Journal of Inequalities in Pure and Applied Mathematics

http://jipam.vu.edu.au/

Volume 5, Issue 4, Article 106, 2004

THE GENERALIZED SINE LAW AND SOME INEQUALITIES FOR SIMPLICES

SHIGUO YANG

DEPARTMENT OF MATHEMATICS
ANHUI INSTITUTE OF EDUCATION
HEFEI 230061
PEOPLE'S REPUBLIC OF CHINA.
sxx@ahieedu.net.cn

Received 31 July, 2003; accepted 23 April, 2004
Communicated by A. Sofo

ABSTRACT. The sines of k-dimensional vertex angles of an n-simplex is defined and the law of sines for k-dimensional vertex angles of an n-simplex is established. Using the generalized sine law for n-simplex, we obtain some inequalities for the sines of k-dimensional vertex angles of an n-simplex. Besides, we obtain inequalities for volumes of n-simplices. As corollaries, the generalizations to several dimensions of the Neuberg-Pedoe inequality and P. Chiakuei inequality, and an inequality for pedal simplex are given.

Key words and phrases: Simplex, k-dimensional vertex angle, Volume, Inequality.

2000 *Mathematics Subject Classification.* 52A40.

1. INTRODUCTION

The law of sines for triangles in E^2 has natural analogues in higher dimensions. In 1978, F. Eriksson [1] defined the n-dimensional sines of the n-dimensional corners of an n-simplex in n-dimensional Euclidean space E^n and obtained the law of sines for the n-dimensional corners of an n-simplex. In this paper, the sines of k-dimensional vertex angles of an n -simplex will be defined, and the law of sines for k-dimensional vertex angles of an n-simplex will be established. Using the generalized sine law for simplices and a known inequality in [2], we get some inequalities for the sines of k-dimensional vertex angles of an n-simplex.

Recently, Yang Lu and Zhang Jingzhong [2, 3], Yang Shiguo [4], Leng Gangson [5, 6] and D. Veljan [7] and V. Volenec et al. [9] have obtained some important inequalities for volumes of n-simplices. In this paper, some interesting new inequalities for volumes of n-simplices will be established. As corollaries, we will obtain an inequality for pedal simplex and a generalization to several dimensions of the Neuberg-Pedoe inequality, which differs from the results in [4], [5] and [6].

ISSN (electronic): 1443-5756

106-03

2. THE GENERALIZED SINE LAW FOR SIMPLICES

Let A_i $(i = 1, 2, \ldots, n+1)$ be the vertices of an n-dimensional simplex Ω_n in the n-dimensional Euclidean space E^n, V the volume of the simplex Ω_n and $F_i(n-1)$-dimensional content of Ω_n. F. Eriksson defined the n-dimensional sines of the n-dimensional corners α_i of the n-simplex Ω_n and obtained the law of sines for n-simplices as follows [1]

$$\frac{F_i}{{}^{n}\sin\alpha_i} = \frac{(n-1)!\prod_{j=1}^{n+1} F_j}{(nV)^{n-1}} \qquad (i = 1, 2, \ldots, n+1). \tag{2.1}$$

In this paper, we will define the sines of the k-dimensional vertex angles of an n-dimensional simplex and establish the law of sines for the k-dimensional vertex angles of an n-simplex. Let $V_{i_1 i_2 \cdots i_k}$ be the $(k-1)$-dimensional content of the $(k-1)$-dimensional face $A_{i_1}A_{i_2}\cdots A_{i_k}$ ($(k-1)$-simplex) of the simplex Ω_n for $k \in \{2, 3, \ldots, n+1\}$ and $i_1, i_2, \ldots, i_k \in \{1, 2, \ldots, n+1\}$, O and R denote the circumcenter and circumradius of the simplex Ω_n respectively. $\overrightarrow{OA_i} = R\boldsymbol{u_i}(i = 1, 2, \ldots, n+1)$, $\boldsymbol{u_i}$ is the unit vector. The sines of the k-dimensional vertex angles of the simplex Ω_n are defined as follows.

Definition 2.1. Let α_{ij} denote the angle formed by the vectors $\boldsymbol{u_i}$ and $\boldsymbol{u_j}$. The sine of a k-dimensional vertex angle $\varphi_{i_1 i_2 \cdots i_k}$ of the simplex Ω_n corresponding the $(k-1)$-dimensional face $A_{i_1}A_{i_2}\cdots A_{i_k}$ is defined as

$$\sin\varphi_{i_1 i_2 \cdots i_k} = (-D_{i_1 i_2 \cdots i_k})^{\frac{1}{2}}, \tag{2.2}$$

where

$$D_{i_1 i_2 \cdots i_k} = \begin{vmatrix} 0 & 1 & \cdots & 1 \\ 1 & & & \\ \vdots & & \boxed{-\frac{1}{2}\sin^2\frac{\alpha_{i_l i_m}}{2}} & \\ 1 & & & \end{vmatrix} \qquad (l, m = 1, 2, \ldots, k). \tag{2.3}$$

We will prove that

$$0 < (-D_{i_1 i_2 \cdots i_k})^{\frac{1}{2}} \le 1. \tag{2.4}$$

If $n = 2$, the sine of the 2-dimensional vertex angle φ_{ij} of the triangle $A_1A_2A_3$ is the sine of the angle formed by two edges A_kA_i and A_kA_j.

With the notation introduced above, we establish the law of sines for the k -dimensional vertex angles of an n-simplex as follows.

Theorem 2.1. *For an n-dimensional simplex Ω_n in E^n and $k \in \{2, 3, \ldots, n+1\}$, we have*

$$\frac{V_{i_1 i_2 \cdots i_k}}{\sin\varphi_{i_1 i_2 \cdots i_k}} = \frac{(2R)^{k-1}}{(k-1)!} \qquad (1 \le i_1 < i_2 < \cdots < i_k \le n+1). \tag{2.5}$$

Put $\varphi_{12\cdots i-1,i+1,\ldots,n+1} = \theta_i$, $V_{12\cdots i-1,i+1,\ldots,n+1} = F_i$ $(i = 1, 2, \ldots, n+1)$, by Theorem 2.1 we obtain the law of sines for the n-dimensional vertex angles of n-simplices as follows.

Corollary 2.2.

$$\frac{F_1}{\sin\theta_1} = \frac{F_2}{\sin\theta_2} = \cdots = \frac{F_{n+1}}{\sin\theta_{n+1}} = \frac{(2R)^{n-1}}{(n-1)!}. \tag{2.6}$$

If we take $n = 2$ in Theorem 2.1 or Corollary 2.2, we obtain the law of sines for a triangle $A_1A_2A_3$ in the form

$$\frac{a_1}{\sin A_1} = \frac{a_2}{\sin A_2} = \frac{a_3}{\sin A_3} = 2R. \tag{2.7}$$

Proof of Theorem 2.1. Let $a_{ij} = |A_iA_j|$ $(i, j = 1, 2, \dots, n+1)$, then

$$a_{ij} = 2R\sin\frac{\alpha_{ij}}{2},$$

$$\sin^2\varphi_{i_1i_2\cdots i_k} = -D_{i_1i_2\cdots i_k} = -\begin{vmatrix} 0 & 1 & \cdots & 1 \\ 1 & & & \\ \vdots & & \boxed{-\frac{1}{8R^2}a^2_{i_li_m}} & \\ 1 & & & \end{vmatrix} \tag{2.8}$$

$$= (-1)^k(8R^2)^{-(k-1)} \cdot \begin{vmatrix} 0 & 1 & \cdots & 1 \\ 1 & & & \\ \vdots & & \boxed{a^2_{i_li_m}} & \\ 1 & & & \end{vmatrix} \qquad (l, m = 1, 2, \dots, k).$$

By the formula for the volume of a simplex, we have

$$\begin{aligned} \sin^2\varphi_{i_1i_2\cdots i_k} &= -D_{i_1i_2\cdots i_k} \\ &= (-1)^k(8R^2)^{-(k-1)}(-1)^k2^{k-1}(k-1)!^2V^2_{i_1i_2\cdots i_k} \\ &= \frac{(k-1)!^2}{(2R)^{2(k-1)}}V^2_{i_1i_2\cdots i_k}. \end{aligned} \tag{2.9}$$

From this equality (2.5) follows.

Now we prove that inequality (2.4) holds. When $k \geq 2$, we have $k \leq 2^{k-1}$. Using the Voljan-Korchmaros inequality [3], we have

$$V_{i_1i_2\cdots i_k} \leq \frac{1}{(k-1)!}\left(\frac{k}{2^{k-1}}\right)^{\frac{1}{2}}\left(\prod_{1\leq l<m\leq k} a_{i_li_m}\right)^{\frac{2}{k}}. \tag{2.10}$$

Equality holds if and only if the simplex $A_{i_1}A_{i_2}\cdots A_{i_k}$ is regular.

Combining inequality (2.10) with equality (2.5), we get

$$\begin{aligned} V_{i_1i_2\cdots i_k} &\leq \frac{(2R)^{k-1}}{(k-1)!}\cdot\left(\frac{k}{2^{k-1}}\right)^{\frac{1}{2}}\left(\prod_{1\leq l<m\leq k}\sin\frac{\alpha_{i_li_m}}{2}\right)^{\frac{2}{k}} \\ &\leq \frac{(2R)^{k-1}}{(k-1)!}\cdot\left(\frac{k}{2^{k-1}}\right)^{\frac{1}{2}} \\ &\leq \frac{(2R)^{k-1}}{(k-1)!}. \end{aligned} \tag{2.11}$$

Using equality (2.5) and inequality (2.11), we get

$$0 < (-D_{i_1i_2\cdots i_k})^{\frac{1}{2}} = \sin\varphi_{i_1i_2\cdots i_k} = \frac{(k-1)!}{(2R)^{k-1}}V_{i_1i_2\cdots i_k} \leq 1.$$

□

For the sines of the k-dimensional vertex angles of an n-simplex, we obtain an inequality as follows.

Theorem 2.3. *Let* $\varphi_{i_1i_2\cdots i_k}$ $(1 \leq i_1 < i_2 < \cdots < i_k \leq n+1)$ *denote the* k*-dimensional vertex angles of an* n*-simplex* Ω_n *in* E^n*, and* $\lambda_i > 0$ $(i = 1, 2, \dots, n+1)$ *be arbitrary real numbers,* *then we have*

$$\sum_{1\le i_1<i_2<\cdots<i_k\le n+1}\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_k}\sin^2\varphi_{i_1i_2\cdots i_k}\le\frac{n!\cdot\left(\sum_{i=1}^{n+1}\lambda_i\right)^k}{(n-k+1)!(k-1)!(4n)^{k-1}}. \tag{2.12}$$

Equality holds if $\lambda_1=\lambda_2=\cdots=\lambda_{n+1}$ *and the simplex* Ω_n *is regular.*

By taking $\lambda_1=\lambda_2=\cdots=\lambda_{n+1}$ in the inequality (2.12), we get:

Corollary 2.4.

$$\sum_{1\le i_1<i_2<\cdots<i_k\le n+1}\sin^2\varphi_{i_1i_2\cdots i_k}\le\frac{n!\cdot(n+1)^k}{(n-k+1)!(k-1)!(4n)^{k-1}}. \tag{2.13}$$

Equality holds if the simplex Ω_n *is regular.*

To prove Theorem 2.3, we need a lemma as follows.

Lemma 2.5. *Let* Ω_n *be an* n*-simplex in* E^n, $x_i>0$ $(i=1,2,\dots,n+1)$ *be real numbers,* $V_{i_1i_2\cdots i_{s+1}}$ *be the* s*-dimensional volume of the* s*-dimensional simplex* $A_{i_1}A_{i_2}\cdots A_{i_{s+1}}$ *for* $i_1,i_2,\dots,i_{s+1}\in\{1,2,\dots,n+1\}$*. Put*

$$M_s=\sum_{1\le i_1<i_2<\cdots<i_k\le n+1}x_{i_1}x_{i_2}\cdots x_{i_{s+1}}V^2_{i_1i_2\cdots i_{s+1}},\ M_0=\sum_{i=1}^{n+1}x_i,$$

then we have

$$M_s^l\ge\frac{[(n-l)!(l!)^3]^s}{[(n-s)!(s!)^3]^l}(n!\cdot M_0)^{l-s}M_l^s(1\le s<l\le n). \tag{2.14}$$

Equality holds if and only if the intertial ellipsoid of the points $A_1,A_2,\dots,A_{n+1}$ *with masses* $x_1,x_2,\dots,x_{n+1}$ *is a sphere.*

For the proof of Lemma 2.5. the reader is referred to [2] or [9].

Proof of Theorem 2.3. By putting $s=1, l=k-1$ and $x_i=\lambda_i$ $(i=1,2,\dots,n+1)$ in the inequality (2.14), we have

$$\left(\sum_{1\le i<j\le n+1}\lambda_i\lambda_j a_{ij}^2\right)^{k-1}\ge\frac{(n-k+1)!\cdot(k-1)!^3}{[(n-1)!]^{k-1}}\left(n!\cdot\sum_{i=1}^{n+1}\lambda_i\right)^{k-2} \times\sum_{1\le i_1<i_2<\cdots<i_k\le n+1}\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_k}V^2_{i_1i_2\cdots i_k}. \tag{2.15}$$

By Theorem 2.1, we have

$$V_{i_1i_2\cdots i_k}=\frac{(2R)^{k-1}}{(k-1)!}\sin\varphi_{i_1i_2\cdots i_k}. \tag{2.16}$$

Using the known inequality [3]

$$\sum_{1\le i<j\le n+1}\lambda_i\lambda_j a_{ij}^2\le\left(\sum_{i=1}^{n+1}\lambda_i\right)^2R^2, \tag{2.17}$$

with equality if and only if the point $P=\sum_{i=1}^{n+1}\lambda_iA_i$ is the circumcenter of simplex Ω_n.

Combining (2.15) with (2.16) and (2.17), we obtain inequality (2.12). It is easy to see that equality holds in (2.12) if $\lambda_1=\lambda_2=\cdots=\lambda_{n+1}$ and simplex Ω_n is regular. □

3. SOME INEQUALITIES FOR VOLUMES OF SIMPLICES

Let P be an arbitrary point inside the simplex Ω_n and B_i the orthogonal projection of the point P on the $(n-1)$-dimensional plane σ_i containing $(n-1)$-simplex $f_i = A_1 \cdots A_{i-1}A_{i+1} \cdots A_{n+1}$. Simplex $\overline{\Omega}_n = B_1 B_2 \cdots B_{n+1}$ is called the pedal simplex of the point P with respect to the simplex Ω_n. Let $r_i = |PB_i|$ $(i = 1, 2, \ldots, n+1)$, $\overline{V}$ be the volume of the pedal simplex $\overline{\Omega}_n$, $V(i)$ and $\overline{V}(i)$ denote the volumes of two n-dimensional simplices $\Omega_n(i) = A_1 \cdots A_{i-1} P A_{i+1} \cdots A_{n+1}$ and $\overline{\Omega}_n(i) = B_1 \cdots B_{i-1} P B_{i+1} \cdots B_{n+1}$, respectively. Then we have an inequality for volumes of just defined n-simplices as follows.

Theorem 3.1. *Let P be an arbitrary point inside n-dimensional simplex Ω_n and λ_i $(i = 1, 2, \ldots, n+1)$ positive real numbers, then we have*

$$\sum_{i=1}^{n+1} \lambda_1 \cdots \lambda_{i-1} \lambda_{i+1} \cdots \lambda_{n+1} \overline{V}(i) \le \frac{1}{n^n} \left[\sum_{i=1}^{n} \lambda_i V(i) \right]^n V^{1-n}, \tag{3.1}$$

with equality if the simplex Ω_n is regular, P is the circumcenter of Ω_n and $\lambda_1 = \lambda_2 = \cdots = \lambda_{n+1}$.

Now we state some applications of Theorem 3.1.

If taking $\lambda_1 = \lambda_2 = \cdots = \lambda_{n+1}$ in inequality (3.1), we have

$$\sum_{i=1}^{n+1} \overline{V}(i) \le \frac{1}{n^n} \left[\sum_{i=1}^{n+1} V(i) \right]^n \cdot V^{1-n}. \tag{3.2}$$

Since the point P is in the interior of the simplex Ω_n, then

$$\sum_{i=1}^{n+1} \overline{V}(i) = \overline{V}, \quad \sum_{i=1}^{n+1} V(i) = V. \tag{3.3}$$

Using (3.2) and (3.3) we obtain an inequality for the volume of the pedal simplex $\overline{\Omega}_n$ of the point P with respect to the simplex Ω_n as follows.

Corollary 3.2. *Let P be an arbitrary point inside the n-simplex Ω_n, then we have*

$$\overline{V} \le \frac{1}{n^n} V, \tag{3.4}$$

with equality if simplex Ω_n is regular and P is the circumcenter of Ω_n.

Corollary 3.3. *Let P be an arbitrary point inside the n-simplex Ω_n, then we have*

$$\sum_{i=1}^{n+1} V(i) \cdot \overline{V}(i) \le \frac{1}{(n+1)n^n} V^2, \tag{3.5}$$

with equality if the simplex Ω_n is regular and P is the circumcenter of Ω_n.

Proof. Let $\lambda_i = [V(i)]^{-1}$ $(i = 1, 2, \ldots, n+1)$ in inequality (3.1); we get

$$\sum_{i=1}^{n+1} V(i) \cdot \overline{V}(i) \le \left(\frac{n+1}{n} \right)^n V^{1-n} \prod_{j=1}^{n+1} V(j). \tag{3.6}$$

Using the arithmetic-geometric mean inequality and equality (3.3), we have

$$\sum_{i=1}^{n+1} V(i) \cdot \overline{V}(i) \le \left(\frac{n+1}{n} \right)^n V^{1-n} \left[\frac{1}{n+1} \sum_{j=1}^{n+1} V(j) \right]^{n+1} = \frac{1}{(n+1)n^n} V^2.$$

It is easy to see that equality in (3.5) holds if the simplex Ω_n is regular and the point P is the circumcenter of Ω_n. □

Proof of Theorem 3.1. Let h_i be the altitude of simplex Ω_n from vertex A_i, $\overrightarrow{PB}_i = r_i \boldsymbol{e_i}$, where $\boldsymbol{e_i}$ is the unit outer normal vector of the i-th side face $f_i = A_1 \cdots A_{i-1} A_{i+1} \cdots A_{n+1}$ of the simplex Ω_n, and ${}^n \sin \alpha_k$ be the n-dimensional sine of the k-th corner α_k of the simplex Ω_n. Wang and Yang [8] proved that

$$(3.7)\qquad {}^n \sin \alpha_n = [\det(e_i \cdot e_j)_{ij \neq k}]^{\frac{1}{2}} \qquad (k = 1, 2, \dots, n+1).$$

By the formula for the volume of an n-simplex and (3.7), we have

$$(3.8)\qquad \overline{V}(i) = \frac{1}{n!} [\det(r_l r_k \boldsymbol{e_l} \cdot \boldsymbol{e_k})_{l,k \neq i}]^{\frac{1}{2}} = \frac{1}{n!} \left(\prod_{\substack{j=1 \\ j \neq i}}^{n+1} r_j \right) \cdot {}^n \sin \alpha_i .$$

Using (3.8), (2.1) and $nV = h_i F_i$, we get that

$$\begin{aligned} &\sum_{i=1}^{n+1} \lambda_1 \cdots \lambda_{i-1} \lambda_{i+1} \cdots \lambda_{n+1} \overline{V}(i) \\ &\qquad = \frac{1}{n!} \sum_{i=1}^{n+1} \left(\prod_{\substack{j=1 \\ j \neq i}}^{n+1} \lambda_j r_j \right) \cdot {}^n \sin \alpha_i \\ &\qquad = \frac{1}{n!} \sum_{i=1}^{n+1} \left\{ \left(\prod_{\substack{j=1 \\ j \neq i}}^{n+1} \lambda_j r_j \right) (nV)^{n-1} \left[(n-1)! \cdot \prod_{\substack{j=1 \\ j \neq i}}^{n+1} F_j \right]^{-1} \right\} \\ &\qquad = [(n!)^2 \cdot V]^{-1} \sum_{i=1}^{n+1} \left(\prod_{\substack{j=1 \\ j \neq i}}^{n+1} \lambda_j r_j h_j \right), \end{aligned}$$

i.e.

$$(3.9)\qquad (n!)^2 V \sum_{i=1}^{n+1} \lambda_1 \cdots \lambda_{i-1} \lambda_{i+1} \cdots \lambda_{n+1} \overline{V}(i) = \sum_{i=1}^{n+1} \left(\prod_{\substack{j=1 \\ j \neq i}}^{n+1} \lambda_j r_j h_j \right).$$

Taking $s = n-1,\ l = n$ in inequality (2.14), we get

$$(3.10)\qquad \left[\sum_{i=1}^{n+1} \left(\prod_{\substack{j=1 \\ j \neq i}}^{n+1} x_j \right) F_i^2 \right]^n \geq \frac{n^{3n}}{(n!)^2} \left(\sum_{i=1}^{n+1} x_i \right) \left(\prod_{i=1}^{n+1} x_i \right)^{n-1} V^{2(n-1)}.$$

Let $x_i = (\lambda_i r_i h_i)^{-1}$ $(i = 1, 2, \dots, n+1)$ in inequality (3.10). Then we have

$$(3.11)\qquad \left(\sum_{i=1}^{n+1} \lambda_i r_i h_i F_i^2 \right)^n \geq \frac{n^{3n}}{(n!)^2} \left[\sum_{i=1}^{n+1} \left(\prod_{\substack{j=1 \\ j \neq i}}^{n+1} \lambda_j r_j h_j \right) F_i^2 \right] \cdot V^{2(n-1)}.$$

Using inequality (3.11) and $r_i F_i = nV(i),\ h_i F_i = nV$, we get

$$V^n \left[\sum_{i=1}^{n+1} \lambda_i V(i) \right]^n \geq \frac{n^n}{(n!)^2} V^{2(n-1)} \sum_{i=1}^{n+1} \left(\prod_{\substack{j=1 \\ j\neq i}}^{n+1} \lambda_j r_j h_j \right). \tag{3.12}$$

Substituting equality (3.9) into inequality (3.12) we get inequality (3.1). It is easy to prove that equality in (3.1) holds if simplex Ω_n is regular, P is the circumcenter of Ω_n and $\lambda_1 = \lambda_2 = \cdots = \lambda_{n+1}$. Theorem 3.1 is proved. □

Finally, we shall establish some inequalities for volumes of two n-simplices. As corollaries, the generalizations to several dimensions of the Neuberg-Pedoe inequality and P.Chiakui inequality will be given.

Let a_i $(i = 1, 2, 3)$ denote the sides of the triangle $A_1A_2A_3$ with area Δ, and a_i' $(i = 1, 2, 3)$ denote the sides of the triangle $A_1'A_2'A_3'$ with area Δ', then

$$\sum_{i=1}^{3} a_i^2 \left(\sum_{j=1}^{3} \left(a_j'\right)^2 - 2\left(a_i'\right)^2 \right) \geq 16\Delta\Delta', \tag{3.13}$$

with equality if and only if $\Delta A_1A_2A_3$ is similar to $\Delta A_1'A_2'A_3'$.

Inequality (3.13) is the well-known Neuberg-Pedoe inequality.

In 1984, P. Chiakui [9] proved the following sharpening of the Neuberg-Pedoe inequality:

$$\sum_{i=1}^{3} a_i^2 \left(\sum_{j=1}^{3} \left(a_j'\right)^2 - 2\left(a_i'\right)^2 \right) \geq 8 \left(\frac{\left(a_1'\right)^2 + \left(a_2'\right)^2 + \left(a_3'\right)^2}{a_1^2 + a_2^2 + a_3^2} \Delta^2 + \frac{a_1^2 + a_2^2 + a_3^2}{\left(a_1'\right)^2 + \left(a_2'\right)^2 + \left(a_3'\right)^2} \left(\Delta'\right)^2 \right), \tag{3.14}$$

with equality if and only if $\Delta A_1A_2A_3$ is similar to $\Delta A_1'A_2'A_3'$.

Recently, Leng Gangson [5] has extended inequality (3.14) to the edge lengths and volumes of two n-simplices. In this paper, we shall extend inequality (3.14) to the volumes of two n-simplices and the contents of their side faces. As a corollary, we get a generalization to several dimensions of the Neuberg-Pedoe inequality. Let A_i $(i = 1, 2, \ldots, n+1)$ be the vertices of n-simplex Ω_n in E^n, V the volume of the simplex Ω_n and $F_i(n-1)$- dimensional content of the $(n-1)$-dimensional face $f_i = A_1 \cdots A_{i-1}A_{i+1} \cdots A_{n+1}$ of Ω_n. For two n-simplices Ω_n and Ω_n' and real numbers $\alpha, \beta \in (0, 1]$, we put

$$\sigma_n(\alpha) = \sum_{i=1}^{n+1} F_i^\alpha, \quad \sigma_n(\beta) = \sum_{i=1}^{n+1} \left(F_i'\right)^\beta, \quad b_n = \frac{n^3}{n+1} \left(\frac{n+1}{n!^2} \right)^{\frac{1}{n}}. \tag{3.15}$$

We obtain an inequality for volumes of two n-simplices as follows.

Theorem 3.4. *For any two n-dimensional simplices Ω_n and Ω_n' and two arbitrary real numbers $\alpha, \beta \in (0, 1]$, we have*

$$\sum_{i=1}^{n+1} F_i^\alpha \left(\sum_{j=1}^{n+1} \left(F_j'\right)^\beta - 2\left(F_i'\right)^\beta \right) \geq \frac{(n-1)^2}{2} \left[b_n^\alpha \frac{\sigma_n(\beta)}{\sigma_n(\alpha)} V^{2(n-1)\alpha/n} + b_n^\beta \frac{\sigma_n(\alpha)}{\sigma_n(\beta)} \left(V'\right)^{2(n-1)\beta/n} \right]. \tag{3.16}$$

Equality holds if and only if simplices Ω_n and Ω_n' are regular.

Using inequality (3.16) and the arithmetic-geometric mean inequality, we get the following corollary.

Corollary 3.5. *For any two n-dimensional simplices Ω_n and Ω_n' and two arbitrary real numbers $\alpha, \beta \in (0,1]$, we have*

$$\sum_{i=1}^{n+1} F_i^{\alpha} \left(\sum_{j=1}^{n+1} \left(F_j'\right)^{\beta} - 2\left(F_i'\right)^{\beta} \right) \geq b_n^{(\alpha+\beta)/2}(n^2-1)(V^{\alpha}\left(V'\right)^{\beta})^{(n-1)/n}. \tag{3.17}$$

Equality holds if and only if simplices Ω_n and Ω_n' are regular.

If we let $\alpha = \beta$ in Corollary 3.5, we get Leng Gangson's inequality [6] as follows. For any $\theta \in (0,1]$ we have

$$\sum_{i=1}^{n+1} F_i^{\theta} \left(\sum_{j=1}^{n+1} \left(F_j'\right)^{\theta} - 2\left(F_i'\right)^{\theta} \right) \geq b_n^{\theta}(n^2-1)(VV')^{(n-1)\theta/n}, \tag{3.18}$$

with equality if and only if simplices Ω_n and Ω_n' are regular.

To prove Theorem 3.4, we need some lemmas as follows.

Lemma 3.6. *For an n-simplex Ω_n and arbitrary number $\alpha \in (0,1]$, we have*

$$\frac{\prod_{i=1}^{n+1} F_i^{2\alpha}}{\sum_{i=1}^{n+1} F_i^{2\alpha}} \geq \frac{1}{(n+1)^{(n-1)\alpha+1}} \left[\frac{n^{3n}}{(n!)^2}\right]^{\alpha} V^{2(n-1)\alpha}, \tag{3.19}$$

with equality if and only if simplex Ω_n is regular.

Proof. If taking $l = n,\ s = n-1$ and $x_i = F_i^2\ (i = 1,2,\dots,n+1)$ in inequality (2.14), we get an inequality as follows

$$\frac{(n+1)^n(n!)^2}{n^{3n}} \prod_{i=1}^{n+1} F_i^2 \geq V^{2(n-1)} \sum_{i=1}^{n+1} F_i^2,$$

or

$$\frac{(n+1)^{n\alpha}(n!)^{2\alpha}}{n^{3n\alpha}} \prod_{i=1}^{n+1} F_i^{2\alpha} \geq V^{2(n-1)\alpha} \left(\sum_{i=1}^{n+1} F_i^2\right)^{\alpha}. \tag{3.20}$$

It is easy to prove that equality in (3.20) holds if and only if simplex Ω_n is regular. From inequality (3.20) we know that inequality (3.19) holds for $\alpha = 1$. For $\alpha \in (0,1)$, using inequality (3.20) and the well-known inequality

$$\sum_{i=1}^{n+1} F_i^2 \geq (n+1)\left(\frac{1}{n+1}\sum_{i=1}^{n+1} F_i^{2\alpha}\right)^{\frac{1}{\alpha}}, \tag{3.21}$$

we get inequality (3.19). It is easy to see that equality in (3.19) holds if and only if the simplex Ω_n is regular. □

Lemma 3.7. *For an n-simplex $\Omega_n (n \geq 3)$ and an arbitrary number $\alpha \in (0,1]$, we have*

$$\left(\sum_{i=1}^{n+1} F_i^{\alpha}\right)^2 - 2\sum_{i=1}^{n+1} F_i^{2\alpha} \geq b_n^{\alpha}(n^2-1)V^{2(n-1)\alpha/n}, \tag{3.22}$$

with equality if and only if the simplex Ω_n is regular.

For the proof of Lemma 3.7, the reader is referred to [6].

Lemma 3.8. *Let* a_i $(i=1,2,3)$ *and* Δ *denote the sides and the area of the triangle* $(A_1A_2A_3)$*, respectively. For arbitrary number* $\alpha \in (0,1]$*, denote by* Δ_α *the area of the triangle* $(A_1A_2A_3)_\alpha$ *with sides* a_i^α $(i=1,2,3)$*, then the following inequality holds*

$$\Delta_\alpha^2 \geq \frac{3}{16}\left(\frac{16}{3}\Delta^2\right)^\alpha. \tag{3.23}$$

For $\alpha \neq 1$*, equality holds if and only if* $a_1 = a_2 = a_3$*.*

For the proof of Lemma 3.8, the reader is referred to [9].

Lemma 3.9. *Let numbers* $x_i > 0, y_i > 0$ $(i = 1,2,\dots,n+1)$*,* $\sigma_n = \sum_{i=1}^{n+1} x_i$*,* $\sigma'_n = \sum_{i=1}^{n+1} y_i$*, then*

$$\sigma_n\sigma'_n - 2\sum_{i=1}^{n+1} x_iy_i \geq \frac{1}{2}\left[\frac{\sigma'_n}{\sigma_n}\left(\sigma_n^2 - 2\sum_{i=1}^{n+1} x_i^2\right) + \frac{\sigma_n}{\sigma'_n}\left((\sigma'_n)^2 - 2\sum_{i=1}^{n+1} y_i^2\right)\right], \tag{3.24}$$

with equality if and only if

$$\frac{y_1}{x_1} = \frac{y_2}{x_2} = \cdots = \frac{y_{n+1}}{x_{n+1}}.$$

Proof. Inequality (3.24) is

$$\frac{\sigma'_n}{\sigma_n}\sum_{i=1}^{n+1} x_i^2 + \frac{\sigma_n}{\sigma'_n}\sum_{i=1}^{n+1} y_i^2 \geq 2\sum_{i=1}^{n+1} x_iy_i. \tag{3.25}$$

Now we prove that inequality (3.25) holds. Using the arithmetic-geometric mean inequality, we have

$$\frac{\sigma'_n}{\sigma_n}x_i^2 + \frac{\sigma_n}{\sigma'_n}y_i^2 \geq 2x_iy_i \qquad (i = 1,2,\dots,n+1).$$

Adding up those $n+1$ inequalities, we get inequality (3.25). Equality in (3.25) holds if and only if $\frac{\sigma'_n}{\sigma_n}x_i^2 = \frac{\sigma_n}{\sigma'_n}y_i^2$ $(i = 1,2,\dots,n+1)$, i.e.

$$\frac{y_1}{x_1} = \frac{y_2}{x_2} = \cdots = \frac{y_{n+1}}{x_{n+1}} = \frac{\sigma'_n}{\sigma_n}.$$

□

Proof of Theorem 3.4. For $n = 2$, consider two triangles $(A_1A_2A_3)_\alpha$ and $(A'_1A'_2A'_3)_\beta$. Using inequality (3.14) and Lemma 3.8, we have

$$\sum_{i=1}^{3} a_i^\alpha\left(\sum_{j=1}^{3}\left(a'_j\right)^\beta - 2\left(a'_i\right)^\beta\right) \geq \frac{1}{2}\left[b_2^\alpha\frac{\sigma'_2(\beta)}{\sigma_2(\alpha)}\Delta^\alpha + b_2^\beta\frac{\sigma_2(\alpha)}{\sigma'_2(\beta)}\left(\Delta'\right)^\beta\right]. \tag{3.26}$$

Equality in (3.26) holds if and only if $a_1 = a_2 = a_3$ and $a'_1 = a'_2 = a'_3$. Hence, inequality (3.16) holds for $n = 2$.

For $n \geq 3$, taking $x_i = F_i^\alpha,\ y_i = \left(F'_i\right)^\beta$ $(i = 1,2,\dots,n+1)$ in inequality (3.24), we get

$$\begin{aligned}\sum_{i=1}^{n+1} F_i^\alpha\left(\sum_{j=1}^{n+1}\left(F'_j\right)^\beta - 2\left(F'_i\right)^\beta\right) \\ = \left(\sum_{i=1}^{n+1} F_i^\alpha\right)\left(\sum_{i=1}^{n+1}\left(F'_i\right)^\beta\right) - 2\sum_{i=1}^{n+1} F_i^\alpha\left(F'_i\right)^\beta\end{aligned} \tag{3.27}$$

$$\geq \frac{1}{2}\left\{\frac{\sigma'_n(\beta)}{\sigma_n(\alpha)}\left[\left(\sum_{i=1}^{n+1} F_i^{\alpha}\right)^2 - 2\sum_{i=1}^{n+1} F_i^{2\alpha}\right] + \frac{\sigma_n(\alpha)}{\sigma'_n(\beta)}\left[\left(\sum_{i=1}^{n+1} (F'_i)^{\beta}\right)^2 - 2\sum_{i=1}^{n+1} F_i^{2\beta}\right]\right\}.$$

Using inequality (3.27) and Lemma 3.7, we get

$$\sum_{i=1}^{n+1} F_i^{\alpha}\left(\sum_{i=1}^{n+1}\left(F'_j\right)^{\beta}\right) \geq \frac{n^2-1}{2}\left[b_n^{\alpha}\frac{\sigma'_n(\beta)}{\sigma_n(\alpha)}V^{2(n-1)\alpha/n} + b_n^{\beta}\frac{\sigma_n(\alpha)}{\sigma'_n(\beta)}V^{2(n-1)\beta/n}\right].$$

Hence, inequality (3.16) is true for $n \geq 3$. For $n \geq 3$, it is easy to see that equality in (3.16) holds if and only if two simplices Ω_n and σ'_n are regular. Theorem 3.4 is proved. □

REFERENCES

[1] F. ERIKSSON, The law of sines for tetrahedra and n-simplics, *Geometriae Dedicata*, **7** (1978), 71–80.

[2] JINGZHONG ZHANG AND LU YANG, A class of geometric inequalities concerning the mass-points system, *J. China. Univ. Sci. Techol.*, **11**(2) (1981), 1–8.

[3] JINGZHONG ZHANG AND LU YANG, A generalization to several dimensions of the Neuberg-Pedoe inequality, *Bull. Austral. Math. Soc.*, **27** (1983), 203–214.

[4] SHIGUO YANG, Three geometric inequalities for a simplex, *Geometriae Dedicata*, **57** (1995), 105–110.

[5] GANGSON LENG, Inequalities for edge-lengths and volumes of two simplices, *Geometriae Dedicata*, **68** (1997), 43–48.

[6] GANGSON LENG, Some generalizations to several dimensions of the Pedoe inequality with applications, *Acta Mathematica Sinica*, **40** (1997), 14–21.

[7] D. VELJAN, The sine theorem and inequalities for volumes of simplices and determinants, *Lin. Alg. Appl.*, **219** (1995), 79–91.

[8] SHIGUO YANG AND JIA WANG, An inequality for n-dimensional sines of vertex angles of a simplex with some aplications, *Journal of Geometry*, **54** (1995), 198–202.

[9] D.S. MITRINOVIĆ, J.E. PEČARIĆ AND V. VOLENEC, *Recent Geometric Inequalities*, Kluwer, Dordrecht, Boston, London, 1989.