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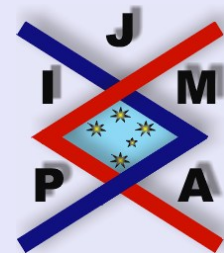
A GENERAL L_2 INEQUALITY OF GRÜSS TYPE

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Abstract

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Abstract

Based on the Euler-Maclaurin formula in the spirit of Wang [1] and sparked by Wang and Han [2], we obtain a general L_2 inequality of Grüss type, which includes some existing results as special cases.

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1. Introduction

Inequalities of the Grüss type have been the subject renewed research interest in the past few years. The monograph [3] has had much impact on the stream of current research in this area.

Inequalities of the Grüss type can be found in e.g. [4, 5, 6, 7, 8, 9, 10] and references therein.

Recently, N. Ujević in [10] proved the following two theorems among others.

Theorem 1.1. *Let $f : [0, 1] \rightarrow \mathbb{R}$ be an absolutely continuous function, whose derivatives $f' \in L_2[0, 1]$. Then,*

$$(1.1) \quad \left| \frac{1}{6} \left[f(0) + 4f\left(\frac{1}{2}\right) + f(1) \right] - \int_0^1 f(t)dt \right| \leq \frac{1}{6} \sqrt{\sigma(f')},$$

where $\sigma(\cdot)$ is defined by

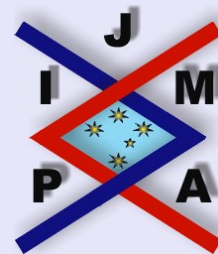
$$(1.2) \quad \sigma(f) = \|f\|_2^2 - \left(\int_0^1 f(t)dt \right)^2.$$

Inequality (1.1) is sharp in the sense that the constant $\frac{1}{6}$ cannot be replaced by a smaller one.

Theorem 1.2. *Under the assumptions of Theorem 1.1, for any $x \in [0, 1]$, we have*

$$(1.3) \quad \left| f(x) - \left(x - \frac{1}{2} \right) [f(1) - f(0)] - \int_0^1 f(t)dt \right| \leq \frac{1}{2\sqrt{3}} \sqrt{\sigma(f')}.$$

Inequality (1.3) is sharp in the sense that the constant $1/(2\sqrt{3})$ cannot be replaced by a smaller one.



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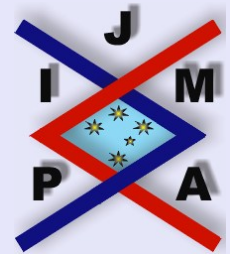
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Based on the Euler-Maclaurin formula in the spirit of Wang [1] and sparked by Wang and Han [2], we obtain a general L_2 inequality of the Grüss type under very natural assumptions. Our results improve and generalize some existing observations.



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2. A L_2 version of Grüss Type Inequality

In what follows, let f be defined on $[0, 1]$,

$$\|f\|_2 = \left(\int_0^1 f(t) dt \right)^2$$

and $L_2[0, 1] = \{f \mid \|f\|_2 < \infty\}$.

Some more notations and the following lemmas are needed before we proceed. In the rest of the paper, a standing assumption is that $x \in [0, 1]$, n is a positive integer and $0 = t_0 < t_1 < \dots < t_n = 1$ is an equidistant subdivision of the interval $[0, 1]$ such that $t_{i+1} - t_i = h = 1/n$, $i = 0, 1, \dots, n - 1$.

We start with the following lemma.

Lemma 2.1 ([1], cf. [11]). *Let $f : [0, 1] \rightarrow \mathbb{R}$ be such that its $(k - 1)$ th derivative $f^{(k-1)}$ is absolutely continuous for some positive integer k . Then for any $x \in [0, 1]$, we have the Euler-Maclaurin formula*

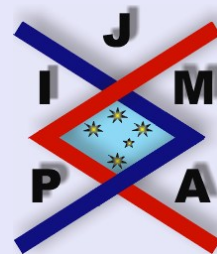
$$(2.1) \quad \int_0^1 f(t) dt = Q_k(f, x) + E_k(Q_k; f, x),$$

where

$$(2.2) \quad Q_k(f, x) = h \sum_{i=0}^{n-1} f(t_i + xh) - \sum_{\nu=1}^k \frac{f^{(\nu-1)}(1) - f^{(\nu-1)}(0)}{\nu!} B_\nu(x) h^\nu,$$

$$E_k(Q_k; f, x) = \frac{h^k}{k!} \int_0^1 \tilde{B}_k(x - nt) f^{(k)}(t) dt,$$

and $\tilde{B}_k(t) := B_k(t - [t])$ where $B_k(t)$ is the k th Bernoulli polynomial.



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Lemma 2.2. For any $x, y \in [0, 1]$, we have

$$(2.3) \quad \int_0^1 \tilde{B}_k(x-t)\tilde{B}_k(y-t)dt = \frac{(-1)^{k-1}(k!)^2}{(2k)!}\tilde{B}_{2k}(x-y).$$

Proof. We use a technique of [2]. Setting $n = 1$, $t_0 = 0$ and

$$f(t) = \frac{(-1)^k k!}{(2k)!}\tilde{B}_{2k}(x-t),$$

in (2.1), then we have

$$f^{(k)}(t) = \tilde{B}_k(x-t).$$

By the periodicity of $\tilde{B}_{2k}(t)$ and the property of $B_{2k}(t)$ (see e.g. [12]), we can easily get

$$(2.4) \quad \int_0^1 \tilde{B}_{2k}(x-t)dt = \int_0^1 B_{2k}(t)dt = 0.$$

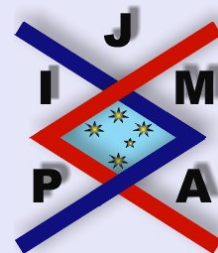
Then we have

$$(2.5) \quad \int_0^1 f(t)dt = 0.$$

From (2.2) and the periodicity of this special function f , we have for any $y \in [0, 1]$

$$(2.6) \quad Q_k(f, y) = f(y) - \sum_{\nu=1}^k \frac{f^{(\nu-1)}(1) - f^{(\nu-1)}(0)}{\nu!} B_\nu(y) = f(y),$$

$$E_k(Q_k; f, y) = \frac{1}{k!} \int_0^1 \tilde{B}_k(y-t)\tilde{B}_k(x-t)dt.$$



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Now from (2.1), (2.5) and (2.6), (2.3) follows. □

By Lemmas 2.1 and 2.2, we have

Corollary 2.3. *Suppose the conditions in Lemma 2.1 hold, then we have*

$$(2.7) \quad E_k(Q_k; f, x) \leq h^k c_k(2) \|f^{(k)}\|_2,$$

where

$$c_k(2) = \sqrt{\frac{(-1)^{k-1}}{(2k)!} B_{2k}}.$$

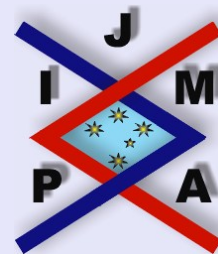
Remark 1. *From Corollary 2.3, the right side of (2.7) is independent of x .*

Lemma 2.4 ([1], cf. [11]). *Suppose that the following quadrature rule*

$$(2.8) \quad \int_0^1 f(t) dt = \sum_{j=0}^{m-1} p_j f(x_j)$$

is exact for any polynomial of degree $\leq k - 1$ for some positive integer k . Let $f : [0, 1] \rightarrow \mathbb{R}$ be such that its $(k - 1)$ th derivative $f^{(k-1)}$ is absolutely continuous. Then we have

$$(2.9) \quad \int_0^1 f(t) dt = \bar{Q}(f) + E_k(\bar{Q}; f),$$



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where

$$(2.10) \quad \bar{Q}(f) = h \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} p_j f(t_i + x_j h),$$

$$E_k(\bar{Q}; f) = \frac{h^k}{k!} \int_0^1 g_k(nt) f^{(k)}(t) dt,$$

and

$$(2.11) \quad g_k(t) = \sum_{j=0}^{m-1} p_j (\tilde{B}_k(x_j - t) - B_k(x_j)).$$

By the Hölder inequality, we have

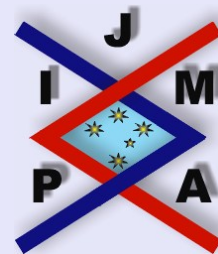
$$(2.12) \quad |E_k(\bar{Q}; f)| \leq \bar{c}_k(2) \|f^{(k)}\|_2,$$

where

$$(2.13) \quad \bar{c}_k(2) = \frac{h^k}{k!} \|g_k\|_2.$$

Remark 2. It is easy to see that (2.12) is sharp in the sense that the constant $\bar{c}_k(2)$ cannot be replaced by a smaller one.

We are now able to find an explicit expression for $\bar{c}_k(2)$.



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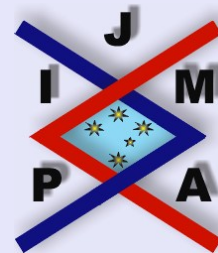
Theorem 2.5. Suppose that the quadrature rule (2.3) is exact for any polynomial of degree $\leq k - 1$ for some positive integer k . Then the following equality is valid.

$$(2.14) \quad \bar{c}_k(2) = \frac{h^k}{k!} \left\{ \sum_{i,j=0}^{m-1} p_i p_j \left(\frac{(-1)^{k-1} (k!)^2}{(2k)!} \tilde{B}_{2k}(x_i - x_j) + B_k(x_i) B_k(x_j) \right) \right\}^{\frac{1}{2}}.$$

Proof. A straightforward computation on using (2.4) and Lemma 2.2 gives

$$\begin{aligned} \|g_k\|_2^2 &= \sum_{i,j=0}^{m-1} p_i p_j \int_0^1 \left(\tilde{B}_k(x_i - t) - B_k(x_i) \right) \left(\tilde{B}_k(x_j - t) - B_k(x_j) \right) dt \\ &= \sum_{i,j=0}^{m-1} p_i p_j \int_0^1 \left(\tilde{B}_k(x_i - t) \tilde{B}_k(x_j - t) + B_k(x_i) B_k(x_j) \right) dt \\ &= \frac{(-1)^{k-1} (k!)^2}{(2k)!} \sum_{i,j=0}^{m-1} p_i p_j \tilde{B}_{2k}(x_i - x_j) + \frac{1}{k!} \sum_{i,j=0}^{m-1} p_i p_j B_k(x_i) B_k(x_j), \end{aligned}$$

which in combination with (2.13) proves the conclusion as desired. \square



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3. Examples

Example 3.1. For the Trapezoid rule, $m = 2, x_0 = 0, x_1 = 1, p_0 = p_1 = 1/2$. It is well known that the Trapezoid rule has degree of precision 1 ($k = 2$). A direct calculation using (2.14) yields

$$\bar{c}_1(2) = \frac{h}{2\sqrt{3}}, \quad \bar{c}_2(2) = \frac{h^2}{2\sqrt{30}}.$$

If f is absolutely continuous, then we can obtain

$$(3.1) \quad \left| \frac{1}{2} [f(0) + f(1)] - \int_0^1 f(t) dt \right| \leq \frac{1}{2\sqrt{3}} \|f'\|_2.$$

Replacing $f(t)$ by $f(t) - t \int_0^1 f(t) dt$ in (3.1), we get

$$(3.2) \quad \left| \frac{1}{2} [f(0) + f(1)] - \int_0^1 f(t) dt \right| \leq \frac{1}{2\sqrt{3}} \sqrt{\sigma(f')},$$

since the Trapezoid rule has degree of precision 1.

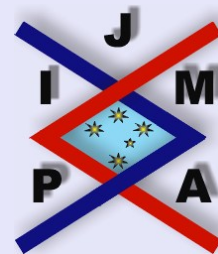
Example 3.2. Consider the following quadrature rule

$$(3.3) \quad \int_0^1 f(t) dt = \left(x - \frac{1}{2}\right) f(0) + f(x) - \left(x - \frac{1}{2}\right) f(1), \quad x \in [0, 1],$$

which has degree of precision 1 ($k = 2$). A direct calculation using Corollary 2.3 gives

$$c_1(2) = \frac{1}{2\sqrt{3}},$$

from which and the similar argument of Example 3.1, follows (1.3).



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Example 3.3. For Simpson's rule, $m = 3$, $x_0 = 0$, $x_1 = 1/2$, $x_2 = 1$, $p_0 = p_2 = 1/6$, $p_1 = 2/3$. It is well known that Simpson's rule has degree of precision 3 ($k = 4$). A direct calculation leads to the following.

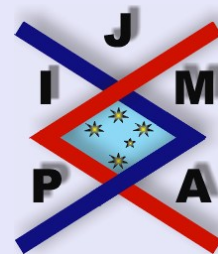
$$\bar{c}_1(2) = \frac{h}{6}; \quad \bar{c}_2(2) = \frac{h^2}{12\sqrt{30}};$$

$$\bar{c}_3(2) = \frac{h^3}{48\sqrt{105}}; \quad \bar{c}_4(2) = \frac{h^4}{576\sqrt{14}}.$$

The inequality (2.12) in combination with $\bar{c}_1(2) = h/6$ yields

$$\left| \frac{1}{6} \left[f(0) + 4f\left(\frac{1}{2}\right) + f(1) \right] - \int_0^1 f(t)dt \right| \leq \frac{1}{6} \|f'\|_2.$$

Again replacing $f(t)$ by $f(t) - t \int_0^1 f(t)dt$ in the above inequality, we easily get Theorem 1.1.



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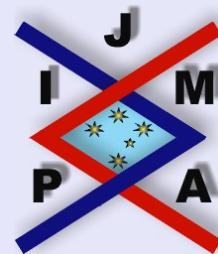
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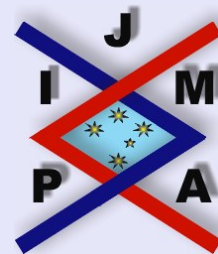
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