d\tau$ and $d\lambda$ such that $f(x) = \int_0^x d\tau$ and $g(x) = \int_0^x d\lambda$. We may assume $f(0) = g(0)$ since adding a constant to a function does not alter (6). Letting $x_+^0 = 0$ if

$x \le 0$ and 1 if $x > 0$ we can rewrite (6) after an interchange of order of integration as

(7) $$\int_0^1 \int_0^1 d\lambda(s)d\tau(t) \left[\int_0^1 d\mu \int_0^1 (x-t)_+^0 (x-s)_+^0 d\mu(x) - \int_0^1 (x-t)_+^0 d\mu(x) \int_0^1 (x-s)_+^0 d\mu(x)\right] \ge 0.$$

Since f, g are arbitrary increasing functions, $d\lambda$ and $d\tau \ge 0$ so (6) holds if and only if the $[\quad] \ge 0$ for each t and s. For example we may take both these measures, $d\tau, d\lambda$ to be point atoms. The equivalent condition then is that

(8) $$\int_0^1 d\mu \int_{t\vee s}^1 d\mu \ge \int_t^1 d\mu \int_s^1 d\mu.$$

By symmetry we may assume that $t \ge s$ so that (8) may be written $\int_0^s d\mu \int_t^1 d\mu \ge 0$. Consequently, if $d\mu \in \widehat{M_0}$ (6) holds and (6) holds for all $f, g \in M_0$ only if $\int_0^s d\mu \int_t^1 d\mu \ge 0$. But for $s = t$ this is the product of two numbers whose sum is positive so each factor must be non-negative, completing the proof. □

Lemma 3. *Suppose f and g are bounded integrable functions on $[0, 1]$. If (6) holds for all $\mu \in \widehat{M_0}$ then f and g are both monotone in the same sense.*

Proof. First let $d\mu = \delta_x + \delta_y$ where δ_x is an atom at x. Then (6) becomes $[f(x) - f(y)][g(x) - g(y)] \ge 0$, i.e. f and g are similarly ordered. If $x < y < z$, take $d\tau = \delta_x - \delta_y + \delta_z$ so that $\mu \in M_0$. To ease the burden of notation let the values of f at x, y, z be a, b, c and the corresponding values of g be A, B, C. By (6) we have

(9) $$aA - bB + cC \ge (a - b + c)(A - B + C).$$

By similar ordering we have

(10) $$(a-b)(A-B) \ge 0,\ (a-c)(A-C) \ge 0, \text{ and } (b-c)(B-C) \ge 0;$$

and (9) may be rewritten as

(11) $$(a-b)(C-B) + (c-b)(A-B) \le 0.$$

Now if one of the two terms in (10) is positive, the other is negative and all the factors are non-zero. By (10) the two terms are the same sign. Thus

(12) $$(a-b)(C-B) \le 0 \text{ and } (c-b)(A-B) \le 0.$$

Now (10) and (12) hold for any triple. We will show that if f is not monotone, then g is a constant.

We say that we have configuration I if $a < b$ and $c < b$, and configuration II if $a > b$ and $c > b$.

We claim that for both configurations I and II we must have $A = B = C$. Take configuration I. Now $b - a > 0$ implies that $B - A \ge 0$ by (10) and $C - B \ge 0$ by (12). Also $b - c > 0$ yields $(B - C) \ge 0$ by (10) and $A - B \ge 0$ by (12). Combining these we have $A = B = C$. The proof for configuration II is the same. □

Assume now that configuration I exists, so $A = B = C$. Let $x < x_0 < y$. If $a_0 < b$ $(a_0 = f(x_0))$ then x_0, y, z form a configuration I and $A_0 = B$. If $a_0 \ge b$, then x, x_0, z form a configuration I and $A_0 = B$. If $x_0 < x$ and $a_0 < b$, then again x_0, y, z form a configuration I and $A_0 = B$. Finally if $a_0 \ge b$ and $x_0 < x$ then x_0, x, b for a configuration II and $A_0 = B$. Thus for $x < y$ $g(0) \equiv g(y)$. The proof for $x > y$ is similar yielding that g is a constant.

If a configuration II exists, then the proof is similar, or alternately we can apply the configuration I argument to the pair $-f, -g$.

Finally if f is not monotone on $[0,1]$ then either a configuration I or II must exist and g is a constant. Consequently, if neither f nor g are constants, then both are monotone and by similar ordering, monotone in the same sense.

Note that if one of f, g is a constant, then (6) is an identity for any measure.

Theorem 4.

i) *Let M be defined as above and $N = \{g | g(0) = 0$ and g is increasing and bounded$\}$. Then for $F(x) \equiv \frac{f(x)}{x}$*

$$\int_0^1 fg d\sigma(x) \geq \left(\int_0^1 x d\sigma(x) \right)^{-1} \left(\int_0^1 F(x) x d\sigma \right) \left(\int_0^1 g(x) x d\sigma(x) \right) \tag{13}$$

holds for all pairs $(f, g) \in M \times N$ if and only if $d\sigma \in \widehat{M}$.

ii) *Let $f(0) = g(0) = 0$ and $\frac{f}{x}$ and g be of bounded variation on $[0,1]$. If (13) holds for all $d\sigma \in \widehat{M}$ then either $\frac{f}{x}$ or g is a constant (in which case (13) is an identity) or $\left(\frac{f}{x}, g\right) \in M \times N$.*

The proof starts with the observation that (13) is in fact a Chebyshev inequality

$$\int_0^1 Fg\, d\tau \int_0^1 d\tau \geq \int_0^1 F\, d\tau \int_0^1 g\, d\tau \tag{14}$$

where $d\tau = x\, d\sigma$; and F, g are the functions. The theorem is a corollary of the two lemmas.

Andersson's inequality (2) now follows by induction, replacing one f by f^* at a time. Note that the case $n = 2$ of Andersson's inequality (2) has the proof

$$\int_0^1 f_1 f_2 \geq \int_0^1 f_1^* f_2 \geq \int_0^1 f_1^* f_2^*$$

and it is only the first one which is best possible! The inequality between the extremes is perhaps 'best possible'.

Remark 5. Of course x can be replaced by any function that is zero at zero and positive elsewhere, i.e. $\frac{f(x)}{x}$ can be replaced by $\frac{f(x)}{p(x)}$ and the measure $d\tau = p(x) d\sigma(x)$.

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