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A NEW PROOF OF THE MONOTONICITY PROPERTY OF POWER MEANS

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ABSTRACT. If M_r is the weighted power mean of the numbers $x_j \in [a,b]$ then $Q_r(a,b,x) = (a^r + b^r - M_r^r)^{1/r}$ is increasing in r. A new proof of this fact is given.

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1. INTRODUCTION

Suppose that $0 < a < b$, $a \le x_1 \le \cdots \le x_n \le b$ and w_i are positive weights with $\sum w_i = 1$. The weighted power means $M_r(x,w)$ of the numbers x_i with weights w_i are defined as

$$M_r(x,w) = \left(\sum w_i x_i^r\right)^{\frac{1}{r}} \quad \text{for } r \neq 0, \quad M_0(x,w) = \exp\left(\sum w_i \log x_i\right).$$

It is well-known (cf. [1, 2, 5]) that M_r increases with r unless or x_i are equal.
In [3] Mercer defined another family of functions

$$Q_r(a,b,x) = (a^r + b^r - M_r^r(x,w))^{1/r} \text{ for } r \neq 0, \quad Q_0(a,b,x) = ab/M_0$$

and proved the following

Theorem 1.1. *For $r < s \quad Q_r(a,b,x) \le Q_s(a,b,x)$.*

The aim of this note is to give another proof of this theorem. We will use the following version of the Jensen inequality ([4])

Lemma 1.2. *If f is convex then*

$$f\left(a + b - \sum w_i x_i\right) \le f(a) + f(b) - \sum w_i f(x_i). \tag{1.1}$$

For concave f the inequality reverses.

Our proof differs from the original one:

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Proof. Let $x_i = \lambda_i a + (1 - \lambda_i b)$. Then

$$\begin{aligned} f(a+b-\sum w_i x_i) &= f\left(\sum w_i[(1-\lambda_i)a + \lambda_i b]\right) \\ &\le \sum w_i f([(1-\lambda_i)a + \lambda_i b]) \\ &\le \sum w_i[(1-\lambda_i)f(a) + \lambda_i f(b)]) \\ &= \sum w_i[f(a) - \lambda_i f(a) + f(b) - (1-\lambda_i)f(b)] \\ &= f(a) + f(b) + \sum w_i[-\lambda_i f(a) - (1-\lambda_i)f(b)] \\ &\le f(a) + f(b) - \sum w_i f(x_i). \end{aligned}$$

□

2. Proof of Theorem 1.1

Proof. Let $\widetilde{a} = a^r/Q_r^r,\ \widetilde{b} = b^r/Q_r^r,\ \widetilde{x}_i = x_i^r/Q_r^r$. Applying (1.1) to the concave function $\log x$ we obtain

$$\begin{aligned} 0 = \log\left(\widetilde{a} + \widetilde{b} - \sum w_i \widetilde{x}_i\right) &\ge \log \widetilde{a} + \log \widetilde{b} - \sum w_i \log \widetilde{x}_i \\ &= r \log \frac{Q_0}{Q_r}, \end{aligned}$$

which shows that for $r > 0 \quad Q_{-r} \le Q_0 \le Q_r$.
If $0 < r < s$ then the function $f(x) = x^{s/r}$ is convex and from (1.1) we have

$$\begin{aligned} 1 = f\left(\widetilde{a} + \widetilde{b} - \sum w_i \widetilde{x}_i\right) &\le \frac{a^s}{Q_r^s} + \frac{b^s}{Q_r^s} - \sum w_i \frac{x_i^s}{Q_r^s} \\ &= \left(\frac{Q_s}{Q_r}\right)^s, \end{aligned}$$

so $Q_r \le Q_s$.
Finally, for $r < s < 0 \quad f$ is concave and we obtain $1 \ge \left(\frac{Q_s}{Q_r}\right)^s$ also equivalent to $Q_r \le Q_s$.
Obviously, equality holds if and only if all x_i's are equal a or all are equal b. □

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