

REFINEMENTS OF THE HERMITE-HADAMARD INEQUALITY FOR CONVEX FUNCTIONS

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Abstract

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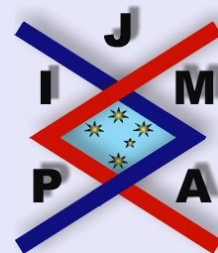


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Abstract

New refinements for the celebrated Hermite-Hadamard inequality for convex functions are obtained. Applications for special means are pointed out as well.

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1. Introduction

The following result is well known in the literature as the Hermite-Hadamard integral inequality:

$$(1.1) \quad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(t) dt \leq \frac{f(a) + f(b)}{2},$$

provided that $f : [a, b] \rightarrow \mathbb{R}$ is a convex function on $[a, b]$.

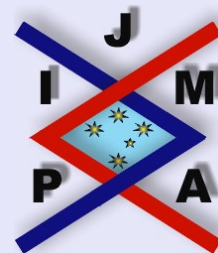
The following refinements of the H - H . inequality were obtained in [2]

$$(1.2) \quad \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) \geq \left| \frac{1}{b-a} \int_a^b \left| \frac{f(x) + f(a+b-x)}{2} \right| dx - f\left(\frac{a+b}{2}\right) \right| \geq 0.$$

and

$$(1.3) \quad \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(t) dt \geq \begin{cases} \left| |f(a)| - \frac{1}{b-a} \int_a^b |f(x)| dx \right| & \text{if } f(a) = f(b) \\ \left| \frac{1}{f(b)-f(a)} \int_{f(a)}^{f(b)} |x| dx - \frac{1}{b-a} \int_a^b |f(x)| dx \right| & \text{if } f(a) \neq f(b) \end{cases}$$

for the general case of convex functions $f : [a, b] \rightarrow \mathbb{R}$.



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If one would assume differentiability of f on (a, b) , then the following bounds in terms of its derivative holds (see [3, pp. 30-31])

$$(1.4) \quad \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(t) dt \geq \max\{|A|, |B|, |C|\} \geq 0$$

where

$$A := \frac{1}{b-a} \int_a^b \left| x - \frac{a+b}{2} \right| |f'(x)| dx - \frac{1}{4} \int_a^b |f'(x)| dx,$$

$$B := \frac{f(b) - f(a)}{4} - \frac{1}{b-a} \left[\int_a^{\frac{a+b}{2}} f(x) dx - \int_{\frac{a+b}{2}}^b f(x) dx \right]$$

and

$$C := \frac{1}{b-a} \int_a^b \left(x - \frac{a+b}{2} \right) |f'(x)| dx.$$

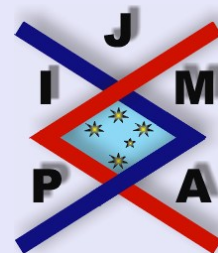
A different approach considered in [1] led to the following lower bounds

$$(1.5) \quad \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(t) dt \geq \max\{|D|, |E|, |F|\} \geq 0,$$

where

$$D := \frac{1}{b-a} \int_a^b |xf'(x)| dx - \frac{1}{b-a} \int_a^b |f'(x)| dx \cdot \frac{1}{b-a} \int_a^b |x| dx,$$

$$E := \frac{1}{b-a} \int_a^b x |f'(x)| dx - \frac{a+b}{2} \cdot \frac{1}{b-a} \int_a^b |f'(x)| dx$$



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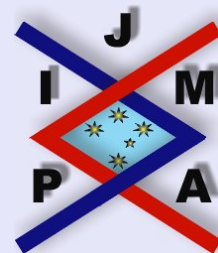
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and

$$F := \frac{1}{b-a} \int_a^b |x| f'(x) dx - \frac{f(b) - f(a)}{b-a} \cdot \frac{1}{b-a} \int_a^b |x| dx.$$

For other results connected to the $H.$ – $H.$ inequality see the recent monograph on line [3].

In the present paper, we use a different method to obtain other refinements of the $H.$ – $H.$ inequality. Applications for special means are pointed out as well.



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2. The Results

The following refinement of the Hermite-Hadamard inequality for differentiable convex functions holds.

Theorem 2.1. Assume that $f : [a, b] \rightarrow \mathbb{R}$ is differentiable convex on (a, b) . Then one has the inequality:

$$(2.1) \quad \frac{1}{b-a} \int_a^b f(t) dt - f\left(\frac{a+b}{2}\right) \geq \left| \frac{1}{b-a} \int_a^b \left| f(x) - f\left(\frac{a+b}{2}\right) \right| dx - \frac{b-a}{4} \cdot \left| f'\left(\frac{a+b}{2}\right) \right| \right| \geq 0.$$

Proof. Since f is differentiable convex on (a, b) , then for each $x, y \in (a, b)$ one has the inequality

$$(2.2) \quad f(x) - f(y) \geq (x - y) f'(y).$$

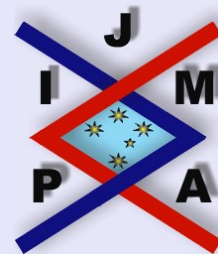
Using the properties of modulus, we have

$$(2.3) \quad f(x) - f(y) - (x - y) f'(y) = |f(x) - f(y) - (x - y) f'(y)| \geq ||f(x) - f(y)| - |x - y| |f'(y)||$$

for each $x, y \in (a, b)$.

If we choose $y = \frac{a+b}{2}$ in (2.3) we get

$$(2.4) \quad f(x) - f\left(\frac{a+b}{2}\right) - \left(x - \frac{a+b}{2}\right) f'\left(\frac{a+b}{2}\right) \geq \left| \left| f(x) - f\left(\frac{a+b}{2}\right) \right| - \left| x - \frac{a+b}{2} \right| \left| f'\left(\frac{a+b}{2}\right) \right| \right|$$



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for any $x \in (a, b)$.

Integrating (2.4) on $[a, b]$, dividing by $(b - a)$ and using the properties of modulus, we have

$$\begin{aligned} & \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) - f'\left(\frac{a+b}{2}\right) \cdot \frac{1}{b-a} \int_a^b \left(x - \frac{a+b}{2}\right) dx \\ & \geq \frac{1}{b-a} \int_a^b \left| \left| f(x) - f\left(\frac{a+b}{2}\right) \right| - \left| x - \frac{a+b}{2} \right| \left| f'\left(\frac{a+b}{2}\right) \right| \right| dx \\ & \geq \left| \frac{1}{b-a} \int_a^b \left| f(x) - f\left(\frac{a+b}{2}\right) \right| dx \right. \\ & \quad \left. - \left| f'\left(\frac{a+b}{2}\right) \right| \frac{1}{b-a} \int_a^b \left| x - \frac{a+b}{2} \right| dx \right| \end{aligned}$$

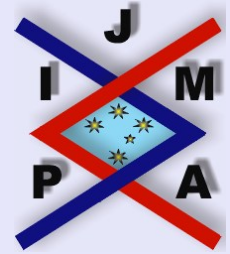
and since

$$(2.5) \quad \int_a^b \left(x - \frac{a+b}{2}\right) dx = 0, \quad \int_a^b \left| x - \frac{a+b}{2} \right| dx = \frac{(b-a)^2}{4},$$

we deduce by (2.5) the desired result (2.1). \square

The second result is embodied in the following theorem.

Theorem 2.2. Assume that $f : [a, b] \rightarrow \mathbb{R}$ is differentiable convex on (a, b) .



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Then one has the inequality

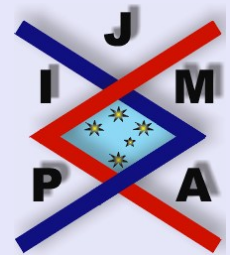
$$(2.6) \quad \frac{1}{2} \left[\frac{f(a) + f(b)}{2} + f\left(\frac{a+b}{2}\right) \right] - \frac{1}{b-a} \int_a^b f(x) dx \\ \geq \frac{1}{2} \left| \frac{1}{b-a} \int_a^b \left| f(x) - f\left(\frac{a+b}{2}\right) \right| dx \right. \\ \left. - \frac{1}{b-a} \int_a^b \left| x - \frac{a+b}{2} \right| |f'(x)| dx \right| \geq 0.$$

Proof. We choose $x = \frac{a+b}{2}$ in (2.3) to get

$$(2.7) \quad f\left(\frac{a+b}{2}\right) - f(y) - \left(\frac{a+b}{2} - y\right) f'(y) \\ \geq \left| \left| f\left(\frac{a+b}{2}\right) - f(y) \right| - \left| \frac{a+b}{2} - y \right| |f'(y)| \right|.$$

Integrating (2.7) over y , dividing by $(b-a)$ and using the modulus properties, we get

$$(2.8) \quad f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(y) dy - \int_a^b \left(\frac{a+b}{2} - y\right) f'(y) dy \\ \geq \left| \frac{1}{b-a} \int_a^b \left| f\left(\frac{a+b}{2}\right) - f(y) \right| dy \right. \\ \left. - \frac{1}{b-a} \int_a^b \left| \frac{a+b}{2} - y \right| |f'(y)| dy \right|.$$



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Since

$$\int_a^b \left(y - \frac{a+b}{2} \right) f'(y) dy = \frac{f(a) + f(b)}{2} (b-a) - \int_a^b f(t) dt,$$

then by (2.8) we deduce

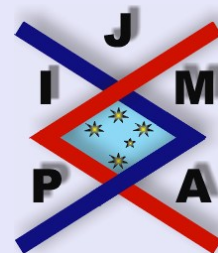
$$\begin{aligned} f\left(\frac{a+b}{2}\right) + \frac{f(a) + f(b)}{2} - \frac{2}{b-a} \int_a^b f(y) dy \\ \geq \left| \frac{1}{b-a} \int_a^b \left| f(y) - f\left(\frac{a+b}{2}\right) \right| dy \right. \\ \left. - \frac{1}{b-a} \int_a^b \left| y - \frac{a+b}{2} \right| |f'(y)| dy \right| \end{aligned}$$

which is clearly equivalent to (2.6). \square

The following result holding for the subclass of monotonic and convex functions is worth to mention.

Theorem 2.3. *Assume that $f : [a, b] \rightarrow \mathbb{R}$ is monotonic and convex on (a, b) . Then we have:*

$$(2.9) \quad \begin{aligned} \frac{1}{2} \left[\frac{f(a) + f(b)}{2} + f\left(\frac{a+b}{2}\right) \right] - \frac{1}{b-a} \int_a^b f(x) dx \\ \geq \left| \frac{1}{4} [f(b) - f(a)] + \frac{1}{b-a} \int_a^b \operatorname{sgn}\left(\frac{a+b}{2} - x\right) f(x) dx \right|. \end{aligned}$$



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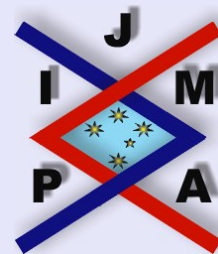
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Proof. Since the class of differentiable convex functions in (a, b) is dense in uniform topology in the class of all convex functions defined on (a, b) , we may assume, without loss of generality, that f is differentiable convex and monotonic on (a, b) .

Firstly, assume that f is monotonic nondecreasing on $[a, b]$. Then

$$\begin{aligned} \int_a^b \left| f(x) - f\left(\frac{a+b}{2}\right) \right| dx &= \int_a^{\frac{a+b}{2}} \left(f\left(\frac{a+b}{2}\right) - f(x) \right) dx \\ &\quad + \int_{\frac{a+b}{2}}^b \left(f(x) - f\left(\frac{a+b}{2}\right) \right) dx \\ &= \int_a^{\frac{a+b}{2}} f(x) dx - \int_a^{\frac{a+b}{2}} f\left(\frac{a+b}{2}\right) dx, \end{aligned}$$

$$\begin{aligned} \int_a^b \left| x - \frac{a+b}{2} \right| |f'(x)| dx &= \int_a^{\frac{a+b}{2}} \left(\frac{a+b}{2} - x \right) f'(x) dx + \int_{\frac{a+b}{2}}^b \left(x - \frac{a+b}{2} \right) f'(x) dx \\ &= \left(\frac{a+b}{2} - x \right) f(x) \Big|_a^{\frac{a+b}{2}} + \int_a^{\frac{a+b}{2}} f(x) dx \\ &\quad + \left(x - \frac{a+b}{2} \right) f(x) \Big|_{\frac{a+b}{2}}^b - \int_{\frac{a+b}{2}}^b f(x) dx \end{aligned}$$



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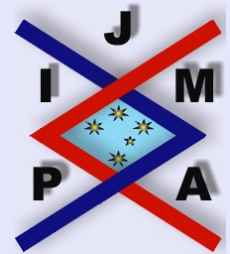
$$= -\frac{b-a}{2}f(a) + \int_a^{\frac{a+b}{2}} f(x) dx + \frac{b-a}{2}f(b) - \int_{\frac{a+b}{2}}^b f(x) dx.$$

Using (2.6) we have

$$\begin{aligned} & \frac{1}{2} \left[\frac{f(a) + f(b)}{2} + f\left(\frac{a+b}{2}\right) \right] - \frac{1}{b-a} \int_a^b f(x) dx \\ & \geq \frac{1}{2(b-a)} \left| \int_{\frac{a+b}{2}}^b f(x) dx - \int_a^{\frac{a+b}{2}} f(x) dx \right. \\ & \quad \left. - \left[\frac{b-a}{2}f(b) - \frac{b-a}{2}f(a) + \int_a^{\frac{a+b}{2}} f(x) dx - \int_{\frac{a+b}{2}}^b f(x) dx \right] \right| \\ & = \frac{1}{2(b-a)} \left| 2 \int_{\frac{a+b}{2}}^b f(x) dx - 2 \int_a^{\frac{a+b}{2}} f(x) dx - \frac{b-a}{2} [f(b) - f(a)] \right|, \end{aligned}$$

which is clearly equivalent to (2.9).

A similar argument may be done if f is monotonic nonincreasing and we omit the details. \square



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3. Applications for Special Means

Let us recall the following means:

a) The *arithmetic mean*

$$A(a, b) := \frac{a + b}{2}, \quad a, b > 0,$$

b) The *geometric mean*

$$G(a, b) := \sqrt{ab}; \quad a, b \geq 0,$$

c) The *harmonic mean*

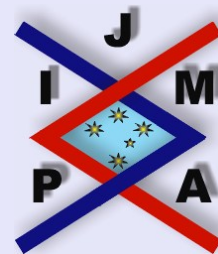
$$H(a, b) := \frac{2}{\frac{1}{a} + \frac{1}{b}}; \quad a, b > 0,$$

d) The *identric mean*

$$I(a, b) := \begin{cases} \frac{1}{e} \left(\frac{b^b}{a^a} \right)^{\frac{1}{b-a}} & \text{if } b \neq a \\ a & \text{if } b = a \end{cases}; \quad a, b > 0$$

e) The *logarithmic mean*

$$L(a, b) := \begin{cases} \frac{b - a}{\ln b - \ln a} & \text{if } b \neq a \\ a & \text{if } b = a \end{cases}; \quad a, b > 0$$



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f) The p -logarithmic mean

$$L_p(a, b) := \begin{cases} \left(\frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)} \right)^{\frac{1}{p}} & \text{if } b \neq a, \quad p \in \mathbb{R} \setminus \{-1, 0\} \\ a & \text{if } b = a \end{cases}; \quad a, b > 0.$$

It is well known that, if, on denoting $L_{-1}(a, b) := L(a, b)$ and $L_0(a, b) := I(a, b)$, then the function $\mathbb{R} \ni p \rightarrow L_p(a, b)$ is strictly monotonic increasing and, in particular, the following classical inequalities are valid

$$(3.1) \quad \min \{a, b\} \leq H(a, b) \leq G(a, b) \\ \leq L(a, b) \leq I(a, b) \leq A(a, b) \leq \max \{a, b\}$$

for any $a, b > 0$.

The following proposition holds:

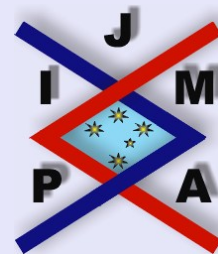
Proposition 3.1. *Let $0 < a < b < \infty$. Then we have the following refinement for the inequality $A \geq L$:*

$$(3.2) \quad A - L \geq \frac{AL}{b-a} \left[\left(\frac{G}{A} \right)^2 - \ln \left(\frac{G}{A} \right)^2 - 1 \right] \geq 0.$$

The proof follows by Theorem 2.1 on choosing $f : [a, b] \rightarrow (0, \infty)$, $f(t) = 1/t$ and we omit the details.

The following proposition contains a refinement of the following well known inequality

$$\frac{1}{2} (A^{-1} + H^{-1}) \geq L^{-1}.$$



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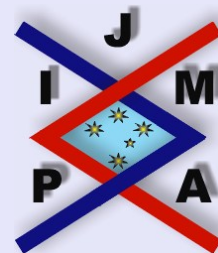
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Proposition 3.2. *With the above assumption for a and b we have*

$$(3.3) \quad \frac{1}{2} (A^{-1} + H^{-1}) - L^{-1} \geq \frac{1}{b-a} \left[\left(\frac{A}{G} \right)^2 - \ln \left(\frac{A}{G} \right)^2 - 1 \right] \geq 0.$$

The proof follows by Theorem 2.3 for the same function $f : [a, b] \rightarrow (0, \infty)$, $f(t) = 1/t$, which is monotonic and convex on $[a, b]$, and the details are omitted.

One may state other similar results that improve classical inequalities for means by choosing appropriate convex functions f . However, they will not be stated below.



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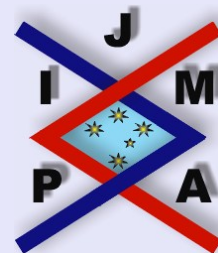
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